Math 2940: Homework 1 Solutions

1.1

25. $$\begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ -2 & 5 & -9 & k \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & -3 & 5 & k+2g \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & 7 & g \\ 0 & 3 & -5 & h \\ 0 & 0 & 0 & k+2g+h \end{bmatrix}$$

Let b denote the number $k + 2g + h$. Then the third equation represented by the augmented matrix above is $0 = b$. This equation is possible if and only if b is zero. So the original system has a solution if and only if $k + 2g + h = 0$.

1.2

12. $$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & -7 & 0 & 6 & 5 \\ 0 & 0 & \textcircled{1} & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Corresponding system:

$$\begin{array}{rcrcrcr} \textcircled{x_1} & - & 7x_2 & & + \; 6x_4 & = & 5 \\ & & & \textcircled{x_3} & - \; 2x_4 & = & -3 \\ & & & & 0 & = & 0 \end{array}$$

Basic variables: x_1 and x_3; free variables: x_2, x_4. General solution: $\begin{cases} x_1 = 5 + 7x_2 - 6x_4 \\ x_2 \text{ is free} \\ x_3 = -3 + 2x_4 \\ x_4 \text{ is free} \end{cases}$

19. $$\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix}$$

a. When $h = 2$ and $k \neq 8$, the augmented column is a pivot column, and the system is inconsistent.

b. When $h \neq 2$, the system is consistent and has a unique solution. There are no free variables.

c. When $h = 2$ and $k = 8$, the system is consistent and has many solutions.

28. Every column in the augmented matrix *except the rightmost column* is a pivot column, and the rightmost column is *not* a pivot column.

1.3

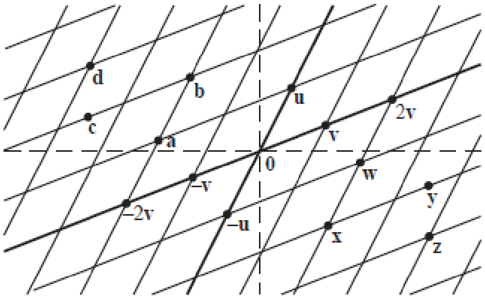

Figure for Exercises 7 and 8

8. See the figure above. Since the grid can be extended in every direction, the figure suggests that every vector in $\mathbf{R}^2$ can be written as a linear combination of **u** and **v**.

w. To reach **w** from the origin, travel –1 units in the **u**-direction (that is, 1 unit in the negative **u**-direction) and travel 2 units in the **v**-direction. Thus, $\mathbf{w} = (-1)\mathbf{u} + 2\mathbf{v}$, or $\mathbf{w} = 2\mathbf{v} - \mathbf{u}$.

x. To reach **x** from the origin, travel 2 units in the **v**-direction and –2 units in the **u**-direction. Thus, $\mathbf{x} = -2\mathbf{u} + 2\mathbf{v}$. Or, use the fact that **x** is –1 units in the **u**-direction from **w**, so that

$$\mathbf{x} = \mathbf{w} - \mathbf{u} = (-\mathbf{u} + 2\mathbf{v}) - \mathbf{u} = -2\mathbf{u} + 2\mathbf{v}$$

y. The vector **y** is 1.5 units from **x** in the **v**-direction, so

$$\mathbf{y} = \mathbf{x} + 1.5\mathbf{v} = (-2\mathbf{u} + 2\mathbf{v}) + 1.5\mathbf{v} = -2\mathbf{u} + 3.5\mathbf{v}$$

z. The map suggests that you can reach **z** if you travel 4 units in the **v**-direction and –3 units in the **u**-direction. So $\mathbf{z} = 4\mathbf{v} - 3\mathbf{u} = -3\mathbf{u} + 4\mathbf{v}$. If you prefer to stay on the paths displayed on the “map,” you might travel from the origin to $-2\mathbf{u}$, then 4 units in the **v**-direction, and finally move –1 unit in the **u**-direction. So

$$\mathbf{z} = -2\mathbf{u} + 4\mathbf{v} - \mathbf{u} = -3\mathbf{u} + 4\mathbf{v}$$

25. **a.** There are only three vectors in the set $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, and $\mathbf{b}$ is not one of them.

b. There are infinitely many vectors in $W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. To determine if $\mathbf{b}$ is in W, use the method of Exercise 13.

$$\begin{array}{c}\begin{bmatrix}1&0&-4&4\\0&3&-2&1\\-2&6&3&-4\end{bmatrix}\\ \begin{matrix}\uparrow&\uparrow&\uparrow&\uparrow\\ \mathbf{a}_1&\mathbf{a}_2&\mathbf{a}_3&\mathbf{b}\end{matrix}\end{array} \sim \begin{bmatrix}1&0&-4&4\\0&3&-2&1\\0&6&-5&4\end{bmatrix} \sim \begin{bmatrix}\textcircled{1}&0&-4&4\\0&\textcircled{3}&-2&1\\0&0&\textcircled{-1}&2\end{bmatrix}$$

The system for this augmented matrix is consistent, so $\mathbf{b}$ is in W.

c. $\mathbf{a}_1 = 1\mathbf{a}_1 + 0\mathbf{a}_2 + 0\mathbf{a}_3$. See the discussion in the text following the definition of $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

1.4

1. The matrix-vector product $A\mathbf{x}$ is not defined because the number of columns (2) in the 3×2 matrix $\begin{bmatrix}-4&2\\1&6\\0&1\end{bmatrix}$ does not match the number of entries (3) in the vector $\begin{bmatrix}3\\-2\\7\end{bmatrix}$.

16. Row reduce the augmented matrix $[A \quad \mathbf{b}]$: $A = \begin{bmatrix}1&-3&-4\\-3&2&6\\5&-1&-8\end{bmatrix}, \mathbf{b} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$.

$$\begin{bmatrix}1&-3&-4&b_1\\-3&2&6&b_2\\5&-1&-8&b_3\end{bmatrix} \sim \begin{bmatrix}1&-3&-4&b_1\\0&-7&-6&b_2+3b_1\\0&14&12&b_3-5b_1\end{bmatrix} \sim \begin{bmatrix}1&-3&-4&b_1\\0&-7&-6&b_2+3b_1\\0&0&0&b_3-5b_1+2(b_2+3b_1)\end{bmatrix}$$

$$= \begin{bmatrix}\textcircled{1}&-3&-4&b_1\\0&\textcircled{-7}&-6&b_2+3b_1\\0&0&0&b_1+2b_2+b_3\end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $b_1 + 2b_2 + b_3 = 0$. The set of such $\mathbf{b}$ is a plane through the origin in $\mathbb{R}^3$.

1.5

7. $\begin{bmatrix} 1 & 3 & -3 & 7 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 9 & -8 & 0 \\ 0 & \textcircled{1} & -4 & 5 & 0 \end{bmatrix}$. $\begin{aligned} \textcircled{x_1} \quad + 9x_3 - 8x_4 &= 0 \\ \textcircled{x_2} - 4x_3 + 5x_4 &= 0 \end{aligned}$

The basic variables are x_1 and x_2, with x_3 and x_4 free. Next, $x_1 = -9x_3 + 8x_4$, and $x_2 = 4x_3 - 5x_4$. The general solution is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 + 8x_4 \\ 4x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -9x_3 \\ 4x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 8x_4 \\ -5x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

31. **a**. When A is a 3×2 matrix with two pivot positions, each column is a pivot column. So the equation $A\mathbf{x} = \mathbf{0}$ has no free variables and hence no nontrivial solution.

b. With two pivot positions and three rows, A cannot have a pivot in every row. So the equation $A\mathbf{x} = \mathbf{b}$ cannot have a solution for every possible $\mathbf{b}$ (in $\mathbb{R}^3$), by Theorem 4 in Section 1.4.