Math 2940 HW 2: Required additional problems

1. Let

$$A = \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ -2 & 1 \end{bmatrix}.$$

The row-reduction of A has two pivots. In class we saw that this has the following consequences.

Since there is a pivot in every column:

(1) The columns of A are linearly independent.

(2) For every $\mathbf{b} \in \mathbf{R}^3$, the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution.

(3) The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.

Since there is not a pivot in every row:

(4) The columns of A do not span all of $\mathbf{R}^3$.

(5) There exist vectors $\mathbf{b} \in \mathbf{R}^3$ such that the equation $A\mathbf{x} = \mathbf{b}$ has no solutions.

(6) The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is not onto.

By row-reducing the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 3 & 2 & b_2 \\ -2 & 1 & b_3 \end{array}\right],$$

show that the equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $-7b_1 + b_2 - 2b_3 = 0$, which verifies (5). Convince yourself that this implies (4) and (6) (you do not need to write an explanation).

Suppose that b_1, b_2, b_3 are given with $-7b_1 + b_2 - 2b_3 = 0$. Use your row-reduction to find all the solutions to $A\mathbf{x} = \mathbf{b}$. (Express x_1 and x_2 in terms of b_1, b_2, b_3.) You should find that there is only one solution, which verifies (2). Explain why this implies (1) and (3).

2. Let

$$A = \begin{bmatrix} -2 & -2 & 1 \\ 3 & 6 & 0 \end{bmatrix}.$$

The row-reduction of A has two pivots.

Since there is not a pivot in every column:

(1) The columns of A are linearly dependent.

(2) There exist vectors $\mathbf{b} \in \mathbf{R}^2$ such that the equation $A\mathbf{x} = \mathbf{b}$ has more than one solution (in particular, $\mathbf{b} = \mathbf{0}$).

(3) The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is not one-to-one.

Since there is a pivot in every row:

(4) The columns of A span all of $\mathbf{R}^2$.

(5) For every $\mathbf{b} \in \mathbf{R}^2$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(6) The linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is onto.

By row-reducing the augmented matrix

$$\left[\begin{array}{ccc|c} -2 & -2 & 1 & b_1 \\ 3 & 6 & 0 & b_2 \end{array}\right],$$

for each vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ find two different solutions to $A\mathbf{x} = \mathbf{b}$. (Remember that you can choose any value you like for the free variable!) Express x_1, x_2, x_3 in terms of b_1, b_2. This verifies (5), which implies (4) and (6), and also verifies (2), which implies (1) and (3). To confirm (1), exhibit a linear dependence relation among the columns of A.