Math 2940 HW 2 — Solutions to Additional Probs.

1. $\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 3 & 2 & b_2 \\ -2 & 1 & b_3 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 2 & b_2 - 3b_1 \\ 0 & 1 & b_3 + 2b_1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_3 + 2b_1 \\ 0 & 2 & b_2 - 3b_1 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_3 + 2b_1 \\ 0 & 0 & -7b_1 + b_2 - 2b_3 \end{array}\right]$. If $-7b_1 + b_2 - 2b_3 \neq 0$, the bottom row gives an inconsistent equation, so no solutions.

If $-7b_1 + b_2 - 2b_3 = 0$, we have $\left[\begin{array}{cc|c} 1 & 0 & b_1 \\ 0 & 1 & b_3 + 2b_1 \\ 0 & 0 & 0 \end{array}\right]$

so $\left.\begin{array}{l} x_1 = b_1 \\ x_2 = b_3 + 2b_1 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} b_1 \\ b_3 + 2b_1 \end{bmatrix}$ is the unique solution to $A\vec{x} = \vec{b}$.

To prove (1) we take $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then the only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \begin{bmatrix} 0 \\ 0 + 2 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Thus, the only linear combination $x_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is $x_1 = x_2 = 0$. Hence A has lin. independent columns.

To prove (3), we must show that if $\vec{x} \neq \vec{x}'$ then $A\vec{x} \neq A\vec{x}'$. Suppose for contradiction that $A\vec{x} = A\vec{x}'$, and call this vector $\vec{b}$. From the above, we get $\vec{x} = \begin{bmatrix} b_1 \\ b_3 + 2b_1 \end{bmatrix}$ and also $\vec{x}' = \begin{bmatrix} b_1 \\ b_3 + 2b_1 \end{bmatrix}$, so $\vec{x} = \vec{x}'$ after all, which is a contradiction.

2. $\left[\begin{array}{ccc|c} -2 & -2 & 1 & b_1 \\ 3 & 6 & 0 & b_2 \end{array}\right] \xrightarrow{(R1 \to R1+R2)} \left[\begin{array}{ccc|c} 1 & 4 & 1 & b_1+b_2 \\ 3 & 6 & 0 & b_2 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 1 & b_1+b_2 \\ 0 & -6 & -3 & -3b_1-2b_2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 1 & b_1+b_2 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2}b_1+\frac{1}{3}b_2 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & -b_1-\frac{1}{3}b_2 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2}b_1+\frac{1}{3}b_2 \end{array}\right]$. (Your intermediate steps might be different, but the rref must be the same.)

Equations:

$$\begin{aligned} x_1 \quad - x_3 &= -b_1 - \tfrac{1}{3}b_2 \\ x_2 + \tfrac{1}{2}x_3 &= \tfrac{1}{2}b_1 + \tfrac{1}{3}b_2 \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} x_1 &= -b_1 - \tfrac{1}{3}b_2 + x_3 \\ x_2 &= \tfrac{1}{2}b_1 + \tfrac{1}{3}b_2 - \tfrac{1}{2}x_3 \\ x_3 &= x_3 . \end{aligned}$$

We can choose any values we like for x_3, so how about 0 and 1.

If $x_3 = 0$ we get the solution $\vec{x} = \begin{bmatrix} -b_1 - \frac{1}{3}b_2 \\ \frac{1}{2}b_1 + \frac{1}{3}b_2 \\ 0 \end{bmatrix}$.

If $x_3 = 1$ we get the solution $\vec{x} = \begin{bmatrix} -b_1 - \frac{1}{3}b_2 + 1 \\ \frac{1}{2}b_1 + \frac{1}{3}b_2 - \frac{1}{2} \\ 1 \end{bmatrix}$.

To confirm (1), we need a nontrivial solution to $A\vec{x} = \vec{0}$. Plugging in $b_1 = b_2 = 0$, we see from above that $\vec{x} = \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}$ works.

Therefore $1 \cdot \begin{bmatrix} -2 \\ 3 \end{bmatrix} - \frac{1}{2} \cdot \begin{bmatrix} -2 \\ 6 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

is a linear dependence relation among the columns of A.