# Math 2940: Homework 3 Solutions

## 2.3

(IMT = Invertible Matrix Theorem)

**15.** If $A$ has two identical columns then its columns are linearly dependent. Part (e) of the IMT shows that $A$ cannot be invertible.

**22.** Statement (g) of the IMT is false for $H$, so statement (d) is false, too. That is, the equation $H\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

**27.** Let $W$ be the inverse of $AB$. Then $ABW = I$ and $A(BW) = I$. Since $A$ is square, $A$ is invertible, by (k) of the IMT.

## 2.5

**4.** $L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$. First, solve $L\mathbf{y} = \mathbf{b}$:

$$[L \ \ \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & -5 \\ 3/2 & -5 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & -5 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -18 \end{bmatrix}, \text{ so } \mathbf{y} = \begin{bmatrix} 0 \\ -5 \\ -18 \end{bmatrix}.$$

Next solve $U\mathbf{x} = \mathbf{y}$, using back-substitution (with matrix notation):

$$[U \ \ \mathbf{y}] = \begin{bmatrix} 2 & -2 & 4 & 0 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & -6 & -18 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & 4 & 0 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & -2 & 0 & -12 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & -2 & 0 & -12 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 0 & -10 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \text{ so } \mathbf{x} = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}.$$

## 3.1

**10.** First expand along the second row, then expand along either the third row or the second column of the remaining matrix.

$$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -4 & -3 & 5 \\ 2 & 0 & 3 & 5 \end{vmatrix} = (-1)^{2+3}\cdot 3\begin{vmatrix} 1 & -2 & 2 \\ 2 & -4 & 5 \\ 2 & 0 & 5 \end{vmatrix}$$

$$= (-3)\left((-1)^{3+1}\cdot 2\begin{vmatrix} -2 & 2 \\ -4 & 5 \end{vmatrix} + (-1)^{3+3}\cdot 5\begin{vmatrix} 1 & -2 \\ 2 & -4 \end{vmatrix}\right) = (-3)(2(-2)+5(0)) = 12$$

or $$\begin{vmatrix} 1 & -2 & 5 & 2 \\ 0 & 0 & 3 & 0 \\ 2 & -4 & -3 & 5 \\ 2 & 0 & 3 & 5 \end{vmatrix} = (-1)^{2+3}\cdot 3\begin{vmatrix} 1 & -2 & 2 \\ 2 & -4 & 5 \\ 2 & 0 & 5 \end{vmatrix}$$

$$= (-3)\left((-1)^{1+2}\cdot(-2)\begin{vmatrix} 2 & 5 \\ 2 & 5 \end{vmatrix} + (-1)^{2+2}\cdot(-4)\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}\right) = (-3)(-4)(1) = 12$$

## 3.2

7. $$\begin{vmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & -4 & 2 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 30 & 27 \\ 0 & 0 & 30 & 27 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & 0 & 30 & 27 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

**24.** Since $\begin{vmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ 2 & 7 & -2 \end{vmatrix} = \begin{vmatrix} 4 & -7 & -3 \\ 6 & 0 & -5 \\ 6 & 0 & -5 \end{vmatrix} = -(-7)(0) = 0$, the columns of the matrix form a linearly dependent set.

**31.** By Theorem 6, $(\det A)(\det A^{-1}) = \det(AA^{-1}) = \det I = 1$, so $\det A^{-1} = 1/\det A$.

**3.3**

**20**. The parallelogram is determined by the columns of $A=\begin{bmatrix}-2 & 4\\ 4 & -5\end{bmatrix}$, so the area of the parallelogram is $|\det A| = |-6| = 6$.

**27**. Since the parallelogram $S$ is determined by the columns of $\begin{bmatrix}-2 & -2\\ 3 & 5\end{bmatrix}$, the area of $S$ is $\left|\det\begin{bmatrix}-2 & -2\\ 3 & 5\end{bmatrix}\right| = |-4| = 4$. The matrix $A$ has $\det A = \begin{vmatrix}6 & -3\\ -3 & 2\end{vmatrix} = 3$. By Theorem 10, the area of $T(S)$ is $|\det A|\{\text{area of } S\} = 3 \cdot 4 = 12$. Alternatively, one may compute the vectors that determine the image, namely, the columns of $A\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix} = \begin{bmatrix}6 & -3\\ -3 & 2\end{bmatrix}\begin{bmatrix}-2 & -2\\ 3 & 5\end{bmatrix} = \begin{bmatrix}-21 & -27\\ 12 & 16\end{bmatrix}$

The determinant of this matrix is –12, so the area of the image is 12.