Math 2940 HW 4: Solutions to additional problems

1. Find a basis for Nul(A):

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 3 & 2 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

which yields the equations

$$\begin{aligned} x_1 &= -2x_3 - x_4 \\ x_2 &= -2x_3 + x_4 \\ x_3 &= x_3 \\ x_4 &= \phantom{-2x_3 + {}} x_4 \end{aligned}$$

so $\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul(A). Therefore Nul(A) = Col(B) where

$$B = \begin{bmatrix} -2 & -1 \\ -2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

2. Row-reduce $\begin{bmatrix} A \mid \mathbf{b} \end{bmatrix}$:

$$\left[\begin{array}{cc|c} 2 & -6 & b_1 \\ -1 & 3 & b_2 \\ -4 & 12 & b_3 \\ 3 & -9 & b_4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & -b_2 \\ 2 & -6 & b_1 \\ -4 & 12 & b_3 \\ 3 & -9 & b_4 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & -b_2 \\ 0 & 0 & b_1 + 2b_2 \\ 0 & 0 & b_3 - 4b_2 \\ 0 & 0 & b_4 + 3b_2 \end{array}\right]$$

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if the bottom three rows are all zero, that is, when

$$\begin{aligned} b_1 + 2b_2 &= 0 \\ b_3 - 4b_2 &= 0 \\ b_4 + 3b_2 &= 0. \end{aligned}$$

(*If* all these equations are true, then the top equation $x_1 - 3x_2 = -b_2$ determines the solutions $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$, which give all the possible ways of writing $\mathbf{b}$ as a linear combination of the columns of A. For this problem, we don't care about *what* the solutions $\mathbf{x}$ are, only *whether* they exist. That question is answered by looking only at the bottom three rows of the augmented matrix.)

(Note also that if you row-reduced in a different way, you would get a different, but equivalent, list of equations that $b_1, \ldots, b_4$ must satisfy.)

This is an implicit characterization of Col(A): the vector $\mathbf{b} \in \mathbf{R}^4$ is a member of Col(A) if and only if the three equations are satisfied. We write the equations suggestively:

$$\begin{aligned} 1b_1 + 2b_2 + 0b_3 + 0b_4 &= 0 \\ 0b_1 - 4b_2 + 1b_3 + 0b_4 &= 0 \\ 0b_1 + 3b_2 + 0b_3 + 1b_4 &= 0 \end{aligned}$$

Therefore, if we let

$$B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix},$$

then Nul(B) is the set of vectors $\mathbf{b} \in \mathbf{R}^4$ satisfying the same three equations. We conclude that $\text{Col}(A) = \text{Nul}(B)$.

You should think of the final step as a way of *encoding* the three equations for the four variables $b_1, \ldots, b_4$ in matrix form.