Math 2940: Homework 4 Solutions

4.1

13. **a.** The vector **w** is not in the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. There are 3 vectors in the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

b. The set $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ contains infinitely many vectors.

c. The vector **w** is in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{w}$ has a solution. Row reducing the augmented matrix for this system of linear equations gives

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so the equation has a solution and **w** is in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

19. Let H be the set of all functions described by $y(t) = c_1\cos\omega t + c_2\sin\omega t$. Then H is a subset of the vector space V of all real-valued functions, and may be written as $H = \text{Span}\{\cos\omega t, \sin\omega t\}$. By Theorem 1, H is a subspace of V and is hence a vector space.

4.2

8. The set W is a subset of $\mathbb{R}^3$. If W were a vector space (under the standard operations in $\mathbb{R}^3$), then it would be a subspace of $\mathbb{R}^3$. But W is not a subspace of $\mathbb{R}^3$ since the zero vector is not in W. Thus W is not a vector space.

16. An element in this set may be written as $b\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} b \\ c \\ d \end{bmatrix}$, where b, c and d are any real numbers. So the set is Col A where $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}$.

4.3

11. Let $A = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$. Then we wish to find a basis for Nul A. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables: $x = -2y - z$ with y and z free. So $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and a basis for Nul A is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

14. Since B is a row echelon form of A, we see that the first, third, and fifth columns of A are its pivot columns. Thus a basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ -5 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ -2 \end{bmatrix} \right\}$.

To find a basis for Nul A, we find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables, mentally completing the row reduction of B to get: $x_1 = -2x_2 - 4x_4$, $x_3 = (7/5)x_4$, $x_5 = 0$, with x_2 and x_4 free. So $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix}$, and a basis for Nul A is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 7/5 \\ 1 \\ 0 \end{bmatrix} \right\}$.

4.4

3. We calculate that $\mathbf{x} = 3\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} + (-1)\begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 9 \end{bmatrix}$.

11. Since $P_{\mathcal{B}}^{-1}$ converts **x** into its $\mathcal{B}$-coordinate vector, we find that

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x} = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -5/2 & -3/2 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

13. We must find c_1, c_2, and c_3 such that $c_1(1+t^2)+c_2(t+t^2)+c_3(1+2t+t^2)=\mathbf{p}(t)=1+4t+7t^2$.

Equating the coefficients of the two polynomials produces the system of equations

$$\begin{array}{ccccccc} c_1 & + & & & c_3 & = & 1 \\ & & c_2 & + & 2c_3 & = & 4 \\ c_1 & + & c_2 & + & c_3 & = & 7 \end{array}$$

. We row reduce the augmented matrix for the system of equations to find

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \text{ so } [\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}.$$

One may also solve this problem using the coordinate vectors of the given polynomials relative to the standard basis $\{1, t, t^2\}$; the same system of linear equations results.

4.5

13. The matrix A is in echelon form. There are three pivot columns, so the dimension of Col A is 3. There are two columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has two free variables. Thus the dimension of Nul A is 2.