Math 2940: Homework 5 Solutions

4.6

2. The matrix B is in echelon form. There are three pivot columns, so the dimension of Col A is 3. There are three pivot rows, so the dimension of Row A is 3. There are two columns without pivots, so the equation $A\mathbf{x}=\mathbf{0}$ has two free variables. Thus the dimension of Nul A is 2. A basis for Col A is the pivot columns of A: $\left\{\begin{bmatrix}1\\-2\\-3\\3\end{bmatrix},\begin{bmatrix}4\\-6\\-6\\4\end{bmatrix},\begin{bmatrix}9\\-10\\-3\\0\end{bmatrix}\right\}$. A basis for Row A is the pivot rows of B: $\{(1,-3,0,5,-7),(0,0,2,-3,8),(0,0,0,0,5)\}$. To find a basis for Nul A row reduce to reduced echelon form: $A\sim\begin{bmatrix}1&-3&0&5&0\\0&0&1&-3/2&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix}$. The solution to $A\mathbf{x}=\mathbf{0}$ in terms of free variables is

$x_1=3x_2-5x_4$, $x_3=(3/2)x_4$, $x_5=0$, with x_2 and x_4 free. Thus a basis for Nul A is $\left\{\begin{bmatrix}3\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-5\\0\\3/2\\1\\0\end{bmatrix}\right\}$.

21. No. Consider the system as $A\mathbf{x}=\mathbf{b}$, where A is a 9×10 matrix. Since the system has a solution for all $\mathbf{b}$ in $\mathbb{R}^9$, A must have a pivot in each row, and so $\operatorname{rank} A=9$. By the Rank Theorem, $\dim \operatorname{Nul} A=10-9=1$. Thus it is impossible to find two linearly independent vectors in Nul A.

4.8

6. Let $y_k = 5^k \cos\frac{k\pi}{2}$. Then

$$y_{k+2} + 25y_k = 5^{k+2}\cos\frac{(k+2)\pi}{2} + 25\left(5^k \cos\frac{k\pi}{2}\right) = 5^k\left(5^2\cos\frac{(k+2)\pi}{2} + 25\cos\frac{k\pi}{2}\right)$$

$$= 25\cdot 5^k\left(\cos\left(\frac{k\pi}{2}+\pi\right) + \cos\frac{k\pi}{2}\right) = 25\cdot 5^k(0) = 0 \text{ for all } k$$

since $\cos(t+\pi) = -\cos t$ for all t. Since the difference equation holds for all k, $5^k\cos\frac{k\pi}{2}$ is in the solution set H.

Let $y_k = 5^k \sin\frac{k\pi}{2}$. Then

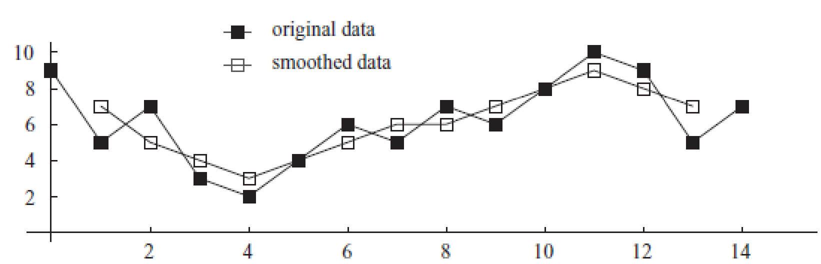
$$y_{k+2} + 25y_k = 5^{k+2}\sin\frac{(k+2)\pi}{2} + 25\left(5^k \sin\frac{k\pi}{2}\right) = 5^k\left(5^2\sin\frac{(k+2)\pi}{2} + 25\sin\frac{k\pi}{2}\right)$$

$$= 25\cdot 5^k\left(\sin\left(\frac{k\pi}{2}+\pi\right) + \sin\frac{k\pi}{2}\right) = 25\cdot 5^k(0) = 0 \text{ for all } k$$

since $\sin(t+\pi) = -\sin t$ for all t. Since the difference equation holds for all k, $5^k\sin\frac{k\pi}{2}$ is in the solution set H.

The signals $5^k\cos\frac{k\pi}{2}$ and $5^k\sin\frac{k\pi}{2}$ are linearly independent because neither is a multiple of the other. By Theorem 17, dim $H = 2$, so the two linearly independent signals $5^k\cos\frac{k\pi}{2}$ and $5^k\sin\frac{k\pi}{2}$ form a basis for H by the Basis Theorem.

21. The smoothed signal z_k has the following values: $z_1 = (9+5+7)/3 = 7$, $z_2 = (5+7+3)/3 = 5$, $z_3 = (7+3+2)/3 = 4$, $z_4 = (3+2+4)/3 = 3$, $z_5 = (2+4+6)/3 = 4$, $z_6 = (4+6+5)/3 = 5$, $z_7 = (6+5+7)/3 = 6$, $z_8 = (5+7+6)/3 = 6$, $z_9 = (7+6+8)/3 = 7$, $z_{10} = (6+8+10)/3 = 8$, $z_{11} = (8+10+9)/3 = 9$, $z_{12} = (10+9+5)/3 = 8$, $z_{13} = (9+5+7)/3 = 7$.



4.9

3. a. Let H stand for "Healthy" and I stand for "Ill." Then the students' conditions are given by the table

From: H	From: I	To:
.95	.45	H
.05	.55	I

so the stochastic matrix is $P = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}$.

b. Since 20% of the students are ill on Monday, the initial state vector is $\mathbf{x}_0 = \begin{bmatrix} .8 \\ .2 \end{bmatrix}$. For Tuesday's percentages, we calculate $\mathbf{x}_1$; for Wednesday's percentages, we calculate $\mathbf{x}_2$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix} \begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .85 \\ .15 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix} \begin{bmatrix} .85 \\ .15 \end{bmatrix} = \begin{bmatrix} .875 \\ .125 \end{bmatrix}$$

Thus 15% of the students are ill on Tuesday, and 12.5% are ill on Wednesday.

c. Since the student is well today, the initial state vector is $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We calculate $\mathbf{x}_2$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .95 \\ .05 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix} \begin{bmatrix} .95 \\ .05 \end{bmatrix} = \begin{bmatrix} .925 \\ .075 \end{bmatrix}$$

Thus the probability that the student is well two days from now is .925.

13. **a**. From Exercise 3, $P=\begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}$, so $P-I=\begin{bmatrix} -.05 & .45 \\ .05 & -.45 \end{bmatrix}$. Solving $(P-I)\mathbf{x}=\mathbf{0}$ by row reducing the augmented matrix gives $\begin{bmatrix} -.05 & .45 & 0 \\ .05 & -.45 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=x_2\begin{bmatrix} 9 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 9 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 9 \\ 1 \end{bmatrix}$ sum to 10, multiply by 1/10 to obtain the steady-state vector $\mathbf{q}=\begin{bmatrix} 9/10 \\ 1/10 \end{bmatrix}=\begin{bmatrix} .9 \\ .1 \end{bmatrix}$.

b. After many days, a specific student is ill with probability .1, and it does not matter whether that student is ill today or not.