Math 2940 HW 6: Required additional problem

1. Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

(a) Verify that the characteristic polynomial of A is $\lambda^2 - \lambda - 1$. By the quadratic formula, the roots of this polynomial are

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.6, \qquad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.6.$$

Note that λ_1 is the so-called golden ratio.

(b) Suppose that λ is either λ_1 or λ_2, so that $\lambda^2 - \lambda - 1 = 0$. Check that

$$(A - \lambda I)\begin{bmatrix} 1 \\ \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\begin{bmatrix} 1 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(c) Use part (b) to find a basis of $\mathbf{R}^2$ consisting of eigenvectors of A.

(d) Consider the discrete dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, with $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Express $\mathbf{x}_0$ as a linear combination of the eigenvectors from part (c), and use this to find a formula for $\mathbf{x}_k$.

(e) Compute $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5, \mathbf{x}_6$ not using the formula from part (d) but simply by repeated multiplication by A. (Do you notice a pattern?)

(f) The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ is defined by $F_0 = 0$, $F_1 = 1$, and then $F_{n+2} = F_n + F_{n+1}$ thereafter. Note that $\mathbf{x}_0 = \begin{bmatrix} F_0 \\ F_1 \end{bmatrix}$. Show that if $\mathbf{x}_k = \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix}$, then $\mathbf{x}_{k+1} = A\mathbf{x}_k = \begin{bmatrix} F_{k+1} \\ F_{k+2} \end{bmatrix}$. This proves by induction that the formula $\mathbf{x}_k = \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix}$ holds for all $k \geq 0$.

(g) In part (d) you computed a formula for $\mathbf{x}_k = \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix}$. Just looking at the first coordinate gives you a formula for F_k. Verify that

$$F_k = \frac{1}{\sqrt{5}} \cdot \lambda_1^k - \frac{1}{\sqrt{5}} \cdot \lambda_2^k.$$

(h) Explain why $F_k \approx \frac{1}{\sqrt{5}} \cdot \lambda_1^k$ is a good approximation when k is large. (As an example, when $k = 10$ this formula gives 55.0036, while $F_{10} = 55$.)