Math 2940 HW 6: Solutions to additional problem

1. (a) The characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = -\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1.$$

(b) Given that $\lambda^2 - \lambda - 1 = 0$,

$$(A - \lambda I)\begin{bmatrix} 1 \\ \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}\begin{bmatrix} 1 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 1+(1-\lambda)\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

since the second entry is $1 + \lambda - \lambda^2 = -(\lambda^2 - \lambda - 1) = 0$.

(c) An eigenvector basis of $\mathbf{R}^2$ is $\left\{ \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}, \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} \right\}$, where the first basis vector has eigenvalue λ_1 and the second basis vector has eigenvalue λ_2.

(d) The general solution to the dynamical system is

$$\mathbf{x}_k = c_1\lambda_1^k \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2\lambda_2^k \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

If $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then

$$c_1 \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \iff \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \iff \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We compute

$$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}^{-1} = \frac{1}{\lambda_2 - \lambda_1}\begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{\lambda_2 - \lambda_1}\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Meanwhile,

$$\lambda_2 - \lambda_1 = \frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2} = -\sqrt{5}.$$

Therefore, $c_1 = 1/\sqrt{5}$ and $c_2 = -1/\sqrt{5}$, so the formula for $\mathbf{x}_k$ is

$$\mathbf{x}_k = \frac{1}{\sqrt{5}} \cdot \lambda_1^k \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} - \frac{1}{\sqrt{5}} \cdot \lambda_2^k \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

(e) Computation:

$$\mathbf{x}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix},$$

$$\mathbf{x}_4 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \quad \mathbf{x}_5 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}, \quad \mathbf{x}_6 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \end{bmatrix}.$$

(f) If $\mathbf{x}_k = \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix}$, then $\mathbf{x}_{k+1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_k \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} F_{k+1} \\ F_k + F_{k+1} \end{bmatrix} = \begin{bmatrix} F_{k+1} \\ F_{k+2} \end{bmatrix}$.

(g) Since the first coordinate of $\mathbf{x}_k$ is F_k, the first coordinate of the formula from part (d) reads

$$F_k = \frac{1}{\sqrt{5}} \cdot \lambda_1^k - \frac{1}{\sqrt{5}} \cdot \lambda_2^k.$$

(h) When k is large, λ_2^k is very small since $|\lambda_2| \approx 0.6 < 1$. Therefore the second term in the formula

$$F_k = \frac{1}{\sqrt{5}} \cdot \lambda_1^k - \frac{1}{\sqrt{5}} \cdot \lambda_2^k$$

tends to zero, and F_k is well approximated by $\frac{1}{\sqrt{5}} \cdot \lambda_1^k$.