Math 2940: Homework 6 Solutions

5.1

12. For $\lambda = 1$: $A - I = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix}$. The augmented matrix for $(A - I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 6 & 4 & 0 \\ -3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 = (-2/3)x_2$ and x_2 is free. A basis for the eigenspace corresponding to 1 is $\begin{bmatrix} -2/3 \\ 1 \end{bmatrix}$. Another choice is $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -3 & -6 \end{bmatrix}$. The augmented matrix for $(A - 5I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 2 & 4 & 0 \\ -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 = -2x_2$ and x_2 is free. The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. A basis for the eigenspace is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

25. If λ is an eigenvalue of A, then there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Since A is invertible, $A^{-1}A\mathbf{x} = A^{-1}(\lambda\mathbf{x})$, and so $\mathbf{x} = \lambda(A^{-1}\mathbf{x})$. Since $\mathbf{x} \neq \mathbf{0}$ (and since A is invertible), λ cannot be zero. Then $\lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x}$, which shows that λ^{-1} is an eigenvalue of A^{-1}.

35. Using the figure in the exercise, plot $T(\mathbf{u})$ as $2\mathbf{u}$, because $\mathbf{u}$ is an eigenvector for the eigenvalue 2 of the standard matrix A. Likewise, plot $T(\mathbf{v})$ as $3\mathbf{v}$, because $\mathbf{v}$ is an eigenvector for the eigenvalue 3. Since T is linear, the image of $\mathbf{w}$ is $T(\mathbf{w}) = T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

5.2

12. Make a cofactor expansion along the third row:

$$\det(A-\lambda I)=\det\begin{bmatrix}-1-\lambda & 0 & 1\\ -3 & 4-\lambda & 1\\ 0 & 0 & 2-\lambda\end{bmatrix}=(2-\lambda)\cdot\det\begin{bmatrix}-1-\lambda & 0\\ -3 & 4-\lambda\end{bmatrix}$$

$$=(2-\lambda)(-1-\lambda)(4-\lambda)=-\lambda^3+5\lambda^2-2\lambda-8$$

5.3

6. As in Exercise 5, inspection of the factorization gives: $\lambda=4:\begin{bmatrix}-1\\ 2\\ 0\end{bmatrix};\lambda=5:\begin{bmatrix}-2\\ 0\\ 1\end{bmatrix},\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}$.

13. The eigenvalues of A are given to be 5 and 1.

For $\lambda=5$: $A-5I=\begin{bmatrix}-3 & 2 & -1\\ 1 & -2 & -1\\ -1 & -2 & -3\end{bmatrix}$, and row reducing $\begin{bmatrix}A-5I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}-1\\ -1\\ 1\end{bmatrix}$.

For $\lambda=1$: $A-1I=\begin{bmatrix}1 & 2 & -1\\ 1 & 2 & -1\\ -1 & -2 & 1\end{bmatrix}$, and row reducing $\begin{bmatrix}A-I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 2 & -1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix}+x_3\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_2,\mathbf{v}_3\}=\left\{\begin{bmatrix}-2\\ 1\\ 0\end{bmatrix},\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}\right\}$.

From $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}=\begin{bmatrix}-1 & -2 & 1\\ -1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}$. Then set $D=\begin{bmatrix}5 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

5.4

8. Since $[3\mathbf{b}_1 - 4\mathbf{b}_2]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$, $[T(3\mathbf{b}_1 - 4\mathbf{b}_2)]_{\mathcal{B}} = [T]_{\mathcal{B}}[3\mathbf{b}_1 - 4\mathbf{b}_2]_{\mathcal{B}} = \begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 24 \\ -20 \\ 11 \end{bmatrix}$ and $T(3\mathbf{b}_1 - 4\mathbf{b}_2) = 24\mathbf{b}_1 - 20\mathbf{b}_2 + 11\mathbf{b}_3$.

17. a. We compute that $A\mathbf{b}_1 = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\mathbf{b}_1$. so $\mathbf{b}_1$ is an eigenvector of A corresponding to the eigenvalue 2. The characteristic polynomial of A is $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$, so 2 is the only eigenvalue for A. Now $A - 2I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$, which implies that the eigenspace corresponding to the eigenvalue 2 is one-dimensional. Thus the matrix A is not diagonalizable.

b. Following Example 4, if $P = [\mathbf{b}_1 \quad \mathbf{b}_2]$, then the $\mathcal{B}$-matrix for T is

$$P^{-1}AP = \begin{bmatrix} -4 & 5 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}.$$

23. If $A\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{x} \neq 0$, then $P^{-1}A\mathbf{x} = \lambda P^{-1}\mathbf{x}$. If $B = P^{-1}AP$, then

$$B(P^{-1}\mathbf{x}) = P^{-1}AP(P^{-1}\mathbf{x}) = P^{-1}A\mathbf{x} = \lambda P^{-1}\mathbf{x} \qquad (*)$$

by the first calculation. Note that $P^{-1}\mathbf{x} \neq 0$, because $\mathbf{x} \neq 0$ and P^{-1} is invertible. Hence (*) shows that $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding to λ. (Of course, λ is an eigenvalue of both A and B because the matrices are similar, by Theorem 4 in Section 5.2.)

5.6

5. $A = \begin{bmatrix} .4 & .3 \\ -.325 & 1.2 \end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 - 1.6\lambda + .5775$. The quadratic formula provides the roots of the characteristic equation: $\lambda = \dfrac{1.6 \pm \sqrt{1.6^2 - 4(.5775)}}{2} = \dfrac{1.6 \pm \sqrt{.25}}{2} = 1.05$ and $.55$.

Because one eigenvalue is larger than one, both populations grow in size. Their relative sizes are determined eventually by the entries in the eigenvector corresponding to 1.05. Solve $(A - 1.05I)\mathbf{x} = \mathbf{0}$: $\begin{bmatrix} -.65 & .3 & 0 \\ -.325 & .15 & 0 \end{bmatrix} \sim \begin{bmatrix} -13 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. An eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 6 \\ 13 \end{bmatrix}$.

Eventually, there will be about 6 spotted owls for every 13 (thousand) flying squirrels.

9. $A = \begin{bmatrix} 1.7 & -.3 \\ -1.2 & .8 \end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 - 2.5\lambda + 1 = 0$,

$\lambda = \dfrac{2.5 \pm \sqrt{2.5^2 - 4(1)}}{2} = \dfrac{2.5 \pm \sqrt{2.25}}{2} = \dfrac{2.5 \pm 1.5}{2} = 2$ and $.5$. The origin is a saddle point because one eigenvalue is greater than 1 and the other eigenvalue is less than 1 in magnitude. The direction of greatest repulsion is through the origin and the eigenvector $\mathbf{v}_1$ found below. Solve $(A - 2I)\mathbf{x} = \mathbf{0}$: $\begin{bmatrix} -.3 & -.3 & 0 \\ -1.2 & -1.2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = -x_2$, and x_2 is free. Take $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The direction of greatest attraction is through the origin and the eigenvector $\mathbf{v}_2$ found below. Solve $(A - .5I)\mathbf{x} = \mathbf{0}$: $\begin{bmatrix} 1.2 & -.3 & 0 \\ -1.2 & .3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.25 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = -.25x_2$, and x_2 is free. Take $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

5.7

5. $A = \begin{bmatrix} 7 & -1 \\ 3 & 3 \end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 - 10\lambda + 24 = (\lambda - 4)(\lambda - 6) = 0$. Eigenvalues: 4 and 6.

<u>For $\lambda = 4$:</u> $\begin{bmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = (1/3)x_2$ with x_2 free. Take $x_2 = 3$ and $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

For $\lambda = 6$: $\begin{bmatrix} 1 & -1 & 0 \\ 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = x_2$ with x_2 free. Take $x_2 = 1$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For the initial condition $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, find c_1 and c_2 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}(0)$:

$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{x}(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 7/2 \end{bmatrix}$. Thus $c_1 = -1/2, c_2 = 7/2$, and

$$\mathbf{x}(t) = -\frac{1}{2}\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{4t} + \frac{7}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}.$$

Since both eigenvalues are positive, the origin is a repellor of the dynamical system described by $\mathbf{x}' = A\mathbf{x}$. The direction of greatest repulsion is the line through $\mathbf{v}_2$ and the origin.