Math 2940 HW 7: Required additional problems

1. This problem covers Exercises 6.4.12 and 6.4.16 in the textbook.

Let $A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}$.

(a) Denote the columns of A in order as $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$. Use the Gram-Schmidt algorithm to find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ for the column space of A. While you are doing this, find constants a_1; b_1, b_2; and c_1, c_2, c_3 such that

$$\begin{aligned} \mathbf{x}_1 &= a_1\mathbf{v}_1, \\ \mathbf{x}_2 &= b_1\mathbf{v}_1 + b_2\mathbf{v}_2, \\ \mathbf{x}_3 &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3. \end{aligned}$$

(b) Let s_1, s_2, s_3 be the lengths of the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Scale by these lengths to find an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for $\mathrm{Col}(A)$. Explain why

$$\begin{aligned} \mathbf{x}_1 &= a_1 s_1\mathbf{u}_1, \\ \mathbf{x}_2 &= b_1 s_1\mathbf{u}_1 + b_2 s_2\mathbf{u}_2, \\ \mathbf{x}_3 &= c_1 s_1\mathbf{u}_1 + c_2 s_2\mathbf{u}_2 + c_3 s_3\mathbf{u}_3. \end{aligned}$$

(c) Let $Q = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3\end{bmatrix}$. Using the equations above, find vectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ in $\mathbf{R}^3$ such that $Q\mathbf{r}_1 = \mathbf{x}_1$, $Q\mathbf{r}_2 = \mathbf{x}_2$, and $Q\mathbf{r}_3 = \mathbf{x}_3$.

(d) If $R = \begin{bmatrix}\mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3\end{bmatrix}$, explain why $A = QR$.

2. Suppose A is a 6×3 matrix with linearly independent columns. Let $A = QR$ be a QR factorization of A.

(a) What are the dimensions of the matrices Q and R?

(b) What can you say about the null spaces and column spaces of the matrices A, Q, R? (Are some of these spaces equal to others? What are the dimensions?)

(c) Consider the linear transformations $T(\mathbf{x}) = Q^TQ\mathbf{x}$ and $S(\mathbf{y}) = QQ^T\mathbf{y}$. If $T : \mathbf{R}^a \to \mathbf{R}^b$ and $S : \mathbf{R}^c \to \mathbf{R}^d$, what are the integers a, b, c, d? Give geometric descriptions of these linear transformations.