Math 2940 HW 7: Solutions to additional problems

1. (a) Given that A has columns

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{bmatrix}, \qquad \mathbf{x}_3 = \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix},$$

we set $\mathbf{v}_1 = \mathbf{x}_1$ (so $\mathbf{x}_1 = a_1\mathbf{v}_1$ with $a_1 = 1$) and compute

$$\mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 = \begin{bmatrix} 3 \\ -3 \\ 2 \\ 5 \\ 5 \end{bmatrix} - \frac{16}{4}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \mathbf{v}_2.$$

Therefore $\mathbf{x}_2 = b_1\mathbf{v}_1 + b_2\mathbf{v}_2$ with $b_1 = 4$, $b_2 = 1$. Finally,

$$\mathbf{x}_3 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix} - \frac{14}{4}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{12}{8}\begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \\ -3 \\ 3 \end{bmatrix}$$

so we can let

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{3}\left(\mathbf{x}_3 - \frac{7}{2}\mathbf{v}_1 - \frac{3}{2}\mathbf{v}_2\right),$$

which means that $\mathbf{x}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ with $c_1 = 7/2$, $c_2 = 3/2$, $c_3 = 3$. In conclusion, the orthogonal basis for $\mathrm{Col}(A)$ is

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

and we have

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{v}_1, \\ \mathbf{x}_2 &= 4\mathbf{v}_1 + \mathbf{v}_2, \\ \mathbf{x}_3 &= (7/2)\mathbf{v}_1 + (3/2)\mathbf{v}_2 + 3\mathbf{v}_3. \end{aligned}$$

(b) The lengths of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are $s_1 = \|\mathbf{v}_1\| = 2$, $s_2 = \|\mathbf{v}_2\| = \sqrt{8}$, $s_3 = \|\mathbf{v}_3\| = 2$. Therefore, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ will be an orthonormal basis for Col(A) if we let $\mathbf{u}_1 = (1/2)\mathbf{v}_1$, $\mathbf{u}_2 = (1/\sqrt{8})\mathbf{v}_2$, $\mathbf{u}_3 = (1/2)\mathbf{v}_3$. The orthonormal basis is

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{8} \\ 1/\sqrt{8} \\ 1/\sqrt{2} \\ 1/\sqrt{8} \\ 1/\sqrt{8} \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ -1/2 \\ 1/2 \end{bmatrix} \right\}.$$

(Note that $\sqrt{8} = 2\sqrt{2}$.) Since $\mathbf{v}_1 = s_1\mathbf{u}_1$, $\mathbf{v}_2 = s_2\mathbf{u}_2$, and $\mathbf{v}_3 = s_3\mathbf{u}_3$, we have

$$\begin{aligned} \mathbf{x}_1 &= s_1\mathbf{u}_1 = 2\mathbf{u}_1, \\ \mathbf{x}_2 &= 4s_1\mathbf{u}_1 + s_2\mathbf{u}_2 = 8\mathbf{u}_1 + \sqrt{8}\mathbf{u}_2, \\ \mathbf{x}_3 &= (7/2)s_1\mathbf{u}_1 + (3/2)s_2\mathbf{u}_2 + 3s_3\mathbf{u}_3 = 7\mathbf{u}_1 + 3\sqrt{2}\mathbf{u}_2 + 6\mathbf{u}_3. \end{aligned}$$

(c) The vector equations above are equivalent to the matrix equations

$$\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \mathbf{x}_1,$$

$$\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} \begin{bmatrix} 8 \\ \sqrt{8} \\ 0 \end{bmatrix} = \mathbf{x}_2,$$

$$\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} \begin{bmatrix} 7 \\ 3\sqrt{2} \\ 6 \end{bmatrix} = \mathbf{x}_3.$$

Again noting that $\sqrt{8} = 2\sqrt{2}$, we let

$$\mathbf{r}_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{r}_2 = \begin{bmatrix} 8 \\ 2\sqrt{2} \\ 0 \end{bmatrix}, \qquad \mathbf{r}_3 = \begin{bmatrix} 7 \\ 3\sqrt{2} \\ 6 \end{bmatrix}.$$

Then $Q\mathbf{r}_1 = \mathbf{x}_1$, $Q\mathbf{r}_2 = \mathbf{x}_2$, and $Q\mathbf{r}_3 = \mathbf{x}_3$.

(d) By the definition of matrix multiplication,

$$QR = Q\begin{bmatrix} \mathbf{r}_1 & \mathbf{r}_2 & \mathbf{r}_3 \end{bmatrix} = \begin{bmatrix} Q\mathbf{r}_1 & Q\mathbf{r}_2 & Q\mathbf{r}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} = A.$$

2. (a) Q is a 6×3 matrix and R is a 3×3 matrix.

(b) Since the columns of A are linearly independent, Nul$(A) = \{\mathbf{0}\} \subseteq \mathbf{R}^3$ and Col(A) is a subspace of dimension 3 of $\mathbf{R}^6$. The columns of Q are an orthonormal basis for Col(A), which implies that Col$(Q) =$ Col(A) and Nul$(Q) = \{\mathbf{0}\} \subseteq \mathbf{R}^3$. Finally, R is a 3×3 invertible matrix, so Nul$(R) = \{\mathbf{0}\} \subseteq \mathbf{R}^3$ and Col$(R) = \mathbf{R}^3$.

(c) Since Q is a 6×3 matrix, $Q^T Q$ is a 3×3 matrix and QQ^T is a 6×6 matrix. Therefore, $T(\mathbf{x}) = Q^T Q\mathbf{x}$ goes from $\mathbf{R}^3$ to $\mathbf{R}^3$, while $S(\mathbf{y}) = QQ^T\mathbf{y}$ goes from $\mathbf{R}^6$ to $\mathbf{R}^6$.

Because Q has orthonormal columns, $Q^T Q = I_3$ (the 3×3 identity matrix). Thus $T(\mathbf{x}) = \mathbf{x}$, the identity function. Meanwhile, $S(\mathbf{y}) = QQ^T\mathbf{y}$ is the orthogonal projection in $\mathbf{R}^6$ onto the 3-dimensional subspace Col$(Q) =$ Col(A). This is true by Theorem 10 in Section 6.3.