Math 2940: Homework 7 Solutions

6.1

14. Since $\mathbf{u}=\begin{bmatrix}0\\-5\\2\end{bmatrix}$ and $\mathbf{z}=\begin{bmatrix}-4\\-1\\8\end{bmatrix}$, $\|\mathbf{u}-\mathbf{z}\|^2=[0-(-4)]^2+[-5-(-1)]^2+[2-8]^2=68$ and dist $(\mathbf{u},\mathbf{z})=\sqrt{68}=2\sqrt{17}$.

28. An arbitrary $\mathbf{w}$ in Span$\{\mathbf{u},\mathbf{v}\}$ has the form $\mathbf{w}=c_1\mathbf{u}+c_2\mathbf{v}$. If $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{u}\cdot\mathbf{y}=\mathbf{v}\cdot\mathbf{y}=0$. By Theorem 1(b) and 1(c),

$$\mathbf{w}\cdot\mathbf{y}=(c_1\mathbf{u}+c_2\mathbf{v})\cdot\mathbf{y}=c_1(\mathbf{u}\cdot\mathbf{y})+c_2(\mathbf{v}\cdot\mathbf{y})=0+0=0$$

30. **a.** If $\mathbf{z}$ is in $W^\perp$, $\mathbf{u}$ is in W, and c is any scalar, then $(c\mathbf{z})\cdot\mathbf{u}=c(\mathbf{z}\cdot\mathbf{u})=c0=0$. Since $\mathbf{u}$ is any element of W, $c\mathbf{z}$ is in $W^\perp$.

b. Let $\mathbf{z}_1$ and $\mathbf{z}_2$ be in $W^\perp$. Then for any $\mathbf{u}$ in W, $(\mathbf{z}_1+\mathbf{z}_2)\cdot\mathbf{u}=\mathbf{z}_1\cdot\mathbf{u}+\mathbf{z}_2\cdot\mathbf{u}=0+0=0$. Thus $\mathbf{z}_1+\mathbf{z}_2$ is in $W^\perp$.

c. Since $\mathbf{0}$ is orthogonal to every vector, $\mathbf{0}$ is in $W^\perp$. Thus $W^\perp$ is a subspace.

6.2

10. Since $\mathbf{u}_1\cdot\mathbf{u}_2=\mathbf{u}_1\cdot\mathbf{u}_3=\mathbf{u}_2\cdot\mathbf{u}_3=0$, $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ is an orthogonal set. Since the vectors are non-zero, $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent by Theorem 4. Three such vectors in $\mathbb{R}^3$ automatically form a basis for $\mathbb{R}^3$. So $\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^3$. By Theorem 5,

$$\mathbf{x}=\frac{\mathbf{x}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1+\frac{\mathbf{x}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2+\frac{\mathbf{x}\cdot\mathbf{u}_3}{\mathbf{u}_3\cdot\mathbf{u}_3}\mathbf{u}_3=\frac{4}{3}\mathbf{u}_1+\frac{1}{3}\mathbf{u}_2+\frac{1}{3}\mathbf{u}_3$$

28. If U is an $n\times n$ orthogonal matrix, then $I=UU^{-1}=UU^T$. Since U is the transpose of U^T, Theorem 6 applied to U^T says that U^T has orthogonal columns. In particular, the columns of U^T are linearly independent and hence form a basis for $\mathbb{R}^n$ by the Invertible Matrix Theorem. That is, the rows of U form a basis (an orthonormal basis) for $\mathbb{R}^n$.

6.3

10. Since $\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0$, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set. By the Orthogonal Decomposition Theorem,

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \frac{\mathbf{y} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 = \frac{1}{3}\mathbf{u}_1 + \frac{14}{3}\mathbf{u}_2 - \frac{5}{3}\mathbf{u}_3 = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}, \mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix} \text{ and } \mathbf{y} = \hat{\mathbf{y}} + \mathbf{z},$$

where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$.

Note: Exercise 6.3.12 was not assigned, but it is an intermediate step in exercise 6.3.16.

12. Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal. The Best Approximation Theorem says that $\hat{\mathbf{y}}$, which is the orthogonal projection of $\mathbf{y}$ onto $W = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, is the closest point to $\mathbf{y}$ in W. This vector is

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = 3\mathbf{v}_1 + 1\mathbf{v}_2 = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}.$$

16. The distance from the point $\mathbf{y}$ in $\mathbb{R}^4$ to a subspace W is defined as the distance from $\mathbf{y}$ to the closest point in W. Since the closest point in W to $\mathbf{y}$ is $\hat{\mathbf{y}} = \text{proj}_W \mathbf{y}$, the desired distance is $\|\mathbf{y} - \hat{\mathbf{y}}\|$. One computes that $\hat{\mathbf{y}} = \begin{bmatrix} -1 \\ -5 \\ -3 \\ 9 \end{bmatrix}$, $\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$, and $\|\mathbf{y} - \hat{\mathbf{y}}\| = 8$.

6.4

2. Set $\mathbf{v}_1 = \mathbf{x}_1$ and compute that $\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - \frac{1}{2}\mathbf{v}_1 = \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}$. Thus an orthogonal basis for W is $\left\{ \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix} \right\}$.