Math 2940: Homework 8 Solutions

6.5

4. To find the normal equations and to find $\hat{\mathbf{x}}$, compute

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}; \; A^T \mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}.$$

a. The normal equations are $(A^T A)\mathbf{x} = A^T \mathbf{b}$: $\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$.

b. Compute $\hat{\mathbf{x}} = (\mathrm{A}^T \mathrm{A})^{-1} \mathrm{A}^T \mathbf{b} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 14 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 14 \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 24 \\ 24 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

8. From Exercise 4, $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$, and $\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Since

$$A\hat{\mathbf{x}} - \mathbf{b} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix},$$ the least squares error is $\| A\hat{\mathbf{x}} - \mathbf{b} \| = \sqrt{6}$.

10. a. Because the columns $\mathbf{a}_1$ and $\mathbf{a}_2$ of A are orthogonal, the method of Example 4 may be used to find $\hat{\mathbf{b}}$, the orthogonal projection of $\mathbf{b}$ onto Col A:

$$\hat{\mathbf{b}} = \frac{\mathbf{b} \cdot \mathbf{a}_1}{\mathbf{a}_1 \cdot \mathbf{a}_1} \mathbf{a}_1 + \frac{\mathbf{b} \cdot \mathbf{a}_2}{\mathbf{a}_2 \cdot \mathbf{a}_2} \mathbf{a}_2 = 3\mathbf{a}_1 + \frac{1}{2}\mathbf{a}_2 = 3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}.$$

b. The vector $\hat{\mathbf{x}}$ contains the weights which must be placed on $\mathbf{a}_1$ and $\mathbf{a}_2$ to produce $\hat{\mathbf{b}}$. These weights are easily read from the above equation, so $\hat{\mathbf{x}} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}$.

6.6

2. The design matrix X and the observation vector $\mathbf{y}$ are $X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix},$ and one can compute $X^T X = \begin{bmatrix} 4 & 12 \\ 12 & 46 \end{bmatrix}, X^T \mathbf{y} = \begin{bmatrix} 6 \\ 25 \end{bmatrix}, \hat{\boldsymbol{\beta}} = (X^T X)^{-1} X^T \mathbf{y} = \begin{bmatrix} -.6 \\ .7 \end{bmatrix}$. The least-squares line $y = \beta_0 + \beta_1 x$ is thus $y = -.6 + .7x$.

6.7

23. The inner product is $\langle f, g \rangle = \int_0^1 f(t) g(t)\, dt$, so $\langle f, f \rangle = \int_0^1 (1 - 3t^2)^2 dt = \int_0^1 9t^4 - 6t^2 + 1\, dt = 4/5$, and $\| f \| = \sqrt{\langle f, f \rangle} = 2/\sqrt{5}$.

25. The inner product is $\langle f, g \rangle = \int_{-1}^1 f(t) g(t) dt$. Then 1 and t are orthogonal because $\langle 1, t \rangle = \int_{-1}^1 t\, dt = 0$. So 1 and t can be in an orthogonal basis for $\text{Span}\{1, t, t^2\}$. By the Gram-Schmidt process, the third basis element in the orthogonal basis can be $t^2 - \frac{\langle t^2, 1 \rangle}{\langle 1, 1 \rangle} 1 - \frac{\langle t^2, t \rangle}{\langle t, t \rangle} t$. Since $\langle t^2, 1 \rangle = \int_{-1}^1 t^2 dt = 2/3$, $\langle 1, 1 \rangle = \int_{-1}^1 1\, dt = 2$, and $\langle t^2, t \rangle = \int_{-1}^1 t^3 dt = 0$, the third basis element can be written as $t^2 - (1/3)$. This element can be scaled by 3, which gives the orthogonal basis as $\{1, t, 3t^2 - 1\}$.

7.1

19. Let $A=\begin{bmatrix}3&-2&4\\-2&6&2\\4&2&3\end{bmatrix}$. The eigenvalues of A are 7 and –2. For $\lambda=7$, one computes that a basis for the eigenspace is $\left\{\begin{bmatrix}-1\\2\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$. This basis may be converted via orthogonal projection to an orthogonal basis for the eigenspace: $\left\{\begin{bmatrix}-1\\2\\0\end{bmatrix},\begin{bmatrix}4\\2\\5\end{bmatrix}\right\}$. These vectors can be normalized to get $\mathbf{u}_1=\begin{bmatrix}-1/\sqrt{5}\\2/\sqrt{5}\\0\end{bmatrix}$, $\mathbf{u}_2=\begin{bmatrix}4/\sqrt{45}\\2/\sqrt{45}\\5/\sqrt{45}\end{bmatrix}$. For $\lambda=-2$, one computes that a basis for the eigenspace is $\begin{bmatrix}-2\\-1\\2\end{bmatrix}$, which can be normalized to get $\mathbf{u}_3=\begin{bmatrix}-2/3\\-1/3\\2/3\end{bmatrix}$. Let $P=[\mathbf{u}_1\quad\mathbf{u}_2\quad\mathbf{u}_3]=\begin{bmatrix}-1/\sqrt{5}&4/\sqrt{45}&-2/3\\2/\sqrt{5}&2/\sqrt{45}&-1/3\\0&5/\sqrt{45}&2/3\end{bmatrix}$ and $D=\begin{bmatrix}7&0&0\\0&7&0\\0&0&-2\end{bmatrix}$. Then P orthogonally diagonalizes A, and $A=PDP^{-1}$.

23. Let $A=\begin{bmatrix}4&-1&-1\\-1&4&-1\\-1&-1&4\end{bmatrix}$. Since each row of A sums to 2, $A\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}4&-1&-1\\-1&4&-1\\-1&-1&4\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}2\\2\\2\end{bmatrix}=2\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is an eigenvector of A with corresponding eigenvalue $\lambda=2$. The eigenvector may be normalized to get $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$. For $\lambda = 5$, one computes that a basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, so $\lambda = 5$ is an eigenvalue of A. This basis may be converted via orthogonal projection to an orthogonal basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$ for the eigenspace, and these vectors can be normalized to get $\mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$. Let

$P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

7.2

6. **a**. The matrix of the quadratic form is $\begin{bmatrix} 3 & 2 & -3 \\ 2 & -2 & 0 \\ -3 & 0 & 5 \end{bmatrix}$.

b. The matrix of the quadratic form is $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 2 \\ 0 & 2 & 4 \end{bmatrix}$.

13. The matrix of the quadratic form is $A=\begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$. The eigenvalues of A are 10 and 0, so the quadratic form is positive semidefinite. An eigenvector for $\lambda = 10$ is $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$, which may be normalized to $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$. An eigenvector for $\lambda = 0$ is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, which may be normalized to $\mathbf{u}_2 = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$. Then $A = PDP^{-1}$, where $P = [\mathbf{u}_1 \quad \mathbf{u}_2] = \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$ and $D = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$. The desired change of variable is $\mathbf{x} = P\mathbf{y}$, and the new quadratic form is

$$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T D\mathbf{y} = 10y_1^2$$