Math 2940: Solutions to Section 7.4 Exercises

7.4

6. Let $A=\begin{bmatrix}-3 & 0\\ 0 & -2\end{bmatrix}$. Then $A^TA=\begin{bmatrix}9 & 0\\ 0 & 4\end{bmatrix}$, and the eigenvalues of A^TA are seen to be (in decreasing order) $\lambda_1=9$ and $\lambda_2=4$. Associated unit eigenvectors may be computed: $\lambda_1=9$: $\begin{bmatrix}1\\ 0\end{bmatrix}$, $\lambda_2=4$: $\begin{bmatrix}0\\ 1\end{bmatrix}$. Thus one choice for V is $V=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. The singular values of A are $\sigma_1=\sqrt{9}=3$ and $\sigma_2=\sqrt{4}=2$. Thus the matrix Σ is $\Sigma=\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}$. Next compute $\mathbf{u}_1=\frac{1}{\sigma_1}A\mathbf{v}_1=\begin{bmatrix}-1\\ 0\end{bmatrix}$,

$\mathbf{u}_2=\frac{1}{\sigma_2}A\mathbf{v}_2=\begin{bmatrix}0\\ -1\end{bmatrix}$. Since $\{\mathbf{u}_1,\mathbf{u}_2\}$ is a basis for $\mathbb{R}^2$, let $U=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$. Thus
$A=U\Sigma V^T=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$.

13. Let $A=\begin{bmatrix}3 & 2 & 2\\ 2 & 3 & -2\end{bmatrix}$. Then $A^T=\begin{bmatrix}3 & 2\\ 2 & 3\\ 2 & -2\end{bmatrix}$, $A^{TT}A^T=AA^T=\begin{bmatrix}17 & 8\\ 8 & 17\end{bmatrix}$, and the eigenvalues of $A^{TT}A^T$ are seen to be (in decreasing order) $\lambda_1=25$ and $\lambda_2=9$. Associated unit eigenvectors may be computed: $\lambda=25$: $\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$, $\lambda=9$: $\begin{bmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$. Thus one choice for V is
$V=\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$. The singular values of A^T are $\sigma_1=\sqrt{25}=5$ and $\sigma_2=\sqrt{9}=3$. Thus the matrix Σ is $\Sigma=\begin{bmatrix}5 & 0\\ 0 & 3\\ 0 & 0\end{bmatrix}$. Next compute $\mathbf{u}_1=\frac{1}{\sigma_1}A^T\mathbf{v}_1=\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\\ 0\end{bmatrix}$, $\mathbf{u}_2=\frac{1}{\sigma_2}A^T\mathbf{v}_2=\begin{bmatrix}-1/\sqrt{18}\\ 1/\sqrt{18}\\ -4/\sqrt{18}\end{bmatrix}$. Since $\{\mathbf{u}_1,\mathbf{u}_2\}$ is not a basis for $\mathbb{R}^3$, we need a unit vector $\mathbf{u}_3$ that is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. The vector $\mathbf{u}_3$ must satisfy the set of equations $\mathbf{u}_1^T\mathbf{x}=0$ and $\mathbf{u}_2^T\mathbf{x}=0$. These are equivalent to the linear

equations $\begin{array}{rl} x_1 + x_2 + 0x_3 &= 0 \\ -x_1 + x_2 - 4x_3 &= 0 \end{array}$, so $\mathbf{x} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$. Therefore let

$U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \\ 0 & -4/\sqrt{18} & 1/3 \end{bmatrix}$. Thus

$A^T = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \\ 0 & -4/\sqrt{18} & 1/3 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. An SVD for A is computed by

taking transposes: $A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{18} & 1/\sqrt{18} & -4/\sqrt{18} \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$.

18. Let $A = U\Sigma V^T = U\Sigma V^{-1}$. Since A is square and invertible, rank $A = n$, and all of the entries on the diagonal of Σ must be nonzero. So $A^{-1} = (U\Sigma V^{-1})^{-1} = V\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^T$.