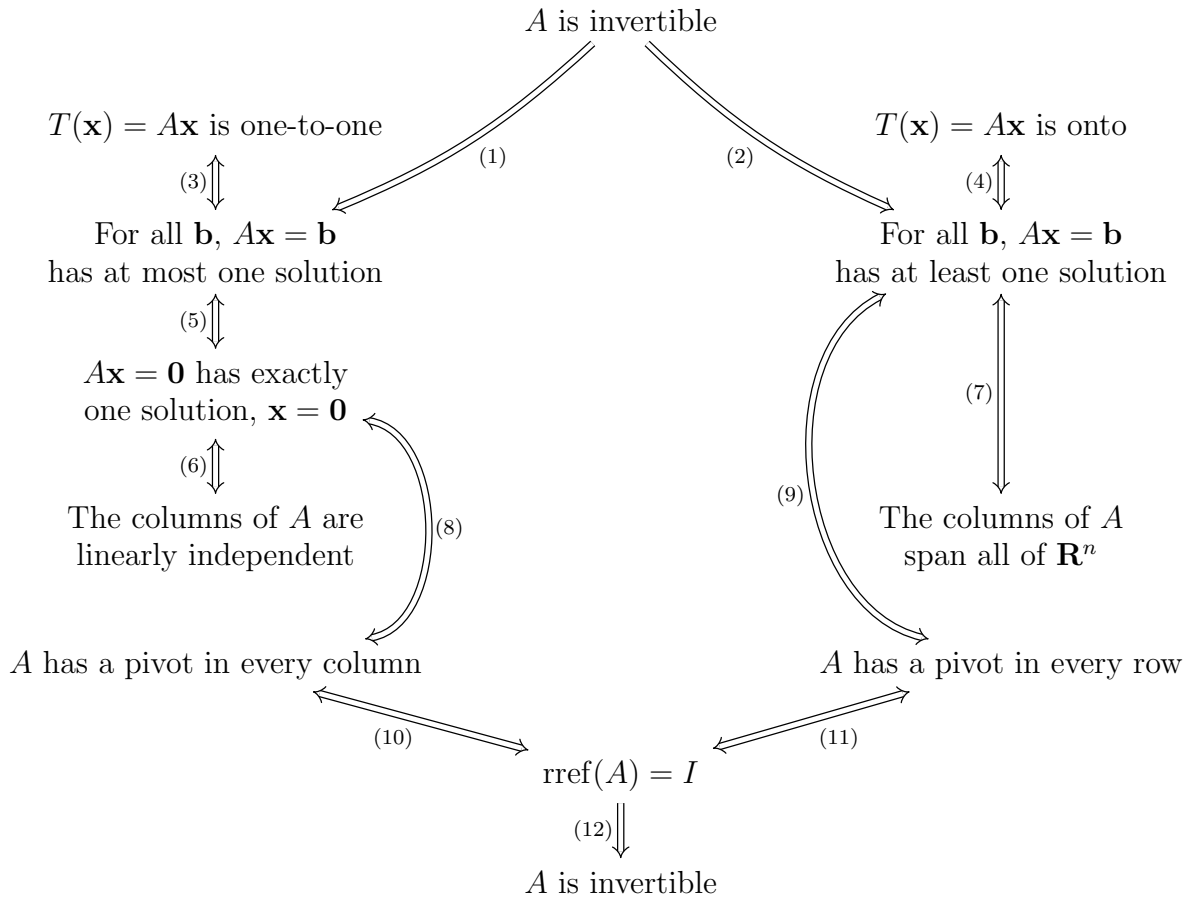


## Math 2940: Invertible matrices

The Invertible Matrix Theorem (Section 2.3, Theorem 8) has many equivalent conditions for a matrix to be invertible. This diagram is intended to help you keep track of the conditions and the relationships between them. The next page has a brief explanation for each numbered arrow.

Let  $A$  be an  $n \times n$  matrix. Then:



Besides the diagram above, there are two other pieces to the theorem.

First: just knowing that  $AB = I$ , or just knowing  $BA = I$ , is enough to conclude that  $B = A^{-1}$  (and therefore all the equivalent statements in the diagram are true). This is somewhat tricky, but it is discussed on the next page.

Second:  $A$  is invertible if and only if  $A^T$  is invertible. This is because of the formula  $(A^T)^{-1} = (A^{-1})^T$ , which can be deduced from the rule  $(AB)^T = B^T A^T$ .

(1,2): Let  $A^{-1}$  be the inverse of  $A$ . Claim:  $A\mathbf{x} = \mathbf{b} \iff \mathbf{x} = A^{-1}\mathbf{b}$ . Proof:

$$\begin{aligned} A\mathbf{x} = \mathbf{b} &\implies A^{-1}A\mathbf{x} = A^{-1}\mathbf{b} \implies I\mathbf{x} = A^{-1}\mathbf{b} \implies \mathbf{x} = A^{-1}\mathbf{b}, \\ \mathbf{x} = A^{-1}\mathbf{b} &\implies A\mathbf{x} = AA^{-1}\mathbf{b} \implies A\mathbf{x} = I\mathbf{b} \implies A\mathbf{x} = \mathbf{b}. \end{aligned}$$

It follows that the equation  $A\mathbf{x} = \mathbf{b}$  has exactly one solution, namely  $\mathbf{x} = A^{-1}\mathbf{b}$ .

(3,4): Definitions of one-to-one and onto.

(5): Downward direction: take  $\mathbf{b} = \mathbf{0}$ . Upward direction: either  $A\mathbf{x} = \mathbf{b}$  has no solutions, or the solutions to  $A\mathbf{x} = \mathbf{b}$  are a translate of the solutions to  $A\mathbf{x} = \mathbf{0}$ .

(6): Let  $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n]$ . Then  $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$  if and only if  $A \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{0}$ .

(7): The vector  $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$  is a solution to  $A\mathbf{x} = \mathbf{b}$  if and only if  $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{b}$ .

Therefore, such  $\mathbf{x}$  exists exactly when  $\mathbf{b}$  is in  $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$ .

(8): Solve  $A\mathbf{x} = \mathbf{0}$  by row reduction. If there are any free variables, there will be infinitely many solutions. If there are no free variables, the only solution will be  $\mathbf{x} = \mathbf{0}$ .

(9): Solve  $A\mathbf{x} = \mathbf{b}$  by row reduction:  $[A \ \mathbf{b}] \rightarrow [U \ \mathbf{d}]$ . If  $U$  has a pivot in every row, the system is consistent. If the bottom row of  $U$  is all zeros, take any  $\mathbf{d}$  whose last entry is nonzero, and reverse the row operations to obtain  $\mathbf{b}$  for which  $A\mathbf{x} = \mathbf{b}$  is inconsistent.

(10,11): Each row and column of  $A$  can have at most one pivot, and  $A$  is a square matrix.

(12): Use the algorithm to compute  $A^{-1}$ : if  $[A \ I]$  reduces to  $[I \ B]$ , then  $B = A^{-1}$ .

Claim: if  $A$  and  $B$  are  $n \times n$  matrices and  $BA = I$ , then  $AB = I$ .

Proof: first we show that for all  $\mathbf{b}$ ,  $A\mathbf{x} = \mathbf{b}$  has at most one solution. If  $\mathbf{x}$  is a solution, then

$$A\mathbf{x} = \mathbf{b} \implies BA\mathbf{x} = B\mathbf{b} \implies \mathbf{x} = B\mathbf{b}.$$

It is not guaranteed yet that  $\mathbf{x} = B\mathbf{b}$  is really a solution! All we know is that either there are no solutions, or  $\mathbf{x} = B\mathbf{b}$  is the only solution.

Now follow arrows (5), (8), (10), (11), (9). This means  $A\mathbf{x} = \mathbf{b}$  does have a solution, which by the previous paragraph must be  $\mathbf{x} = B\mathbf{b}$ . Thus for all  $\mathbf{b}$ ,  $A(B\mathbf{b}) = \mathbf{b}$ .

The first column of  $AB$  is  $AB\mathbf{e}_1 = \mathbf{e}_1$ . The second column of  $AB$  is  $AB\mathbf{e}_2 = \mathbf{e}_2$ , and so forth. Therefore  $AB = I$ .