

Math 2940: Prelim 1 Practice Solutions

1. Find all solutions $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ to the following system of equations:

$$\begin{aligned} 2x_1 + 4x_2 + 2x_3 + 2x_4 &= 6 \\ x_1 + 2x_2 + x_3 + x_4 &= 3 \\ -3x_1 - 6x_2 + x_3 + 5x_4 &= -5 \end{aligned}$$

Write your answer in parametric vector form, that is,

$$\mathbf{x} = \mathbf{v}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k,$$

where the vectors $\mathbf{v}_0, \dots, \mathbf{v}_k$ are specific numerical vectors in \mathbf{R}^4 that you must find, and any choice of values for the real numbers c_1, \dots, c_k yields a valid solution to the system.

Solution: Row-reduce the augmented matrix:

$$\begin{aligned} &\left[\begin{array}{cccc|c} 2 & 4 & 2 & 2 & 6 \\ 1 & 2 & 1 & 1 & 3 \\ -3 & -6 & 1 & 5 & -5 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 3 \\ 1 & 2 & 1 & 1 & 3 \\ -3 & -6 & 1 & 5 & -5 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 8 & 4 \end{array} \right] \\ &\rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \end{aligned}$$

The new equations are:

$$\begin{aligned} x_1 + 2x_2 - x_4 &= 2 \\ x_3 + 2x_4 &= 1 \end{aligned}$$

This yields

$$\begin{aligned} x_1 &= 2 - 2x_2 + x_4 \\ x_2 &= x_2 \\ x_3 &= 1 - 2x_4 \\ x_4 &= x_4 \end{aligned}$$

so the solutions are

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \quad \text{or, if you like,} \quad \mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

2. Determine whether or not the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 9 \\ 4 \\ -12 \end{bmatrix}$$

are linearly independent. If not, provide a linear dependence relation that they satisfy.

Solution: Row-reduce the matrix $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$.

$$\begin{bmatrix} 5 & 7 & 9 \\ 0 & 2 & 4 \\ 0 & -6 & -12 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 7 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since there are only two pivots, there will be a nontrivial solution to $A\mathbf{x} = \mathbf{0}$, which we can use to find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Continue the row-reduction:

$$\begin{bmatrix} 5 & 7 & 9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The solutions $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ to $A\mathbf{x} = \mathbf{0}$ satisfy the equations

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

All we need is a single nontrivial solution, so we can choose any value we like for x_3 (except $x_3 = 0$, which leads to the trivial solution). Say $x_3 = 1$, then $x_1 = 1$ and $x_2 = -2$. The matrix equation $A\mathbf{x} = \mathbf{0}$ is equivalent to the linear dependence relation

$$1 \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix} + 1 \begin{bmatrix} 9 \\ 4 \\ -12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

3. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be in \mathbf{R}^n , and let $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$. Determine whether each if-then statement is true or false. If true, explain why. If false, provide a specific numerical example of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ for which the “if” holds but the “then” does not.

(a) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then \mathbf{v}_3 is not in $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$.

Answer: True. Suppose $\mathbf{v}_3 \in \text{Span}(\mathbf{v}_1, \mathbf{v}_2)$, then one can write \mathbf{v}_3 as a linear combination of \mathbf{v}_1 and \mathbf{v}_2 , say $\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. This means

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$$

is a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. (Note that the coefficient of \mathbf{v}_3 is nonzero.)

(b) If \mathbf{v}_3 is not in $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Answer: False. Say $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then $\mathbf{v}_3 \notin \text{Span}(\mathbf{v}_1, \mathbf{v}_2)$ but $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.

(c) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

Answer: True, because if $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{0}$ with x_1, x_2, x_3 not all zero, then $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ is a linear dependence relation among the \mathbf{v}_i .

(d) If the equation $A\mathbf{x} = \mathbf{0}$ has at least one solution, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

Answer: False. The correct statement would be “If the equation $A\mathbf{x} = \mathbf{0}$ has at least one *nontrivial* solution, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.” For an easy counterexample, let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be respectively $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbf{R}^3$. The equation $A\mathbf{x} = \mathbf{0}$ does have at least one solution, namely $\mathbf{x} = \mathbf{0}$, but the \mathbf{v}_i are linearly independent.

(e) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then so is $\{3\mathbf{v}_1, 5\mathbf{v}_2, -2\mathbf{v}_3\}$.

Answer: True. We'll show that the only choice of c_1, c_2, c_3 that makes

$$c_1(3\mathbf{v}_1) + c_2(5\mathbf{v}_2) + c_3(-2\mathbf{v}_3) = \mathbf{0}$$

is $c_1 = 0, c_2 = 0, c_3 = 0$. Indeed, if the above equation is true, then

$$(3c_1)\mathbf{v}_1 + (5c_2)\mathbf{v}_2 + (-2c_3)\mathbf{v}_3 = \mathbf{0}.$$

This is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to get the zero vector. Since we are given that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, it must be true that

$$3c_1 = 0, \quad 5c_2 = 0, \quad -2c_3 = 0$$

which immediately implies that $c_1 = c_2 = c_3 = 0$.

4. (a) Let A be an $n \times n$ matrix. Suppose that for a particular \mathbf{b} in \mathbf{R}^n , the equation $A\mathbf{x} = \mathbf{b}$ has no solutions. Is A invertible? What can you say about the number of solutions to $A\mathbf{x} = \mathbf{0}$? Explain your reasoning.

Solution: If A were invertible, then for every $\mathbf{b} \in \mathbf{R}^n$ the equation $A\mathbf{x} = \mathbf{b}$ would have exactly one solution. Therefore A isn't invertible. In particular, A

has fewer than n pivots, so when we solve $A\mathbf{x} = \mathbf{0}$ there will be at least one free variable, leading to infinitely many solutions.

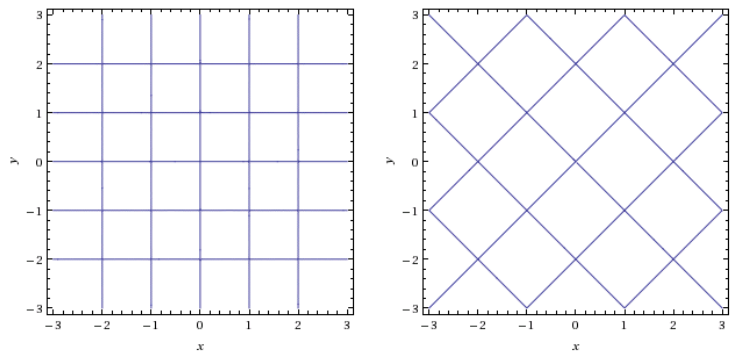
(b) Let A be an $n \times n$ matrix. Suppose that for a particular \mathbf{b} in \mathbf{R}^n , the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution. What can you say about the number of solutions to $A\mathbf{x} = \mathbf{0}$? Is A invertible? Explain your reasoning.

Solution: Since $A\mathbf{x} = \mathbf{b}$ has at least one solution, the solutions to $A\mathbf{x} = \mathbf{b}$ are a parallel translate of the solutions to $A\mathbf{x} = \mathbf{0}$. In this case there is exactly one solution to $A\mathbf{x} = \mathbf{b}$, so there's also exactly one solution to $A\mathbf{x} = \mathbf{0}$ (which has to be $\mathbf{x} = \mathbf{0}$). This implies that the columns of A are linearly independent, so A is invertible.

5. Consider the linear transformation $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$, $T(\mathbf{x}) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{x}$.

(a) Fill in the blanks: T represents a clockwise rotation of _____ degrees followed by a dilation by a factor of _____.

Since $T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the standard square grid determined by $\mathbf{e}_1, \mathbf{e}_2$ is sent by T to the diagonal grid shown below.



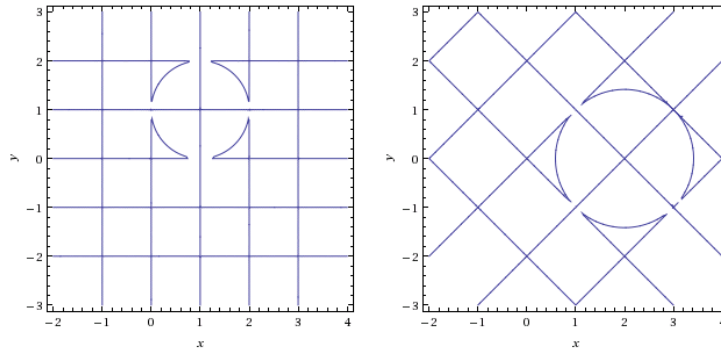
Answer: The blanks are 45 degrees and $\sqrt{2}$.

(b) Let C be the circle of radius 1 centered at $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Use a determinant to find the area of $T(C)$.

Solution: Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. The area of $T(C)$ equals $|\det A|$ multiplied by the area of C . Since $\det A = 2$ and C has area π , the area of $T(C)$ is 2π .

(c) Draw C and $T(C)$ on two separate graphs. Use the graph of $T(C)$ to verify your answer from part (b).

Solution: The grid lines are optional but they help you draw the graph of $T(C)$. (The circular curves are supposed to be fully connected.)



$T(C)$ is a circle centered at $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ passing through $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Therefore its radius is $\sqrt{2}$ and its area is 2π .

(d) What is the area of $T(T(C))$?

Solution: The area of $T(T(C))$ is $|\det A|$ multiplied by the area of $T(C)$, which is $2 \cdot 2\pi = 4\pi$.

6. (TRICK QUESTION) Consider the invertible matrix

$$A = \begin{bmatrix} 2 & 4 & -1 \\ 0 & -5 & 3 \\ 1 & 1 & 7 \end{bmatrix}.$$

Find all solutions to the equation $A^{-1}\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Solution: $A^{-1}\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ if and only if $\mathbf{x} = A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Therefore the unique solution

$$\text{is } \mathbf{x} = A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}.$$

7. Given the following determinants:

$$\begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 1 \\ -2 & 1 & -4 \end{vmatrix} = 19, \quad \begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -6$$

Compute the determinant

$$\begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 1 \\ -20 & 10 & -38 \end{vmatrix}.$$

Solution: Since

$$[-20 \ 10 \ -38] = 10[-2 \ 1 \ -4] + 2[0 \ 0 \ 1],$$

we have

$$\begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 1 \\ -20 & 10 & -38 \end{vmatrix} = 10 \begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 1 \\ -2 & 1 & -4 \end{vmatrix} + 2 \begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 190 - 12 = 178.$$

8. Let A be a 3×4 matrix. Determine whether each statement is true or false. Explain your answers.

(a) If \mathbf{b} is in the span of the columns of A , then the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

Answer: True. Label the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$. If $\mathbf{b} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 +$

$c_3\mathbf{a}_3 + c_4\mathbf{a}_4$, then $A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \mathbf{b}$.

(b) If the reduced row echelon form of A has two pivots, then the equation

$A\mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ has a solution if and only if $b_3 = 0$.

Answer: False. When solving $A\mathbf{x} = \mathbf{b}$, we set up the augmented matrix $[A \mid \mathbf{b}]$. Suppose that after row-reduction, we get the augmented matrix $[U \mid \mathbf{d}]$, where $U = \text{rref}(A)$. The first and second rows of U have pivots, while the third row is all zeros. Therefore, the original equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if the third coordinate of \mathbf{d} is zero. However, the third coordinate of \mathbf{d} will usually be different from the third coordinate of \mathbf{b} , since we performed various row operations to get from \mathbf{b} to \mathbf{d} .

(c) The columns of A are linearly dependent.

Answer: True. Any list of 4 vectors in \mathbf{R}^3 is linearly dependent, since $4 > 3$. Alternatively, the row-reduction of A must have at least one free column, which produces a nontrivial solution to $A\mathbf{x} = \mathbf{0}$.

(d) The columns of A span all of \mathbf{R}^3 .

Answer: False. (That is, it *could* be true, but it is not *necessarily* true.) For example, we could have

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

9. Use the LU factorization

$$\begin{bmatrix} 4 & 3 & -5 \\ -4 & -5 & 7 \\ 8 & 6 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

to solve the equation

$$\begin{bmatrix} 4 & 3 & -5 \\ -4 & -5 & 7 \\ 8 & 6 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}.$$

Solution: First solve

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}.$$

The equations are

$$\begin{array}{lll} y_1 = 2 & \implies & y_1 = 2 \\ -y_1 + y_2 = -4 & \implies & y_2 = -2 \\ 2y_1 + y_3 = 6 & \implies & y_3 = 2. \end{array}$$

Then solve

$$\begin{bmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}.$$

Writing the equations and working from the bottom up,

$$\begin{array}{lll} 4x_1 + 3x_2 - 5x_3 = 2 & \implies & x_1 = 1/4 \\ -2x_2 + 2x_3 = -2 & \implies & x_2 = 2 \\ 2x_3 = 2 & \implies & x_3 = 1. \end{array}$$

10. Let $S : \mathbf{R}^k \rightarrow \mathbf{R}^n$ and $T : \mathbf{R}^n \rightarrow \mathbf{R}^m$ be linear transformations given by $S(\mathbf{x}) = B\mathbf{x}$ and $T(\mathbf{y}) = A\mathbf{y}$, where B is an $n \times k$ matrix and A is an $m \times n$ matrix. Define $R : \mathbf{R}^k \rightarrow \mathbf{R}^m$ by $R(\mathbf{x}) = T(S(\mathbf{x})) = AB\mathbf{x}$.

(a) Explain why the range of R is contained in the range of T .

$$\mathbf{R}^k \xrightarrow{S} \mathbf{R}^n \xrightarrow{T} \mathbf{R}^m$$

Solution: The range of R is the set of all possible values of $T(S(\mathbf{x}))$ for $\mathbf{x} \in \mathbf{R}^k$. The range of T is the set of all possible values of $T(\mathbf{y})$ for $\mathbf{y} \in \mathbf{R}^n$. Suppose $\mathbf{b} \in \mathbf{R}^m$ is in the range of R . Then we can write $\mathbf{b} = T(S(\mathbf{x}))$ for some \mathbf{x} . If we set $\mathbf{y} = S(\mathbf{x})$, then $\mathbf{b} = T(\mathbf{y})$, which confirms that \mathbf{b} is in the range of T .

(By the way: If S is onto, so that the range of S is all of \mathbf{R}^n , then the range of R is equal to the range of T . Even if S isn't onto, it is still possible for the ranges of R and T to be equal. The question is whether the extra values of $\mathbf{y} \in \mathbf{R}^n$ that can't be expressed as $S(\mathbf{x})$ lead to new values of $T(\mathbf{y})$.)

(b) Denote the columns of A by $\mathbf{a}_1, \dots, \mathbf{a}_n$, and denote the columns of AB by $\mathbf{v}_1, \dots, \mathbf{v}_k$. What can you say about the relationship between $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n)$ and $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$?

Solution: $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n)$ equals the range of T , and $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ equals the range of R . Thus, $\text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ is contained within $\text{Span}(\mathbf{a}_1, \dots, \mathbf{a}_n)$.

Another way of seeing this is to note that every $\mathbf{b} \in \text{Span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ can be written as $\mathbf{b} = AB\mathbf{x}$ for some $\mathbf{x} \in \mathbf{R}^k$. If we write $\mathbf{b} = A(B\mathbf{x})$ and define $\mathbf{y} = B\mathbf{x}$, then the equation $\mathbf{b} = A\mathbf{y}$ expresses \mathbf{b} as a linear combination of the columns of A .

(c) Suppose that $k = n = m$, so that A , B , and AB are all square matrices. Prove that if AB is invertible, then A is invertible.

Solution: By part (b), the span of the columns of AB is contained within the span of the columns of A . If AB is invertible, then the columns of AB span all of \mathbf{R}^m , which means that the columns of A must also span all of \mathbf{R}^m . This proves that A is invertible (by the Invertible Matrix Theorem).

An alternative solution is to write $(AB)(AB)^{-1} = I$, therefore $A[B(AB)^{-1}] = I$. Let $C = B(AB)^{-1}$, so that $AC = I$. The Invertible Matrix Theorem implies that also $CA = I$, so $C = A^{-1}$.