Math 2940: Prelim 1 Practice Problems

1. Find all solutions $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ to the following system of equations:

$$\begin{aligned} 2x_1 + 4x_2 + 2x_3 + 2x_4 &= 6 \\ x_1 + 2x_2 + x_3 + x_4 &= 3 \\ -3x_1 - 6x_2 + x_3 + 5x_4 &= -5 \end{aligned}$$

Write your answer in parametric vector form, that is,

$$\mathbf{x} = \mathbf{v}_0 + c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k,$$

where the vectors $\mathbf{v}_0, \ldots, \mathbf{v}_k$ are specific numerical vectors in $\mathbf{R}^4$ that you must find, and any choice of values for the real numbers $c_1, \ldots, c_k$ yields a valid solution to the system.

2. Determine whether or not the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 9 \\ 4 \\ -12 \end{bmatrix}$$

are linearly independent. If not, provide a linear dependence relation that they satisfy.

3. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be in $\mathbf{R}^n$, and let $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$. Determine whether each if-then statement is true or false. If true, explain why. If false, provide a specific numerical example of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ for which the "if" holds but the "then" does not.

(a) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then $\mathbf{v}_3$ is not in $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$.

(b) If $\mathbf{v}_3$ is not in $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

(c) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.

(d) If the equation $A\mathbf{x} = \mathbf{0}$ has at least one solution, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

(e) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then so is $\{3\mathbf{v}_1, 5\mathbf{v}_2, -2\mathbf{v}_3\}$.

4. (a) Let A be an $n \times n$ matrix. Suppose that for a particular $\mathbf{b}$ in $\mathbf{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has no solutions. Is A invertible? What can you say about the number of solutions to $A\mathbf{x} = \mathbf{0}$? Explain your reasoning.

(b) Let A be an $n \times n$ matrix. Suppose that for a particular $\mathbf{b}$ in $\mathbf{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution. What can you say about the number of solutions to $A\mathbf{x} = \mathbf{0}$? Is A invertible? Explain your reasoning.

5. Consider the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$, $T(\mathbf{x}) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{x}$.

(a) Fill in the blanks: T represents a clockwise rotation of ________ degrees followed by a dilation by a factor of ________.

(b) Let C be the circle of radius 1 centered at $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Use a determinant to find the area of $T(C)$.

(c) Draw C and $T(C)$ on two separate graphs. Use the graph of $T(C)$ to verify your answer from part (b).

(d) What is the area of $T(T(C))$?

6. (TRICK QUESTION) Consider the invertible matrix

$$A = \begin{bmatrix} 2 & 4 & -1 \\ 0 & -5 & 3 \\ 1 & 1 & 7 \end{bmatrix}.$$

Find all solutions to the equation $A^{-1}\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

7. Given the following determinants:

$$\begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 1 \\ -2 & 1 & -4 \end{vmatrix} = 19, \qquad \begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = -6$$

Compute the determinant

$$\begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 1 \\ -20 & 10 & -38 \end{vmatrix}.$$

8. Let A be a 3×4 matrix. Determine whether each statement is true or false. Explain your answers.

(a) If $\mathbf{b}$ is in the span of the columns of A, then the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) If the reduced row echelon form of A has two pivots, then the equation $A\mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ has a solution if and only if $b_3 = 0$.

(c) The columns of A are linearly dependent.

(d) The columns of A span all of $\mathbf{R}^3$.

9. Use the LU factorization

$$\begin{bmatrix} 4 & 3 & -5 \\ -4 & -5 & 7 \\ 8 & 6 & -8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

to solve the equation

$$\begin{bmatrix} 4 & 3 & -5 \\ -4 & -5 & 7 \\ 8 & 6 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}.$$

10. Let $S : \mathbf{R}^k \to \mathbf{R}^n$ and $T : \mathbf{R}^n \to \mathbf{R}^m$ be linear transformations given by $S(\mathbf{x}) = B\mathbf{x}$ and $T(\mathbf{y}) = A\mathbf{y}$, where B is an $n \times k$ matrix and A is an $m \times n$ matrix. Define $R : \mathbf{R}^k \to \mathbf{R}^m$ by $R(\mathbf{x}) = T(S(\mathbf{x})) = AB\mathbf{x}$.

(a) Explain why the range of R is contained in the range of T.

(b) Denote the columns of A by $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and denote the columns of AB by $\mathbf{v}_1, \ldots, \mathbf{v}_k$. What can you say about the relationship between $\text{Span}(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ and $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$?

(c) Suppose that $k = n = m$, so that A, B, and AB are all square matrices. Prove that if AB is invertible, then A is invertible.