Math 2940: Prelim 1 Solutions

July 12, 2016

1. (28 points) Let

$$A = \begin{bmatrix} -2 & 4 & -2 & 1 \\ 3 & -6 & 1 & 1 \\ 1 & -2 & 0 & 1 \end{bmatrix}.$$

Denote the columns of A in order by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$.

(a) (10 points) Find all solutions to the equation $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Write your answer in parametric vector form.

Solution: Row-reduce the matrix A:

$$\begin{bmatrix} -2 & 4 & -2 & 1 \\ 3 & -6 & 1 & 1 \\ 1 & -2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 1 \\ 3 & -6 & 1 & 1 \\ -2 & 4 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The new equations are: $x_1 - 2x_2 = 0$, $x_3 = 0$, $x_4 = 0$. The only free variable is x_2, so

$$\begin{aligned} x_1 &= 2x_2 \\ x_2 &= x_2 \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

and

$$\mathbf{x} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(b) (5 points) Use part (a) to find a linear dependence relation among the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$.

Solution: Choose any nonzero value for x_2, say $x_2 = 1$. Then $A\begin{bmatrix}2\\1\\0\\0\end{bmatrix} = \mathbf{0}$, which means that $2\mathbf{a}_1 + \mathbf{a}_2 + 0\mathbf{a}_3 + 0\mathbf{a}_4 = \mathbf{0}$.

(c) (5 points) Do the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ span all of $\mathbf{R}^3$? Explain briefly why or why not.

Solution: Yes. Since the reduced row echelon form of A has a pivot in every row, for any vector $\mathbf{b} \in \mathbf{R}^3$ we can reduce the augmented matrix $\left[A \mid \mathbf{b}\right]$ and find a solution (actually infinitely many solutions) to $A\mathbf{x} = \mathbf{b}$.

(d) (2 points) Suppose that $b\mathbf{a}_2 + c\mathbf{a}_3 + d\mathbf{a}_4 = \mathbf{0}$ for some $b, c, d \in \mathbf{R}$. Find a vector $\mathbf{x} \in \mathbf{R}^4$ whose four entries can be expressed in terms of b, c, d such that $A\mathbf{x} = \mathbf{0}$.

Solution: The vector equation $b\mathbf{a}_2 + c\mathbf{a}_3 + d\mathbf{a}_4 = \mathbf{0}$ is equivalent to the matrix equation

$$\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4\end{bmatrix}\begin{bmatrix}0\\b\\c\\d\end{bmatrix} = \mathbf{0}.$$

Therefore $\mathbf{x} = \begin{bmatrix}0\\b\\c\\d\end{bmatrix}$ works.

(e) (6 points) Use parts (a) and (d) to show that $\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are linearly independent.

Solution: There are many different arguments that work for this question. Here is the one that was intended.

Suppose $b\mathbf{a}_2 + c\mathbf{a}_3 + d\mathbf{a}_4 = \mathbf{0}$. The goal is to show that $b = c = d = 0$. By part (d), we know that $A\mathbf{x} = \mathbf{0}$ where $\mathbf{x} = \begin{bmatrix} 0 \\ b \\ c \\ d \end{bmatrix}$. Also from part (a), we know that every solution to $A\mathbf{x} = \mathbf{0}$ must take the form $\mathbf{x} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Therefore,

$$\begin{bmatrix} 0 \\ b \\ c \\ d \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix}.$$

Looking at the first coordinate, $2x_2 = 0$, which implies that $b = c = d = 0$. This proves that $\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are linearly independent.

2. (20 points) Let

$$A = \begin{bmatrix} 3 & -2 & -3 \\ 1 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix}.$$

(a) (10 points) Compute A^{-1}.

Solution: Row-reduce $\begin{bmatrix} A \mid I \end{bmatrix}$ to $\begin{bmatrix} I \mid A^{-1} \end{bmatrix}$.

$$\begin{bmatrix} 3 & -2 & -3 & 1 & 0 & 0 \\ 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & -2 & -4 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -2 & 0 & 1 & 0 \\ 3 & -2 & -3 & 1 & 0 & 0 \\ 0 & -2 & -4 & 0 & 0 & 1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & -1 & -2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & -3 & 0 \\ 0 & -2 & -4 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 1 & -3 & 0 \\ 0 & 0 & 2 & 2 & -6 & 1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 1 & 1 & -2 & 0 \\ 0 & 1 & 3 & 1 & -3 & 0 \\ 0 & 0 & 1 & 1 & -3 & 1/2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1/2 \\ 0 & 1 & 0 & -2 & 6 & -3/2 \\ 0 & 0 & 1 & 1 & -3 & 1/2 \end{bmatrix}.$$

Therefore,

$$A^{-1} = \begin{bmatrix} 0 & 1 & -1/2 \\ -2 & 6 & -3/2 \\ 1 & -3 & 1/2 \end{bmatrix}.$$

(b) (5 points) Use part (a) to find all solutions to the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$.

Solution:

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \Longleftrightarrow \mathbf{x} = A^{-1} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1/2 \\ -2 & 6 & -3/2 \\ 1 & -3 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -8 \\ 3 \end{bmatrix}.$$

(c) (5 points) Compute the determinant of A by examining the row operations that you used in part (a).

Solution: Following the row operations we performed in part (a): The first one interchanged two rows, which multiplied the determinant by -1. The second and third were row replacement operations, which did not change the determinant. At that point we had an upper triangular matrix with determinant $1 \cdot 1 \cdot 2 = 2$. Therefore, $\det A = -2$.

3. (18 points) Let $S : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates each vector clockwise by 90°, and let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects each vector across the line $x_2 = -x_1$.

(a) (6 points) If $S(\mathbf{x}) = A\mathbf{x}$ and $T(\mathbf{x}) = B\mathbf{x}$, find the matrices A and B.

Solution: We have

$$S(\mathbf{e}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \qquad S(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad T(\mathbf{e}_1) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \qquad T(\mathbf{e}_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Therefore,

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

(b) (8 points) Compute $S(T(\mathbf{e}_1)), S(T(\mathbf{e}_2)), T(S(\mathbf{e}_1)), T(S(\mathbf{e}_2))$. Use these to find the matrices AB and BA. You may find it helpful to draw graphs.

Solution: Building on the work from part (a),

$$S(T(\mathbf{e}_1)) = S\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \qquad S(T(\mathbf{e}_2)) = S\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$T(S(\mathbf{e}_1)) = T\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad T(S(\mathbf{e}_2)) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Therefore,

$$AB = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad BA = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

(c) (4 points) Use geometric reasoning to explain why $A^4 = I$ and $B^2 = I$.

Solution: Given any vector $\mathbf{x} \in \mathbf{R}^2$, multiplication by A^4 rotates $\mathbf{x}$ four times, each time by 90° clockwise. This makes a full circle, so $A^4\mathbf{x} = \mathbf{x}$. It follows that $A^4 = I$. Multiplication by B^2 reflects $\mathbf{x}$ across the line $x_2 = -x_1$ and then back across, again returning to $\mathbf{x}$, so $B^2\mathbf{x} = \mathbf{x}$ and $B^2 = I$.

4. (18 points; 3 each) Given the factorization $A = LU$ below,

$$\begin{bmatrix} 2 & 1 & 2 \\ 4 & 1 & 1 \\ -4 & -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix},$$

provide short answers to the following questions. No explanations are necessary.

[However, these solutions do include brief explanations for your benefit.]

(a) Is A invertible?

Solution: No. One way to see this is that $\det A = (\det L)(\det U)$ and $\det U = 0$, so $\det A = 0$.

(b) Give a solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$. (Hint: Don't use the factorization.)

Solution: Since $\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$ is the second column of A, $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ works.

(c) How many solutions are there to $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$?

Solution: Infinitely many solutions. Since there is at least one solution from part (b), the solutions are a parallel translate of the solutions to $A\mathbf{x} = \mathbf{0}$. Since A isn't invertible, there are infinitely many of these.

(d) Is $\text{rref}(A) = \text{rref}(L)$?

Solution: No. L is invertible, so its reduced row echelon form will be the identity matrix. A isn't invertible, so its reduced row echelon form will have fewer than 3 pivots (in fact, exactly 2 pivots, because of part (e)).

(e) Is $\text{rref}(A) = \text{rref}(U)$?

Solution: Yes. The LU factorization arises as a consequence of row-reducing A: the elementary matrices encoding the row operations are collected into L, and the matrix U is an intermediate step along the reduction of A to reduced row echelon form. In fact it is true in general that if $A = BC$ and B is invertible, then B can be written as a product of elementary matrices, so C can be converted into A (and vice versa) by row operations.

(f) Can U be written as a product of elementary matrices?

Solution: No. All elementary matrices are invertible, so any product of elementary matrices must itself be invertible.

5. (16 points) Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix, so that AB is an $m \times k$ matrix. Label the columns of B in order by $\mathbf{b}_1, \ldots, \mathbf{b}_k$ and the columns of AB in order by $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

(a) (8 points) Suppose that for a particular choice of coefficients $c_1, \ldots, c_k$,

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_k\mathbf{b}_k = \mathbf{0} \in \mathbf{R}^n.$$

Show that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \in \mathbf{R}^m.$$

Solution: Turn the vector equations into matrix equations. We are given that

$$\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k\end{bmatrix}\begin{bmatrix}c_1\\c_2\\\vdots\\c_k\end{bmatrix} = \mathbf{0},$$

so if we let $\mathbf{c} = \begin{bmatrix}c_1\\\vdots\\c_k\end{bmatrix}$, we have $B\mathbf{c} = \mathbf{0}$. We are asked to prove that

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k\end{bmatrix}\begin{bmatrix}c_1\\c_2\\\vdots\\c_k\end{bmatrix} = \mathbf{0},$$

which is the same as $(AB)\mathbf{c} = \mathbf{0}$. Now the proof is very short:

$$B\mathbf{c} = \mathbf{0} \Longrightarrow AB\mathbf{c} = A\mathbf{0} = \mathbf{0}.$$

An alternative argument is to recall the rule for matrix multiplication:

$$A\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k\end{bmatrix} = \begin{bmatrix}A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k\end{bmatrix}.$$

Therefore, $\mathbf{v}_1 = A\mathbf{b}_1, \ldots, \mathbf{v}_k = A\mathbf{b}_k$. Given that

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_k\mathbf{b}_k = \mathbf{0},$$

we multiply by A:

$$\begin{aligned} A(c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_k\mathbf{b}_k) &= A\mathbf{0} \\ c_1(A\mathbf{b}_1) + c_2(A\mathbf{b}_2) + \cdots + c_k(A\mathbf{b}_k) &= \mathbf{0} \\ c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k &= \mathbf{0}. \end{aligned}$$

(b) (8 points) If the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one, show that the converse of part (a) holds: If

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0},$$

then

$$c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_k\mathbf{b}_k = \mathbf{0}.$$

Solution: From part (a), we rephrase the vector equations as matrix equations. Given that $AB\mathbf{c} = \mathbf{0}$ and that $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one, we want to show that $B\mathbf{c} = \mathbf{0}$. Of course it is not true in general that $AB\mathbf{c} = \mathbf{0} \implies B\mathbf{c} = \mathbf{0}$, since it's possible that $B\mathbf{c}$ could be some nonzero vector $\mathbf{x}$ for which $A\mathbf{x} = \mathbf{0}$. But, since T is one-to-one, the only vector $\mathbf{x} \in \mathbf{R}^n$ for which $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. So the argument goes like this: Given that $A(B\mathbf{c}) = \mathbf{0}$, and because $A\mathbf{x} = \mathbf{0}$ implies that $\mathbf{x} = \mathbf{0}$, it follows that $B\mathbf{c} = \mathbf{0}$.