

## Math 2940: Prelim 2 Practice Solutions

1. Given the following matrix with its reduced row echelon form:

$$A = \begin{bmatrix} 1 & 0 & -2 & 2 \\ -2 & 1 & 5 & -7 \\ 0 & -1 & -1 & 3 \\ 3 & -2 & -8 & 12 \\ 2 & -2 & -6 & 10 \end{bmatrix}, \quad \text{rref}(A) = U = \begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What is the rank of  $A$ ? Find bases for  $\text{Col}(A)$ ,  $\text{Nul}(A)$ , and  $\text{Row}(A)$ .

Solution:  $\text{rank}(A) = 2$ . A basis for  $\text{Col}(A)$  is  $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} \right\}$ . A basis for

$\text{Row}(A)$  is  $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -3 \end{bmatrix} \right\}$ . For  $\text{Nul}(A)$ , put the solution to  $A\mathbf{x} = \mathbf{0}$  in parametric vector form:

$$\begin{aligned} x_1 &= 2x_3 - 2x_4 \\ x_2 &= -x_3 + 3x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

so  $\mathbf{x} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \end{bmatrix}$ , and  $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$  is a basis for  $\text{Nul}(A)$ .

(b) Is  $\text{Col}(A) = \text{Col}(U)$ ? Is  $\text{Nul}(A) = \text{Nul}(U)$ ? Is  $\text{Row}(A) = \text{Row}(U)$ ? Explain your answers.

Solution: No,  $\text{Col}(A)$  is different from  $\text{Col}(U)$ . Both are two-dimensional sub-

spaces of  $\mathbf{R}^5$ , but the vectors in  $\text{Col}(U)$  have the form  $\begin{bmatrix} x_1 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ .

Yes,  $\text{Nul}(A) = \text{Nul}(U)$ . Indeed, row operations preserve the solutions to the equation  $A\mathbf{x} = \mathbf{0}$ , so  $A\mathbf{x} = \mathbf{0}$  and  $U\mathbf{x} = \mathbf{0}$  have the same solution set.

Yes,  $\text{Row}(A) = \text{Row}(U)$ . Row operations do not change the span of the rows of the matrix.

2. Let  $\mathbb{P}_3$  be the space of all polynomials  $p(t) = c_0 + c_1t + c_2t^2 + c_3t^3$  with degree at most 3 and real coefficients.

(a) Prove that  $\mathcal{B} = \{1, t, t^2, t^3\}$  is a basis for  $\mathbb{P}_3$  by verifying directly that the elements of  $\mathcal{B}$  are linearly independent and span all of  $\mathbb{P}_3$ .

Solution: For linear independence, suppose that

$$a_1(1) + a_2(t) + a_3(t^2) + a_4(t^3) = 0 + 0t + 0t^2 + 0t^3$$

is a linear combination of  $1, t, t^2, t^3$  that equals the zero polynomial. It is immediate that  $a_1 = a_2 = a_3 = a_4 = 0$ . Therefore, the set  $\{1, t, t^2, t^3\}$  is linearly independent.

For spanning, we must be able to express any polynomial  $p(t) = c_0 + c_1t + c_2t^2 + c_3t^3 \in \mathbb{P}_3$  as a linear combination of  $1, t, t^2, t^3$ . This is easily done:

$$p(t) = c_0(1) + c_1(t) + c_2(t^2) + c_3(t^3)$$

is the appropriate linear combination.

(b) Let  $T : \mathbb{P}_3 \rightarrow \mathbb{P}_3$  be the “differentiation operator”: if  $p(t) = c_0 + c_1t + c_2t^2 + c_3t^3$ , then

$$(T(p))(t) = p'(t) = c_1 + 2c_2t + 3c_3t^2.$$

Prove that  $T$  is a linear transformation.

Solution: We must check that  $T(p + q) = T(p) + T(q)$  and  $T(rp) = rT(p)$  for  $p, q \in \mathbb{P}_3$  and  $r \in \mathbf{R}$ . Suppose that

$$\begin{aligned} p(t) &= c_0 + c_1t + c_2t^2 + c_3t^3, \\ q(t) &= d_0 + d_1t + d_2t^2 + d_3t^3. \end{aligned}$$

Then

$$\begin{aligned} (T(p) + T(q))(t) &= (c_1 + d_1) + (2c_2 + 2d_2)t + (3c_3 + 3d_3)t^2, \\ (T(p))(t) + (T(q))(t) &= (c_1 + 2c_2t + 3c_3t^2) + (d_1 + 2d_2t + 3d_3t^2) \end{aligned}$$

and these are equal. Likewise,

$$\begin{aligned} (T(rp))(t) &= (rc_1) + (2rc_2)t + (3rc_3)t^2, \\ (rT(p))(t) &= r(c_1 + 2c_2t + 3c_3t^2) \end{aligned}$$

and these are equal.

Another valid argument is to note that  $(p + q)'(t) = p'(t) + q'(t)$  and  $(rp)'(t) = r \cdot p'(t)$ . This is a quick proof that avoids writing out all the coefficients.

(c) The kernel and range of  $T$  are both subspaces of  $\mathbb{P}_3$ . Find a basis for each of these subspaces.

Solution: The kernel of  $T$  is the space of polynomials in  $\mathbb{P}_3$  whose derivative is zero. These are exactly the constant polynomials,  $p(t) = c_0$ . Therefore the kernel of  $T$  is the 1-dimensional subspace spanned by the polynomial  $p(t) = 1$ , and a basis is  $\{1\}$ .

The range of  $T$  is the space of all possible polynomials  $p'(t)$  where  $p(t) \in \mathbb{P}_3$ . These are the polynomials of degree at most 2. A basis for the range is therefore  $\{1, t, t^2\}$ .

(d) Find the matrix of  $T$  with respect to the basis  $\mathcal{B}$  from part (a).

Solution: Since  $T(1) = 0$ ,  $T(t) = 1$ ,  $T(t^2) = 2t$ , and  $T(t^3) = 3t^2$ , the matrix of  $T$  with respect to  $\mathcal{B}$  is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. Let  $A$  be a  $3 \times 4$  matrix. Assume that the equation  $A\mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$  has no solutions.

(a) What are the possible values of  $\dim \text{Nul}(A)$ ? List all possible ranks of  $A$ .

Solution: Since the equation  $A\mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$  has no solutions, the column space of  $A$  cannot be all of  $\mathbf{R}^3$ . Therefore the rank of  $A$  could be 0, 1, or 2. The dimension of the null space is 4 minus the rank, so it could be 4, 3, or 2.

Note that the particular coordinates of the vector  $\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$  are irrelevant: it could be any nonzero vector. Also, the only way that the rank could be 0 and the null space could have dimension 4 is if  $A$  is the zero matrix, but this is indeed a possibility.

(b) Explain why the three rows of  $A$  cannot be linearly independent vectors in

$\mathbf{R}^4$ .

Solution: If this were true, then the row space of  $A$  would have dimension 3, but the dimension of the row space is the rank of  $A$ , which we saw in part (a) is at most 2.

4. The space  $\text{Span} \left( \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right)$  is a plane through the origin in  $\mathbf{R}^3$ . Find constants  $c_1, c_2, c_3$  such that the equation  $c_1x_1 + c_2x_2 + c_3x_3 = 0$  describes this plane.

Solution: An explicit description of the plane is given, and we want an implicit

description. A vector  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$  is contained in the plane if it can be written as a linear combination of  $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$  and  $\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ . To determine when this is possible, row-reduce the augmented matrix

$$\left[ \begin{array}{cc|c} 1 & 2 & x_1 \\ -2 & 0 & x_2 \\ -1 & 2 & x_3 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 2 & x_1 \\ 0 & 4 & x_2 + 2x_1 \\ 0 & 4 & x_3 + x_1 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 2 & x_1 \\ 0 & 4 & x_2 + 2x_1 \\ 0 & 0 & (x_3 + x_1) - (x_2 + 2x_1) \end{array} \right].$$

A solution will exist if and only if  $(x_3 + x_1) - (x_2 + 2x_1) = 0$ . Therefore the plane is described by the equation  $-x_1 - x_2 + x_3 = 0$ .

5. Let

$$A = \begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{bmatrix}$$

be the transition matrix of a Markov chain. The eigenvalues of  $A$  are 1 and  $-0.2$ ; you do not need to show this.

(a) Find a basis  $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$  for  $\mathbf{R}^2$  consisting of eigenvectors of  $A$ . Make it so that  $\mathbf{b}_1$  is the steady-state vector for the Markov chain.

Solution: For  $\lambda = 1$ , find a basis for the null space of

$$A - I = \begin{bmatrix} -0.9 & 0.3 \\ 0.9 & -0.3 \end{bmatrix} \rightarrow \begin{bmatrix} -0.9 & 0.3 \\ 0 & 0 \end{bmatrix}.$$

Since there is one free variable, the null space has dimension 1. It contains the nonzero vector  $\begin{bmatrix} 0.3 \\ 0.9 \end{bmatrix}$ . Therefore

$$A \begin{bmatrix} 0.3 \\ 0.9 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.9 \end{bmatrix};$$

this can also be checked directly. In order for  $\mathbf{b}_1$  to be the steady-state vector for the Markov chain, we require not just  $A\mathbf{b}_1 = \mathbf{b}_1$  but also that  $\mathbf{b}_1$  is a probability vector, that is, the entries are nonnegative and sum to 1. Thus,

$$\mathbf{b}_1 = \frac{1}{1.2} \begin{bmatrix} 0.3 \\ 0.9 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}.$$

For  $\lambda = -0.2$ , find a basis for the null space of

$$A + 0.2I = \begin{bmatrix} 0.3 & 0.3 \\ 0.9 & 0.9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

This null space also has dimension 1 and contains the nonzero vector  $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .

(b) Find  $2 \times 2$  matrices  $P$  and  $D$  such that  $A = PDP^{-1}$ .

Solution: Let

$$P = [\mathbf{b}_1 \quad \mathbf{b}_2] = \begin{bmatrix} 0.25 & 1 \\ 0.75 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -0.2 \end{bmatrix}.$$

Since  $A\mathbf{b}_1 = 1\mathbf{b}_1$  and  $A\mathbf{b}_2 = -0.2\mathbf{b}_2$ , it follows that  $A = PDP^{-1}$ .

(c) Suppose that at time 0, the Markov chain has probability  $x_1$  of being in state 1 and probability  $x_2$  of being in state 2. This means that  $x_1 + x_2 = 1$ . If  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$ , show that  $c_1 = 1$ .

Solution: Given that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.25c_1 + c_2 \\ 0.75c_1 - c_2 \end{bmatrix},$$

we have

$$x_1 + x_2 = (0.25c_1 + c_2) + (0.75c_1 - c_2) = c_1.$$

If  $x_1 + x_2 = 1$ , then  $c_1 = 1$ .

(d) Express the vector equation  $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$  as a matrix equation involving the matrix  $P$  from part (b).

Solution:  $\mathbf{x} = P \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ .

(e) After  $n$  steps of the Markov chain, the probabilities of being in each state are given by the vector  $A^n\mathbf{x}$ . Use the diagonalization  $A = PDP^{-1}$  to write a formula for  $A^n$ . Then use parts (c) and (d) to fill in the blanks:

$$A^n\mathbf{x} = \_ \mathbf{b}_1 + \_ \mathbf{b}_2.$$

The expressions in the blanks should be written in terms of  $n$  and  $c_2$  (recall that  $c_1 = 1$ ).

Solution: First,

$$A^n = (PDP^{-1})^n = (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) = PD^nP^{-1}.$$

Now, using part (d),

$$A^n \mathbf{x} = (PD^nP^{-1})P \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = PD^n \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Since  $D^n = \begin{bmatrix} 1^n & 0 \\ 0 & (-0.2)^n \end{bmatrix}$  and  $P = [\mathbf{b}_1 \ \mathbf{b}_2]$ ,

$$PD^n \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = [\mathbf{b}_1 \ \mathbf{b}_2] \begin{bmatrix} c_1 \\ (-0.2)^n c_2 \end{bmatrix} = c_1 \mathbf{b}_1 + c_2 (-0.2)^n \mathbf{b}_2 = \mathbf{b}_1 + c_2 (-0.2)^n \mathbf{b}_2.$$

(f) Use part (e) to argue that  $\lim_{n \rightarrow \infty} A^n \mathbf{x} = \mathbf{b}_1$ , that is, the Markov chain converges to the steady-state vector.

Solution: The limit as  $n \rightarrow \infty$  of  $c_2(-0.2)^n$  is 0. Therefore, the limit as  $n \rightarrow \infty$  of  $A^n \mathbf{x} = \mathbf{b}_1 + c_2(-0.2)^n \mathbf{b}_2$  is  $\mathbf{b}_1$ .

6. (a) Find the characteristic polynomial and the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & -4 & 0 & 3 \end{bmatrix}.$$

Solution: Since  $A$  is lower triangular, the characteristic polynomial is

$$\det(A - \lambda I) = (1 - \lambda)(3 - \lambda)(-1 - \lambda)(3 - \lambda)$$

and the eigenvalues are  $-1, 1, 3$  (where 3 has algebraic multiplicity 2).

(b) In order to determine whether  $A$  is diagonalizable, you would need to find the dimension of the null space of a particular matrix. What is that matrix? What are the possible values for the dimension of its null space, and under what circumstances will  $A$  be diagonalizable? Note: Do not actually find the null space of the matrix!

Solution: We know that the  $\lambda = -1$  eigenspace will have dimension 1 and the  $\lambda = 1$  eigenspace will also have dimension 1. The  $\lambda = 3$  eigenspace will have dimension either 1 or 2. In the first case,  $A$  will not be diagonalizable, and in the

second case,  $A$  will be diagonalizable. Therefore we should find the dimension of the null space of

$$A - 3I = \begin{bmatrix} -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & -1 & -4 & 0 \\ 1 & -4 & 0 & 0 \end{bmatrix}.$$

If the null space has dimension 1, then  $A$  isn't diagonalizable. If the null space has dimension 2, then  $A$  is diagonalizable.

7. (a) Give an example of a  $2 \times 2$  matrix that is diagonalizable but not invertible.

Solution: Take for example  $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ . It is diagonalizable since it is already diagonal! But it is not invertible. In general, any  $2 \times 2$  matrix with two linearly independent eigenvectors, one of which has the eigenvalue 0, will work.

(b) Give an example of a  $2 \times 2$  matrix that is invertible but not diagonalizable.

Solution: As seen in class,  $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$  is not diagonalizable. It is invertible since its determinant is nonzero; alternatively, it's easy to see that the columns are linearly independent.

8. Construct a  $2 \times 2$  matrix  $A$  with eigenvalues  $-3$  and  $2$  such that the  $\lambda = -3$  eigenspace is  $\text{Span} \left( \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right)$  and the  $\lambda = 2$  eigenspace is  $\text{Span} \left( \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right)$ . You may write your final answer as a product of three  $2 \times 2$  matrices.

Solution: We are given a basis of eigenvectors:

$$A \begin{bmatrix} 3 \\ 1 \end{bmatrix} = -3 \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

Therefore  $A = PDP^{-1}$  with

$$P = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}.$$

9. (a) Suppose that the  $n \times n$  matrix  $A$  has eigenvalues  $\lambda_1, \dots, \lambda_n$  (not necessarily distinct). For fixed  $k \geq 1$ , what are the eigenvalues of  $A^k$ ?

Solution: If  $A\mathbf{v}_j = \lambda_j\mathbf{v}_j$ , then  $A^k\mathbf{v}_j = \lambda_j^k\mathbf{v}_j$ . Therefore  $\mathbf{v}_j$  is an eigenvector of  $A^k$  with eigenvalue  $\lambda_j^k$ . The eigenvalues of  $A^k$  are  $\lambda_1^k, \dots, \lambda_n^k$ .

(b) If  $A^4$  is the zero matrix, what are the eigenvalues of  $A$ ?

Solution: The only eigenvalue of  $A$  is  $\lambda = 0$ . If any nonzero  $\lambda$  were an eigenvalue of  $A$ , then  $\lambda^4$  (which is also nonzero) would be an eigenvalue of  $A^4$ , which is impossible since  $A^4$  multiplied by any vector gives the zero vector.

(c) As is well-known, if you start with a polynomial of degree 3 and take four derivatives, you always get zero. Explain what this means about the eigenvalues of the matrix in Problem 2(d).

Solution: Let  $A$  be the matrix in Problem 2(d). When we multiply  $A$  by a vector

$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$ , which corresponds to the polynomial  $c_0 + c_1t + c_2t^2 + c_3t^3$ , we get the vector  $\begin{bmatrix} c_1 \\ 2c_2 \\ 3c_3 \\ 0 \end{bmatrix}$ , which corresponds to the polynomial  $c_1 + 2c_2t + 3c_3t^2$  that is the

derivative of the original polynomial. Therefore, multiplying  $A^4$  by any vector

$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$  has the effect of taking four derivatives, which yields the zero vector. We

conclude that  $A^4$  is the zero matrix. By part (b), the only eigenvalue of  $A$  is zero. (This could be seen directly since  $A$  is upper triangular, so its eigenvalues are the diagonal entries, which are all zero.)

10. (a) Solve the initial-value problem  $\mathbf{x}'(t) = A\mathbf{x}(t)$  with

$$A = \begin{bmatrix} -2 & -5 \\ 1 & 4 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Solution: First find the eigenvalues and eigenvectors of  $A$ . Characteristic polynomial:

$$\begin{vmatrix} -2 - \lambda & -5 \\ 1 & 4 - \lambda \end{vmatrix} = (-2 - \lambda)(4 - \lambda) + 5 = \lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3)$$

so the eigenvalues are  $-1$  and  $3$ . For  $\lambda = -1$ ,

$$A + I = \begin{bmatrix} -1 & -5 \\ 1 & 5 \end{bmatrix}, \quad \text{Nul}(A + I) = \text{Span} \left( \begin{bmatrix} 5 \\ -1 \end{bmatrix} \right).$$

For  $\lambda = 3$ ,

$$A - 3I = \begin{bmatrix} -5 & -5 \\ 1 & 1 \end{bmatrix}, \quad \text{Nul}(A - 3I) = \text{Span} \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right).$$

Therefore, a fundamental set of solutions to the differential equation is

$$\mathbf{x}_1(t) = e^{-t} \begin{bmatrix} 5 \\ -1 \end{bmatrix}, \quad \mathbf{x}_2(t) = e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The general solution  $\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t)$  satisfies

$$\mathbf{x}(0) = c_1 \begin{bmatrix} 5 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

If we set this equal to  $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ , we must solve the equation  $\begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ .

The solution is

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} -1 & -1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Therefore, the initial-value problem has solution  $\mathbf{x}(t) = e^{-t} \begin{bmatrix} 5 \\ -1 \end{bmatrix} - 2e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .

(b) Is the origin an attractor, repeller, or saddle point for this dynamical system? Draw a graph with the directions of greatest attraction/repulsion along with some typical trajectories.

Solution: The origin is a saddle point since the matrix  $A$  has one positive and one negative eigenvalue. Graph:

