Math 2940: Prelim 2 Practice Problems

1. Given the following matrix with its reduced row echelon form:

$$A = \begin{bmatrix} 1 & 0 & -2 & 2 \\ -2 & 1 & 5 & -7 \\ 0 & -1 & -1 & 3 \\ 3 & -2 & -8 & 12 \\ 2 & -2 & -6 & 10 \end{bmatrix}, \qquad \text{rref}(A) = U = \begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) What is the rank of A? Find bases for $\text{Col}(A)$, $\text{Nul}(A)$, and $\text{Row}(A)$.

(b) Is $\text{Col}(A) = \text{Col}(U)$? Is $\text{Nul}(A) = \text{Nul}(U)$? Is $\text{Row}(A) = \text{Row}(U)$? Explain your answers.

2. Let $\mathbb{P}_3$ be the space of all polynomials $p(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$ with degree at most 3 and real coefficients.

(a) Prove that $\mathcal{B} = \{1, t, t^2, t^3\}$ is a basis for $\mathbb{P}_3$ by verifying directly that the elements of $\mathcal{B}$ are linearly independent and span all of $\mathbb{P}_3$.

(b) Let $T : \mathbb{P}_3 \to \mathbb{P}_3$ be the "differentiation operator": if $p(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$, then

$$(T(p))(t) = p'(t) = c_1 + 2c_2 t + 3c_3 t^2.$$

Prove that T is a linear transformation.

(c) The kernel and range of T are both subspaces of $\mathbb{P}_3$. Find a basis for each of these subspaces.

(d) Find the matrix of T with respect to the basis $\mathcal{B}$ from part (a).

3. Let A be a 3×4 matrix. Assume that the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$ has no solutions.

(a) What are the possible values of $\dim \text{Nul}(A)$? List all possible ranks of A.

(b) Explain why the three rows of A cannot be linearly independent vectors in $\mathbf{R}^4$.

4. The space $\text{Span}\left(\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right)$ is a plane through the origin in $\mathbf{R}^3$. Find constants c_1, c_2, c_3 such that the equation $c_1 x_1 + c_2 x_2 + c_3 x_3 = 0$ describes this plane.

5. Let

$$A = \begin{bmatrix} 0.1 & 0.3 \\ 0.9 & 0.7 \end{bmatrix}$$

be the transition matrix of a Markov chain. The eigenvalues of A are 1 and -0.2; you do not need to show this.

(a) Find a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ for $\mathbf{R}^2$ consisting of eigenvectors of A. Make it so that $\mathbf{b}_1$ is the steady-state vector for the Markov chain.

(b) Find 2×2 matrices P and D such that $A = PDP^{-1}$.

(c) Suppose that at time 0, the Markov chain has probability x_1 of being in state 1 and probability x_2 of being in state 2. This means that $x_1 + x_2 = 1$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$, show that $c_1 = 1$.

(d) Express the vector equation $\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$ as a matrix equation involving the matrix P from part (b).

(e) After n steps of the Markov chain, the probabilities of being in each state are given by the vector $A^n\mathbf{x}$. Use the diagonalization $A = PDP^{-1}$ to write a formula for A^n. Then use parts (c) and (d) to fill in the blanks:

$$A^n\mathbf{x} = \underline{\quad}\mathbf{b}_1 + \underline{\quad}\mathbf{b}_2.$$

The expressions in the blanks should be written in terms of n and c_2 (recall that $c_1 = 1$).

(f) Use part (e) to argue that $\lim_{n\to\infty} A^n\mathbf{x} = \mathbf{b}_1$, that is, the Markov chain converges to the steady-state vector.

6. (a) Find the characteristic polynomial and the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & -4 & 0 & 3 \end{bmatrix}.$$

(b) In order to determine whether A is diagonalizable, you would need to find the dimension of the null space of a particular matrix. What is that matrix? What are the possible values for the dimension of its null space, and under what circumstances will A be diagonalizable? Note: Do not actually find the null space of the matrix!

7. (a) Give an example of a 2×2 matrix that is diagonalizable but not invertible.

(b) Give an example of a 2×2 matrix that is invertible but not diagonalizable.

8. Construct a 2×2 matrix A with eigenvalues -3 and 2 such that the $\lambda = -3$ eigenspace is Span $\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)$ and the $\lambda = 2$ eigenspace is Span $\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\right)$. You may write your final answer as a product of three 2×2 matrices.

9. (a) Suppose that the $n \times n$ matrix A has eigenvalues $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct). For fixed $k \geq 1$, what are the eigenvalues of A^k?

(b) If A^4 is the zero matrix, what are the eigenvalues of A?

(c) As is well-known, if you start with a polynomial of degree 3 and take four derivatives, you always get zero. Explain what this means about the eigenvalues of the matrix in Problem 2(d).

10. (a) Solve the initial-value problem $\mathbf{x}'(t) = A\mathbf{x}(t)$ with

$$A = \begin{bmatrix} -2 & -5 \\ 1 & 4 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

(b) Is the origin an attractor, repeller, or saddle point for this dynamical system? Draw a graph with the directions of greatest attraction/repulsion along with some typical trajectories.