Math 2940: Prelim 2 Solutions

July 26, 2016

1. (10 points) Let

$$A = \begin{bmatrix} 3 & 8 & -2 & 1 & 1 \\ -2 & 0 & 4 & 1 & 5 \end{bmatrix}.$$

Find the dimensions of Col(A), Row(A), and Nul(A). (It's possible to do this without any row operations, but you must explain your reasoning.)

Solution: Col(A) is a subspace of $\mathbf{R}^2$. Since the first two columns of A, $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ 0 \end{bmatrix}$, are already linearly independent, $\text{Col}(A) = \mathbf{R}^2$ and has dimension 2. It follows that $\dim \text{Row}(A) = \dim \text{Col}(A) = 2$ and $\dim \text{Nul}(A) = 5 - \dim \text{Col}(A) = 3$.

2. (16 points) Let W be the set of all vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbf{R}^4$ such that $x_1 = 5x_3$ and $x_1 + x_2 = x_4$.

(a) (6 points) Prove that W is a subspace of $\mathbf{R}^4$.

Solution 1: We must verify that (1) $\mathbf{0} \in W$, (2) W is closed under addition, and (3) W is closed under scalar multiplication. For (1), the zero vector $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ satisfies $x_1 = 5x_3$ and $x_1 + x_2 = x_4$, so $\mathbf{0} \in W$. For (2), suppose that $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$ are both in W. This means:

$$\begin{aligned} x_1 &= 5x_3 & \qquad x_1 + x_2 &= x_4 \\ y_1 &= 5y_3 & \qquad y_1 + y_2 &= y_4. \end{aligned}$$

It follows that $x_1 + y_1 = 5(x_3 + y_3)$ and $(x_1 + y_1) + (x_2 + y_2) = (x_4 + y_4)$, which proves that $\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{bmatrix} \in W$.

For (3), suppose that $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in W$ and $c \in \mathbf{R}$. Then $x_1 = 5x_3$ and $x_1 + x_2 = x_4$, so $cx_1 = 5cx_3$ and $cx_1 + cx_2 = cx_4$. This shows that $c\mathbf{x} = \begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \\ cx_4 \end{bmatrix}$ is in W.

Solution 2: Let

$$A = \begin{bmatrix} 1 & 0 & -5 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}.$$

A vector $\mathbf{x} \in \mathbf{R}^4$ is in $\text{Nul}(A)$ if and only if $A\mathbf{x} = \mathbf{0}$, which happens exactly when $x_1 - 5x_3 = 0$ and $x_1 + x_2 - x_4 = 0$. Therefore, $W = \text{Nul}(A)$. The null space of an $m \times n$ matrix is always a subspace of $\mathbf{R}^n$, so W is a subspace of $\mathbf{R}^4$.

(b) (10 points) Find a basis for W.

Solution: There are a few possible approaches for this problem, but the most foolproof method is to continue along the lines of Solution 2 to part (a). Since $W = \text{Nul}(A)$, we can use the established technique to find a basis for $\text{Nul}(A)$. First row-reduce:

$$\begin{bmatrix} 1 & 0 & -5 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 5 & -1 \end{bmatrix}$$

The equations become:

$$\begin{aligned} x_1 &= 5x_3 \\ x_2 &= -5x_3 + x_4 \\ x_3 &= x_3 \\ x_4 &= \qquad\quad x_4 \end{aligned}$$

so

$$\mathbf{x} = x_3 \begin{bmatrix} 5 \\ -5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

and therefore $\left\{ \begin{bmatrix} 5 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for W.

Problem-solving note: Once you have written down a basis for W, it is straightforward to see whether your basis vectors are actually elements of W or not. This provides a good way to check your work.

3. (22 points) Let A be a 3×3 matrix whose characteristic polynomial is $(4-\lambda)(2-\lambda)^2$.

(a) (6 points) Is A invertible? Explain.

Solution: The eigenvalues of A are 4 (with algebraic multiplicity 1) and 2 (with algebraic multiplicity 2). Therefore 0 is not an eigenvalue of A, so $\mathrm{Nul}(A) = \{\mathbf{0}\}$ and A is invertible.

(b) (8 points) What is $\mathrm{Nul}(A-I)$? What is $\det(A-I)$?

Solution: Since 1 is not an eigenvalue of A, there are no nonzero vectors $\mathbf{x}$ such that $(A-I)\mathbf{x} = \mathbf{0}$. Hence $\mathrm{Nul}(A-I) = \{\mathbf{0}\}$. To compute $\det(A-I)$, plug $\lambda = 1$ into the characteristic polynomial:

$$\det(A - 1I) = (4-1)(2-1)^2 = 3.$$

(c) (8 points) Suppose that

$$A\begin{bmatrix}1\\3\\-3\end{bmatrix}=\begin{bmatrix}2\\6\\-6\end{bmatrix},\quad A\begin{bmatrix}0\\4\\1\end{bmatrix}=\begin{bmatrix}0\\8\\2\end{bmatrix},\quad A\begin{bmatrix}1\\7\\1\end{bmatrix}=\begin{bmatrix}4\\28\\4\end{bmatrix}.$$

Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution: If we let

$$\mathbf{v}_1=\begin{bmatrix}1\\3\\-3\end{bmatrix},\quad \mathbf{v}_2=\begin{bmatrix}0\\4\\1\end{bmatrix},\quad \mathbf{v}_3=\begin{bmatrix}1\\7\\1\end{bmatrix}$$

then $A\mathbf{v}_1 = 2\mathbf{v}_1$, $A\mathbf{v}_2 = 2\mathbf{v}_2$, and $A\mathbf{v}_3 = 4\mathbf{v}_3$. It follows that $A = PDP^{-1}$ where

$$P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}=\begin{bmatrix}1&0&1\\3&4&7\\-3&1&1\end{bmatrix},\qquad D=\begin{bmatrix}2&0&0\\0&2&0\\0&0&4\end{bmatrix}.$$

4. (14 points) Let

$$\mathcal{B}=\left\{\begin{bmatrix}3\\4\\1\end{bmatrix},\begin{bmatrix}-2\\2\\0\end{bmatrix},\begin{bmatrix}1\\-5\\3\end{bmatrix}\right\}$$

be a basis for $\mathbf{R}^3$, and let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear transformation whose matrix with respect to $\mathcal{B}$ is

$$[T]_{\mathcal{B}}=\begin{bmatrix}1&0&0\\2&3&-2\\-1&3&-4\end{bmatrix}.$$

(a) (7 points) Let

$$\mathbf{x}=\begin{bmatrix}4\\-1\\4\end{bmatrix}=\begin{bmatrix}3\\4\\1\end{bmatrix}+\begin{bmatrix}1\\-5\\3\end{bmatrix}.$$

Compute $[\mathbf{x}]_{\mathcal{B}}$ and $[T(\mathbf{x})]_{\mathcal{B}}$.

Solution: Let $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ denote the vectors in $\mathcal{B}$ (in order). It is given that $\mathbf{x} = \mathbf{b}_1 + \mathbf{b}_3 = 1\mathbf{b}_1 + 0\mathbf{b}_2 + 1\mathbf{b}_3$. Therefore,

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Next,

$$[T(\mathbf{x})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & -2 \\ -1 & 3 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}.$$

(b) (7 points) Compute $T\left(\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}\right)$.

Solution: We are asked to compute $T(\mathbf{b}_2)$. We know that $[\mathbf{b}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, so

$$[T(\mathbf{b}_2)]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{b}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}.$$

This implies that

$$T(\mathbf{b}_2) = 0\mathbf{b}_1 + 3\mathbf{b}_2 + 3\mathbf{b}_3 = 3\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -9 \\ 9 \end{bmatrix}.$$

5. (18 points) Let

$$A = \begin{bmatrix} 0.68 & 0.08 \\ 0.12 & 0.72 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0.8 & 0 \\ 0 & 0.6 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}^{-1}$$

and consider the discrete dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

(a) (10 points) If $\mathbf{x}_0 = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$, find a formula for $\mathbf{x}_k$.

Solution: We are given a diagonalization of A, so we can read off the general formula for $\mathbf{x}_k$:

$$\mathbf{x}_k = c_1(0.8)^k \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2(0.6)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The values of c_1, c_2 depend on $\mathbf{x}_0$. By plugging in $k = 0$ we get

$$c_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -6 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -6 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

It follows that

$$\mathbf{x}_k = -(0.8)^k \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 4(0.6)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(b) (8 points) Is the origin a saddle point, attractor, or repeller for this dynamical system? Draw a graphical description of the system, indicating the directions of greatest attraction/repulsion and showing several typical trajectories.

Solution: The origin is an attractor since both eigenvalues are between 0 and 1. The line of greatest attraction is the span of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, since it corresponds to the eigenvalue 0.6 (which is less than 0.8). There is no repulsion.

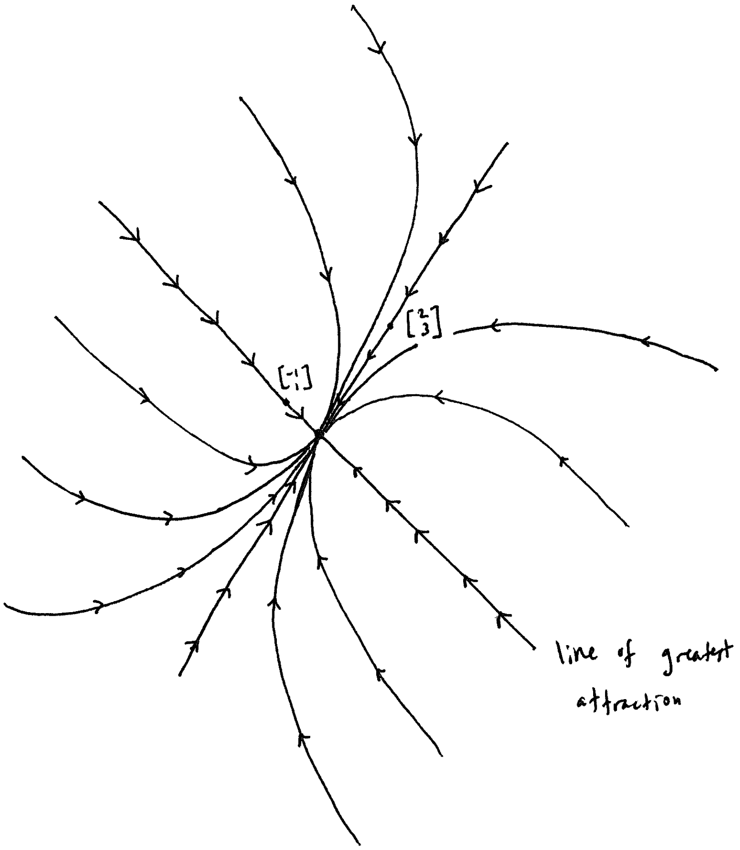
To draw the trajectories, keep in mind that if

$$\mathbf{x}_k = a \begin{bmatrix} 2 \\ 3 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

then

$$\mathbf{x}_{k+1} = 0.8a \begin{bmatrix} 2 \\ 3 \end{bmatrix} + 0.6b \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

That is, the $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ component of $\mathbf{x}_k$ disappears at a faster exponential rate than the $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ component. After a long time, both components will be very close to zero, but the $\begin{bmatrix}2\\3\end{bmatrix}$ component will be relatively larger, since 0.8^k is large in relation to 0.6^k.



6. (20 points) Let A be a 4×5 matrix whose columns are $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$ in order. Suppose that $5\mathbf{a}_1 - 3\mathbf{a}_2 + \mathbf{a}_4 = \mathbf{0}$, and that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b} \in \mathbf{R}^4$.

(a) (7 points) Find the rank of A and the dimension of $\mathrm{Nul}(A)$.

Solution: For every $\mathbf{b} \in \mathbf{R}^4$ there is at least one $\mathbf{x} \in \mathbf{R}^5$ such that $A\mathbf{x} = \mathbf{b}$. Thus $\mathrm{Col}(A) = \mathbf{R}^4$ and $\mathrm{rank}(A) = 4$. The sum of the rank and the dimension of the null space is 5, so $\dim \mathrm{Nul}(A) = 1$.

(b) (5 points) Find a nonzero vector in $\mathrm{Nul}(A)$. What is a basis for $\mathrm{Nul}(A)$?

Solution: The vector equation $5\mathbf{a}_1 - 3\mathbf{a}_2 + \mathbf{a}_4 = \mathbf{0}$ can be rewritten as the matrix equation

$$A \begin{bmatrix} 5 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, $\begin{bmatrix} 5 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in \operatorname{Nul}(A)$. Since $\dim \operatorname{Nul}(A) = 1$, a basis is $\left\{ \begin{bmatrix} 5 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(c) (8 points) What is $\operatorname{rref}(A)$? (*Hint:* The columns of $\operatorname{rref}(A)$ satisfy the same relations as the columns of A.)

Solution: Suppose that

$$\operatorname{rref}(A) = U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 & \mathbf{u}_4 & \mathbf{u}_5 \end{bmatrix}.$$

The vectors $\mathbf{u}_1, \ldots, \mathbf{u}_5$ satisfy the same relations as $\mathbf{a}_1, \ldots, \mathbf{a}_5$. In particular, $5\mathbf{u}_1 - 3\mathbf{u}_2 + \mathbf{u}_4 = \mathbf{0}$, so $\mathbf{u}_4 = -5\mathbf{u}_1 + 3\mathbf{u}_2$.

Since $\operatorname{rank}(A) = 4$, U has four pivot columns. Each pivot column is a standard basis vector (like $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$) that is, by construction, linearly independent of the preceding columns. Therefore $\mathbf{u}_4$ cannot be a pivot column. The pivots are $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_5$ while $\mathbf{u}_4$ is determined by taking $-5\mathbf{u}_1 + 3\mathbf{u}_2$. Putting it all together,

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -5 & 0 \\ 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$