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Homological Criteria for Finiteness

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Let R be a ring. Bieri and Eckmann give in [2] a homological criterion (involving the functors $\mathrm{Tor}_R(-, -)$) for deciding whether an R-module admits a resolution by finitely generated projectives, and they use this criterion to prove that "duality groups" (see §2 for the definition) have a number of finiteness properties. The purpose of this note is to extend these results in two directions.

On the one hand, we show that the criterion of Bieri and Eckmann has an analogue involving Ext instead of Tor (see the corollary of Theorem 1 in §1), and we use this result in §2 to show that duality groups satisfy a stronger finiteness condition than those given in [2], namely, *duality groups are of type* (FP).

Secondly, we give criteria (of which those above are special cases) for deciding whether a chain complex is equivalent to a complex of finitely generated projectives (see Theorem 1). This leads to criteria (given in §3) for deciding whether a CW-complex with finitely presented fundamental group has the homotopy type of a complex with finitely many cells in each dimension. In particular, we recover from these criteria some finiteness results due to Browder [3] for spaces which satisfy Poincaré duality with local coefficients.

§1. Finiteness Criteria for Chain Complexes

Let R be a ring with identity and let $C=(C_i)_{i\geqslant 0}$ be a chain complex of projective left R-modules. If M is a left R-module and N is a right R-module, we set

$$H^n(C, M)=H^n(\mathrm{Hom}_R(C., M)) \quad \text{and} \quad H_n(C, N)=H_n(N \underset{R}{\otimes} C.).$$

If $\{M_i\}$ is a direct system of left R-modules (indexed by a directed set) and n is a fixed integer, then the natural maps $M_i \to \varinjlim M_i$ induce a compatible family of maps $H^n(C, M_i) \to H^n(C, \varinjlim M_i)$, and hence a map

$$\varinjlim_i H^n(C, M_i) \to H^n(C, \varinjlim_i M_i).$$

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We will say that the functor $H^n(C, -)$ *preserves direct limits* if this map is an isomorphism for every direct system $\{M_i\}$.

Similarly, if $\{N_j\}$ is a family of right R-modules (indexed by an arbitrary set), then we have for each n a natural map

$$H_n(C, \prod_j N_j) \to \prod_j H_n(C, N_j),$$

and we say that the functor $H_n(C, -)$ *preserves products* if this map is an isomorphism for every family $\{N_j\}$.

We can now state the main result of this section:

THEOREM 1. *The following conditions on C are equivalent:*
(i) *C is homotopy equivalent to a complex of finitely generated projectives.*
(ii) *The functors $H^n(C, -)$ preserve direct limits for all n.*
(iii) *The functors $H_n(C, -)$ preserve products for all n.*

In particular, specializing to the case where C is a projective resolution of an R-module M, we obtain:

COROLLARY. *The following conditions on an R-module M are equivalent:*
(i) *M admits a resolution by finitely generated projectives.*
(ii) *The functors $\mathrm{Ext}_R^n(M, -)$ preserve direct limits for all n.*
(iii) *The functors $\mathrm{Tor}_n^R(-, M)$ preserve products for all n.*

Theorem 1 will be deduced from the following more precise result:

THEOREM 2. *Let C be a chain complex of projective left R-modules and let k be a non-negative integer. The following conditions are equivalent:*

(i) *There exists a complex C' of finitely generated projectives with $C'_i = 0$ for $i > k$, together with a map $f: C' \to C$ inducing homology isomorphisms in dimensions less then k and a homology epimorphism in dimension k.*

(ii) *For any direct system $\{M_i\}$ of left R-modules, the natural map*

$$\varinjlim H^n(C, M_i) \to H^n(C, \varinjlim M_i)$$

is an isomorphism for $n < k$ and a monomorphism for $n = k$.

(iii) *For any direct system $\{M_i\}$ with $\varinjlim M_i = 0$, we have $\varinjlim H^n(C, M_i) = 0$ for all $n \leqslant k$.*

(iv) *For any family $\{N_j\}$ of right R-modules, the natural map*

$$H_n(C, \prod N_j) \to \prod H_n(C, N_j)$$

is an isomorphism for $n < k$ and an epimorphism for $n = k$.

(v) *For any index set J, the natural map*

$$H_n(C, \prod_J R) \to \prod_J H_n(C)$$

is an isomorphism for $n<k$ and an epimorphism for $n=k$.

Remark. In case C is a projective resolution of an R-module M, the equivalence of (i) and (v) is due to Bieri and Eckmann [2].

The proof of Theorem 2 will use the following three lemmas:

LEMMA 1. *A left R-module M which satisfies either of the following two conditions is finitely generated:*

(a) $\varinjlim \operatorname{Hom}_R(M, M_i)=0$ *for any direct system* $\{M_i\}$ *with* $\varinjlim M_i=0$.

(b) *The natural map* $(\prod_J R)\otimes_R M \to \prod_J M$ *is surjective for any index set J.*

The fact that (b) implies that M is finitely generated is proved in [2]. Assuming now that (a) holds, consider the direct system $\{M/M'\}$ where M' ranges over the finitely generated submodules of M. Then $\varinjlim M/M'=0$, so we have $\varinjlim \operatorname{Hom}(M, M/M')=0$. In particular, the identity map $\mathrm{id}_M \in \operatorname{Hom}(M, M)$ must vanish in some $\operatorname{Hom}(M, M/M')$. But this implies that $M/M'=0$, so M is finitely generated, as required.

LEMMA 2. *Let C be a chain complex of projectives and assume that $H_i(C)=0$ for $i<k$. Then for arbitrary coefficient modules M and N, $H^i(C, M)$ and $H_i(C, N)$ vanish for $i<k$, and there are natural isomorphisms*

$$H^k(C, M) \xrightarrow{\approx} \operatorname{Hom}_R(H_k(C), M) \quad \textit{and} \quad N \underset{R}{\otimes} H_k(C) \xrightarrow{\approx} H_k(C, N).$$

This follows from the universal coefficient spectral sequences (cf. [4], Chapter I, Theorems 5.4.1 and 5.5.1),

$$\operatorname{Ext}_R^p(H_q(C), M) \Rightarrow H^{p+q}(C, M) \quad \text{and} \quad \operatorname{Tor}_p^R(N, H_q(C)) \Rightarrow H_{p+q}(C, N).$$

Alternatively, one can give a direct argument, based on the fact that for $i \leqslant k$ the module of i-cycles of C is a direct summand of C_i.

The next lemma formalizes a standard "cell-attaching" procedure for killing homology classes.

LEMMA 3. *Let $f: C \to D$ be a map of chain complexes, let $C(f)$ be the mapping cone, and let $\varphi: P \to H_k(C(f))$ be a homomorphism, where P is a projective module and k is an arbitrary integer. Define a graded module $\bar{C}$ by*

$$\bar{C}_i = \begin{cases} C_i & i \neq k \\ C_k \oplus P & i=k. \end{cases}$$

Then one can extend the differential of C to a differential on $\bar{C}$ in such a way that f extends to a chain map $\bar{f}:\bar{C}\to D$ satisfying the following two conditions:

(a) $H_i(C(\bar{f}))\approx H_i(C(f))$ *for* $i\neq k,\ k+1$.

(b) *There is an exact sequence*

$$0\to H_{k+1}(C(f))\to H_{k+1}(C(\bar{f}))\to P\xrightarrow{\varphi} H_k(C(f))\to H_k(C(\bar{f}))\to 0.$$

Lift φ to a map $\tilde{\varphi}:P\to Z_k(C(f))$, where $Z_k(-)$ denotes k-cycles. Examining the definition of the mapping cone, we see that $Z_k(C(f))$ is isomorphic to the fibred product $D_k\times_{D_{k-1}}Z_{k-1}(C)$, so we obtain from $\tilde{\varphi}$ a commutative diagram

$$\begin{array}{ccc} P & \to & Z_{k-1}(C) \\ \downarrow & & \downarrow f \\ D_k & \xrightarrow[\partial]{} & D_{k-1}. \end{array}$$

We now use the unlabelled maps in this diagram to extend the differential from C to $\bar{C}$ and to extend f from C to $\bar{C}$, and the assertions (a) and (b) follow from the exact sequence of the pair $(C(\bar{f}), C(f))$.

Proof of Theorem 2

We will only prove the equivalence of (i), (ii), and (iii); the equivalence of (i), (iv), and (v) is proved similarly.

(i)$\Rightarrow$(ii): Let C'' be the mapping cone of f. Then $H_n(C'')=0$ for $n\leqslant k$, hence by Lemma 2 we have $H^n(C'', M)=0$ for all M and all $n\leqslant k$. Therefore the map $H^n(C, M)\to H^n(C', M)$ induced by f is an isomorphism for $n<k$ and a monomorphism for $n=k$, and (ii) now follows from the elementary fact that $H^n(C', -)$ preserves directlimits for all n. (It suffices to check that $\mathrm{Hom}(P, -)$ preserves direct limits if P is finitely generated and projective; one reduces easily to the case where P is free, in which case the assertion is obvious.)

(ii)$\Rightarrow$(iii): Trivial.

(iii)$\Rightarrow$(i): We argue by induction on k. If (iii) holds then (iii) also holds with k replaced by $k-1$, so by the induction hypothesis we can find a $(k-1)$-dimensional complex $C'_{(k-1)}$ of finitely generated projectives, together with a map $f_{(k-1)}:C'_{(k-1)}\to C$ whose mapping cone C'' has vanishing homology in dimensions less than k. (We include here the possibility that $k=0$, in which case we take $C'_{(k-1)}=0$.)

I claim that $H_k(C'')$ is finitely generated. In view of Lemmas 1 and 2, it suffices to show that $\varinjlim H^k(C'', M_i)=0$ whenever $\{M_i\}$ is a direct system such that $\varinjlim M_i=0$. But this follows at once (via the cohomology exact sequence associated to $f_{(k-1)}$) from condition (iii) on C together with the fact that $H^n(C'_{(k-1)}, -)$ preserves direct limits for all n.

We now apply Lemma 3 to the map $f_{(k-1)}$, taking $\varphi: P \to H_k(C'')$ to be any surjection with P finitely generated and projective. We obtain in this way an embedding of $C'_{(k-1)}$ as the $(k-1)$-skeleton of a k-dimensional complex $C'_{(k)}$ of finitely generated projectives, together with an extension of $f_{(k-1)}$ to a map $f_{(k)}: C'_{(k)} \to C$ such that $H_n(C(f_{(k)}))=0$ for $n \leqslant k$. This establishes (i) and completes the proof of the equivalence of (i), (ii), and (iii).

Proof of Theorem 1

Theorem 1 follows from Theorem 2 together with the observation that if the conditions of Theorem 2 hold for all k, then C is homotopy equivalent to a complex C' of finitely generated projectives. This can be seen, for example, by examining the above proof that (iii) implies (i) in Theorem 2. In fact, using the notation of that proof, we can take $C' = \bigcup_{k \geqslant 0} C'_{(k)}$, and we can combine the maps $f_{(k)}: C'_{(k)} \to C$ to obtain a homotopy equivalence $f: C' \to C$.

§2. Groups of Type (FP)

Let Γ be a group and let $\mathbf{Z}[\Gamma]$ be its integral group ring. We say that Γ is of *type* $(\overline{FP})$ (resp. of *type* (FP)) if the $\mathbf{Z}[\Gamma]$-module $\mathbf{Z}$, with trivial Γ-action, admits a resolution (resp. a resolution of finite length) by finitely generated projective $\mathbf{Z}[\Gamma]$-modules.

According to the corollary of Theorem 1, applied with $R = \mathbf{Z}[\Gamma]$ and $M = \mathbf{Z}$, we have:

THEOREM 3. *The following conditions on Γ are equivalent:*

(i) *Γ is of type $(\overline{FP})$.*

(ii) *The cohomology functors $H^i(\Gamma, -)$, regarded as functors on the category of Γ-modules, preserve direct limits.*

(iii) *The homology functors $H_i(\Gamma, -)$, regarded as functors on the category of Γ-modules, preserve products.*

Recall from [1] that Γ is called a *duality group* of dimension n if there is a Γ-module C (called the *dualizing module*) and an element $e \in H_n(\Gamma, C)$ such that cap product with e yields isomorphisms

$$e \cap - : H^i(\Gamma, M) \xrightarrow{\approx} H_{n-i}(\Gamma, C \otimes M)$$

for all integers i and all Γ-modules M. Since the functor $H_{n-i}(\Gamma, C \otimes -)$ clearly preserves direct limits, a duality group automatically satisfies condition (ii) of Theorem 3. Since it is also clear that a duality group has finite cohomological dimension, we obtain the following corollary of Theorem 3:

COROLLARY 1. *If Γ is a duality group then Γ is of type* (FP).

The six finiteness properties given on p. 74 of [2] follow easily from Corollary 1, and we can sharpen property (3), as follows:

COROLLARY 2. *If Γ is a duality group of dimension n then the dualizing module C admits a resolution of length n by finitely generated projective* $\mathbf{Z}[\Gamma]$*-modules.*

In fact, since $C \approx H^n(\Gamma, \mathbf{Z}[\Gamma])$ and $H^i(\Gamma, \mathbf{Z}[\Gamma])=0$ for $i \neq n$, any finite projective resolution of $\mathbf{Z}$ of length n,

$$0 \to P_n \to \cdots \to P_0 \to \mathbf{Z} \to 0,$$

gives rise to a finite projective resolution of C of length n:

$$0 \to P_0^* \to \cdots \to P_n^* \to C \to 0.$$

(Here $P^* = \mathrm{Hom}_{\mathbf{Z}[\Gamma]}(P, \mathbf{Z}[\Gamma])$.)

§3. Spaces of Finite Type

We will say that a CW-complex X is of *finite type* if X has only finitely many cells in each dimension.

THEOREM 4. *Let X be a* CW*-complex such that $\pi_1(X, x)$ is finitely presented for all $x \in X$. The following conditions are equivalent:*

(i) *X is homotopy equivalent to a complex of finite type.*

(ii) *The cohomology functors $H^i(X, -)$, regarded as functors on the category of local coefficient systems on X, preserve direct limits.*

(iii) *The homology functors $H_i(X, -)$, regarded as functors on the category of local coefficient systems on X, preserve products.*

Note first that (ii) and (iii) each imply that $\pi_0 X$ is finite; this follows, for example, from Lemma 1 of §1, applied with $R = \mathbf{Z}$ and $M = H_0(X)$. Using this observation, we reduce easily to the case where X is connected. In this case it is known from the work of Wall (cf. [6], Theorem 2) that X is equivalent to a complex of finite type if and only if the chain complex of the universal cover of X, regarded as a chain complex of $\mathbf{Z}[\pi_1 X]$-modules, is homotopy equivalent to a complex of finitely generated projectives. The theorem now follows at once from Theorem 1.

Remark. I do not know whether one can drop the assumption on $\pi_1 X$, i.e., whether conditions (ii) and (iii) imply that $\pi_1 X$ is finitely presented. They do imply, however, that $\pi_1 X$ is "almost finitely presented", in the sense that there is an exact sequence

$$P_2 \to P_1 \to P_0 \to \mathbf{Z} \to 0$$

of $\mathbf{Z}[\pi_1 X]$-modules, with each P_i finitely generated and projective. (Take $P_2 \to P_1 \to P_0$ to be the 2-skeleton of a complex of finitely generated projectives equivalent to $C(\tilde{X})$.)

We will call a CW-complex X a *duality space* of formal dimension n if there is a local coefficient system C on X and an element $e \in H_n(X, C)$ such that cap product with e yields isomorphisms

$$e \cap - : H^i(X, M) \to H_{n-i}(X, C \otimes M)$$

for every integer i and every local coefficient system M.

COROLLARY. *If X is a duality space then $\pi_1(X, x)$ is almost finitely presented for each x. If each $\pi_1(X, x)$ is finitely presented, then X is dominated by a finite complex.*

In fact, it is clear that a duality space satisfies condition (ii) of Theorem 4, so the first assertion follows from the above remark. If $\pi_1(X, x)$ is finitely presented for each x, then Theorem 4 implies that X is equivalent to a complex of finite type. But clearly X has finite cohomological dimension (with respect to arbitrary local coefficient systems), so we conclude that X is finitely dominated (cf. [5], first assertion of Theorem F).

In case $C = \mathbf{Z}$, the corollary yields the finiteness results of Browder ([3], Corollaries 1 and 2 of Theorem A) referred to in the introduction.

Note added in proof: R. Strebel (unpublished) has independently proved the corollary of Theorem 1 and the results of §2.

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