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2 · Groups of virtually finite dimension

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The purpose of this paper is to give an exposition of two topics in the theory of groups of finite virtual cohomological dimension: (a) the theory of Euler characteristics and (b) the recently developed Farrell cohomology theory. These are treated in Parts II and III, respectively. Part I is devoted to a review of the necessary background material.

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PART I. REVIEW

Good references for the material of Part I are [5] and [30].

§1. Finiteness conditions

Recall that the homology and cohomology of a group Γ can be defined algebraically, in terms of projective resolutions, as follows. Regard $\mathbb{Z}$ as a module (with trivial Γ-action) over the integral group ring $\mathbb{Z}\Gamma$, and choose a projective resolution $P = (P_i)_{i\geq 0}$:

$$\dots \to P_1 \to P_0 \to \mathbb{Z} \to 0 .$$

One then defines, for any Γ-module M,

$$H_*(\Gamma, M) = H_*(P \otimes_{\mathbb{Z}\Gamma} M) \text{ and } H^*(\Gamma, M) = H^*(\mathrm{Hom}_{\mathbb{Z}\Gamma}(P, M)).$$

[Note: We have been sloppy here about the distinction between left modules and right modules. To avoid ambiguity, let us agree that all modules in this paper are to be understood as left modules unless the contrary is explicitly stated. But then in order to make sense out of the tensor product above, one must convert P to a complex of right modules in the usual way, by setting $x\gamma = \gamma^{-1}x$ for $x \in P_i$, $\gamma \in \Gamma$.]

Alternatively, the homology and cohomology groups $H(\Gamma, M)$ can be defined topologically, in terms of Eilenberg-MacLane complexes. One chooses an Eilenberg-MacLane complex of type $K(\Gamma, 1)$, i.e., a connected CW-complex Y such that $\pi_1 Y = \Gamma$ and $\pi_i Y = 0$ for $i > 1$, and one sets

$$H(\Gamma, M) = H(Y, M),$$

where the groups on the right are to be interpreted as homology and cohomology groups with local coefficients. [The equivalence of the algebraic and topological definitions follows from the fact that the universal cover $\tilde{Y}$ of Y is contractible, so that its chain complex $C(\tilde{Y})$ provides a free resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$.]

(1.1) **Example.** Suppose Γ is a discrete subgroup of a Lie group G which has only finitely many connected components. Let K be a maximal compact subgroup of G and let X be the homogeneous space G/K. One knows that X is diffeomorphic to Euclidean space $\mathbb{R}^d$ ($d = \dim G - \dim K$) and that Γ acts properly on X (i.e., every point $x \in X$ has a neighbourhood U such that $\gamma U \cap U \neq \emptyset$ for only finitely many $\gamma \in \Gamma$). In particular, every isotropy group Γ_x is finite. If we now assume that Γ is torsion-free, then these isotropy groups are trivial, so that Γ acts freely on X and the projection $X \to \Gamma\backslash X$ is a covering map. Since X is contractible, it follows that the manifold $\Gamma\backslash X$ is a $K(\Gamma, 1)$, hence

$$H(\Gamma, M) \approx H(\Gamma\backslash X, M).$$

In the definitions of $H(\Gamma, M)$ above, one is free to choose the resolution P or the Eilenberg-MacLane complex Y. It is therefore natural to try to take them to be as 'small' as possible, and this leads to various finiteness notions. For example, if we interpret 'small' in terms of dimension, then we arrive at the notion of cohomological dimension:

one says that Γ has finite cohomological dimension if the following conditions, which are known to be equivalent, are satisfied:

(i) $\mathbb{Z}$ admits a projective resolution over $\mathbb{Z}\Gamma$ of finite length, i. e. a resolution of the form

$$0 \to P_n \to \ldots \to P_0 \to \mathbb{Z} \to 0.$$

(Such a resolution is said to have length $\leq n$.)

(ii) $\mathbb{Z}$ admits a free resolution over $\mathbb{Z}\Gamma$ of finite length.

(iii) There is an integer n such that $H^i(\Gamma, M) = 0$ for $i > n$ and all Γ-modules M.

(iv) There exists a finite dimensional $K(\Gamma, 1)$-complex.

If these conditions are satisfied then we define the cohomological dimension of Γ (denoted $\mathrm{cd}\,\Gamma$) to be the minimal length of a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$; otherwise we set $\mathrm{cd}\,\Gamma = \infty$. It is known that $\mathrm{cd}\,\Gamma$ is also equal to the minimal length of a free resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$, as well as to the smallest integer n satisfying (iii). If $\mathrm{cd}\,\Gamma \neq 2$ then $\mathrm{cd}\,\Gamma$ can also be described topologically, as the minimal dimension of a $K(\Gamma, 1)$-complex, but it is not known whether this is true if $\mathrm{cd}\,\Gamma = 2$; in this case one knows only that there exists a $K(\Gamma, 1)$ of dimension ≤ 3.

The torsion-free discrete subgroups of Lie groups as in 1.1 provide examples of groups of finite cohomological dimension. On the other hand, any group with torsion has infinite cohomological dimension. (In case Γ is a non-trivial finite cyclic group, this is proved by a direct calculation of $H^*(\Gamma)$, which is non-trivial in arbitrarily high dimensions; the general case follows from the elementary fact that $\mathrm{cd}\,\Gamma' \leq \mathrm{cd}\,\Gamma$ whenever $\Gamma' \subset \Gamma$.)

Further finiteness conditions are obtained by requiring not only that the projective resolution P be of finite length, but also that each module P_i be finitely generated. Such a resolution is said to be finite, and we will say that Γ is of type (FP) (resp. (FL)) if $\mathbb{Z}$ admits a finite projective (resp. free) resolution over $\mathbb{Z}\Gamma$. Unlike the situation in the definition of $\mathrm{cd}\,\Gamma$ above, where we allowed infinitely generated modules, there is no reason to expect that a group of type (FP) is necessarily of type (FL). Nevertheless, the surprising fact is that there are no known examples of groups of type (FP) which are not of type (FL). Indeed, such a group would necessarily have a non-trivial projective class group $\tilde{K}_0(\mathbb{Z}\Gamma)$, and there are no known examples of torsion-free groups Γ such that $\tilde{K}_0(\mathbb{Z}\Gamma) \neq 0$. In spite of this lack of examples, however, we will see below (cf. 5.4) that the theory of groups of type (FP) has concrete applications.

The (FP) and (FL) conditions have reasonable topological interpretations, at least if we assume that Γ is finitely presented: If Γ is finitely presented then Γ is of type (FL) if and only if there exists a $K(\Gamma, 1)$ which is a finite CW-complex, and Γ is of type (FP) if and only if some (and hence every) $K(\Gamma, 1)$ is finitely dominated, i.e. is a retract, in the homotopy category, of a finite complex. [Note: It is not known whether the (FP) (or (FL)) condition implies that Γ is finitely presented; if so, then the finite presentation assumption above can be dropped.]

For example, if Γ is a torsion-free subgroup of a Lie group G as in 1.1, and if Γ is co-compact (i.e. G/Γ is compact), then Γ is of type (FL). More interestingly, all torsion-free arithmetic groups are of type (FL) even though they are rarely co-compact (see Serre's lectures [32]).

We close this section by discussing the behavior of the finiteness conditions with respect to passage to subgroups of finite index.

(1.2) **Theorem** (Serre [30]). *Let Γ be a torsion-free group and Γ' a subgroup of finite index. Then* $\operatorname{cd} \Gamma' = \operatorname{cd} \Gamma$.

(1.3) **Corollary.** *If Γ and Γ' are as in 1.2, then Γ is of type (FP) if and only if Γ' is of type (FP).*

Remark. It is not known whether the analogous statement for groups of type (FL) is true.

We will now sketch the proof of the theorem; the corollary is left as an exercise for the reader. Assuming first that $\operatorname{cd} \Gamma < \infty$, it is easy to prove $\operatorname{cd} \Gamma' = \operatorname{cd} \Gamma$. For if $\operatorname{cd} \Gamma = n$ then the functor $H^n(\Gamma, -)$ is right exact, hence $H^n(\Gamma, F) \neq 0$ for some free $\mathbb{Z}\Gamma$-module F; letting F' be the free $\mathbb{Z}\Gamma'$-module of the same rank, we have $H^n(\Gamma', F') \approx H^n(\Gamma, F)$ (this is a special case of 'Shapiro's lemma'), so $\operatorname{cd} \Gamma' \geq n = \operatorname{cd} \Gamma$. The opposite inequality is trivial.

It remains to prove that if $\operatorname{cd}\Gamma' < \infty$ then $\operatorname{cd}\Gamma < \infty$. Let X' be a finite-dimensional contractible simplicial complex on which Γ' acts freely (and simplicially), i.e. X' is the universal cover of a finite-dimensional simplicial $K(\Gamma', 1)$. By a 'multiplicative induction' construction (see [30], 1.7, or [25], II, §16) one produces a simplicial Γ-complex X whose underlying simplicial complex is isomorphic to the product of $(\Gamma : \Gamma')$ copies of X'; in particular, X is contractible and finite dimensional. Moreover, the action of Γ on X is proper. Using now the hypothesis that Γ is torsion-free, we see that Γ acts freely on X, so that X/Γ is a finite dimensional $K(\Gamma, 1)$ and $\operatorname{cd}\Gamma < \infty$.

§2. Virtual notions

Groups Γ with torsion, as we have seen, cannot satisfy any of the finiteness conditions of §1. There will, however, often be torsion-free subgroups Γ' of finite index which do satisfy the finiteness conditions. (For example, in the arithmetic case we have the congruence subgroups.) We are thus led to introduce 'virtual' finiteness conditions.

Let Γ be a group which is virtually torsion-free, i.e. which has a torsion-free subgroup of finite index. By Serre's theorem (1.2), all such subgroups have the same cohomological dimension, and this common dimension is called the virtual cohomological dimension of Γ, denoted $\operatorname{vcd}\Gamma$. Similarly, we say that Γ is of type (VFP) (resp. (VFL)) if Γ has a subgroup of finite index of type (FP) (resp. (FL)). If Γ is of type (VFP) then Corollary 1.3 implies that every torsion-free subgroup of finite index is of type (FP). The analogous statement for groups of type (VFL) is not known, and one therefore introduces the following apparent strengthening of the (VFL) condition: A virtually torsion-free group is said to be of type (WFL) if every torsion-free subgroup of finite index is of type (FL). The main examples of groups of type (WFL) are the arithmetic groups, as well as the S-arithmetic groups in the reductive case (cf. [8], [9], [32]).

This paper is concerned with groups Γ such that $\operatorname{vcd}\Gamma < \infty$. This condition has the following topological interpretation, which follows immediately from the proof of Theorem 1.2:

(2.1) **Proposition.** Let Γ be a virtually torsion-free group. Then $\mathrm{vcd}\,\Gamma < \infty$ if and only if there exists a finite-dimensional contractible simplicial complex X on which Γ acts properly (and simplicially).

One should think of X as an analogue of the homogeneous space G/K which is available if Γ is a discrete subgroup of a Lie group, cf. 1.1.

For future reference we record the following fact, which comes from an examination of Serre's construction used in the proof of Theorem 1.2:

(2.2) **Addendum.** If $\mathrm{vcd}\,\Gamma < \infty$ then the space X in 2.1 can be chosen so that the fixed-point set X^H is contractible for every finite subgroup $H \subseteq \Gamma$.

Questions. 1. Can X always be chosen so that $\dim X = \mathrm{vcd}\,\Gamma$? We will see in the examples below a number of cases where this is known to be true, but the general case remains open, even if Γ is arithmetic. Note, in particular, that if Γ has torsion then the space X constructed by Serre in the proof of Theorem 1.2 always has $\dim X \geq 2\cdot \mathrm{vcd}\,\Gamma$, except in the trivial case where Γ is finite and X is a point.

2. What algebraic finiteness conditions on Γ will guarantee that X can be chosen so that X/Γ is compact? For arithmetic groups such an X exists by Borel-Serre [8] and the equivariant triangulation theorem [21]. Even for S-arithmetic groups, however, the question seems to be open, the problem being the existence of an equivariant triangulation (cf. [9, p231]). Note, again, that Serre's construction in the proof of 1.2 will never produce an X with compact quotient, unless Γ is finite.

Examples. 1. $\mathrm{vcd}\,\Gamma = 0$ if and only if Γ is finite.

2. $\mathrm{vcd}\,\Gamma \leq 1$ if and only if Γ is the fundamental group of a graph of finite groups of bounded order. This result is a generalization of the theorem of Stallings [35] and Swan [40] that groups of cohomological dimension 1 are free. See [28] for a proof and further references; see

also [31], ch. II, 2.6. (Note, in this case, that one does have a contractible 1-dimensional complex on which Γ operates properly.)

3. If Γ is a (finitely generated) 1-relator group then Γ is of type (WFL) and vcd $\Gamma \leq 2$. To prove this we use the Lyndon exact sequence [23]

$$0 \to \mathbb{Z}[\Gamma/C] \to \mathbb{Z}[\Gamma]^n \to \mathbb{Z}[\Gamma] \to \mathbb{Z} \to 0,$$

where n is the number of generators in some 1-relator presentation of Γ, and C is a finite cyclic subgroup of Γ. It is known [19] that Γ is virtually torsion-free, and clearly the above exact sequence provides a finite free resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma'$ of length 2 for any torsion-free subgroup $\Gamma' \subseteq \Gamma$ of finite index, whence our assertion. We remark that it is easy to realize Lyndon's exact sequence topologically as the cellular chain complex of a 2-dimensional CW-complex on which Γ operates properly and with compact quotient.

4. If Γ is a finitely generated nilpotent group then Γ is of type (WFL) and vcd Γ is equal to the rank (or Hirsch number) of Γ.

5. $GL_n(\mathbb{Z})$ is of type (WFL) and has virtual cohomological dimension $n(n-1)/2$. This is, of course, a special case of the Borel-Serre results on arithmetic groups ([8], [32]), but we will indicate here a different proof due to Ash [1], based on the reduction theory of Voronoi [43]. Let X be the space of positive-definite real quadratic forms in n variables, modulo multiplication by positive scalars. The group $GL_n(\mathbb{Z})$ acts properly on X (but with non-compact quotient). We have $\dim X = \frac{n(n+1)}{2} - 1$. According to Voronoi [43] (see also [22] and the references cited there), X can be enlarged to a space X^* with the following properties:

(a) The action of $GL_n(\mathbb{Z})$ on X extends to X^*, and $X^*/GL_n(\mathbb{Z})$ is compact. (This extended action, however, is not proper.)

(b) X^* admits a cell-decomposition compatible with the action of $GL_n(\mathbb{Z})$.

Let $\partial X^* = X^* - X$. A glance at Voronoi's definition of the cells of X^* shows:

(c) ∂X^* is a subcomplex and contains the $(n-2)$-skeleton of X^*.

It is easy to see that X^* admits a barycentric subdivision compatible with the $GL_n(\mathbb{Z})$-action, and we denote by X' the 'simplicial complement' of ∂X^* in this subdivision, i. e. the union of all closed simplices which are disjoint from ∂X^*. Then X' inherits a simplicial action of $GL_n(\mathbb{Z})$, and this action is proper since $X' \subset X$. Moreover, $X = X^* - \partial X^*$ admits a canonical deformation retraction onto X' (cf. [34], ch. 3, sec. 3, proof of Cor. 11), so X' is contractible. From (a) we see that $X'/GL_n(\mathbb{Z})$ is compact (whence $GL_n(\mathbb{Z})$ is of type (WFL)), and from (c) we see that X' has codimension at least $n - 1$ in X^*, so that

$$\text{vcd } GL_n(\mathbb{Z}) \leq \dim X' \leq \frac{n(n+1)}{2} - 1 - (n-1) = \frac{n(n-1)}{2} .$$

Finally, to show that these inequalities are in fact equalities, we need only note that $GL_n(\mathbb{Z})$ contains the strict upper triangular group, which is a finitely generated nilpotent group of rank $n(n - 1)/2$; thus $\text{vcd } GL_n(\mathbb{Z}) \geq n(n - 1)/2$.

Remark. The fact that X retracts onto a $GL_n(\mathbb{Z})$-invariant subspace X' of dimension $n(n - 1)/2$ was first proved by Serre for $n = 2$ (cf. [32], or [31], Ch. I, 4. 2), by Soulé [33] for $n = 3$, and by Ash [1] for arbitrary n. More generally, Ash proves the analogous statement for a class of arithmetic groups including the groups $GL_n(\mathbb{Z})$, using a generalization of Voronoi's theory.

§3. Duality groups and virtual duality groups

References: [5], [6].

For any group Γ we may regard $\mathbb{Z}\Gamma$ as a left Γ-module and define $H^*(\Gamma, \mathbb{Z}\Gamma)$. (The group $H^1(\Gamma, \mathbb{Z}\Gamma)$, for example, arises in the theory of ends of groups.) Since $\mathbb{Z}\Gamma$ is also a right Γ-module and the left and right actions commute, the groups $H^i(\Gamma, \mathbb{Z}\Gamma)$ inherit a right Γ-module structure. These modules play a special role in the theory of groups of type (FP).

Definition. Γ is called a <u>duality group</u> if the following two conditions are satisfied:

(i) Γ is of type (FP).

(ii) There is an integer n such that $H^i(\Gamma, \mathbb{Z}\Gamma) = 0$ for $i \neq n$ and $H^n(\Gamma, \mathbb{Z}\Gamma)$ is $\mathbb{Z}$-torsion-free.

The integer n here is necessarily equal to cd Γ; in fact, it is easy to see for any group Γ of type (FP) that cd Γ is equal to the largest integer i such that $H^i(\Gamma, \mathbb{Z}\Gamma) \neq 0$.

If Γ is a duality group then the Γ-module $D = H^n(\Gamma, \mathbb{Z}\Gamma)$ is called the dualizing module of Γ. The terminology 'duality group' and 'dualizing module' is justified by the existence of a duality isomorphism

$$(3.1)\quad H^i(\Gamma, M) \approx H_{n-i}(\Gamma, D \otimes M)$$

for any integer i and Γ-module M, where the tensor product is over $\mathbb{Z}$ and is given the diagonal Γ-action: $\gamma \cdot (d \otimes m) = d\gamma^{-1} \otimes \gamma m$ for $\gamma \in \Gamma$, $d \in D$, $m \in M$.

To prove 3.1, choose a finite projective resolution P of length n of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ and let P' be the dual complex of projective right $\mathbb{Z}\Gamma$-modules, i.e. $P' = \mathrm{Hom}_{\mathbb{Z}\Gamma}(P, \mathbb{Z}\Gamma)$. Since $H^i(\Gamma, \mathbb{Z}\Gamma) = 0$ for $i \neq n$, P' provides a projective resolution of D over $\mathbb{Z}\Gamma$:

$$0 \to P'_0 \to \dots P'_n \to D \to 0.$$

Using the canonical isomorphism $\mathrm{Hom}_{\mathbb{Z}\Gamma}(P, M) \approx P' \otimes_{\mathbb{Z}\Gamma} M$, we deduce

$$H^i(\Gamma, M) \approx \mathrm{Tor}^{\mathbb{Z}\Gamma}_{n-i}(D, M).$$

Finally, since D is $\mathbb{Z}$-torsion-free we have

$$\mathrm{Tor}^{\mathbb{Z}\Gamma}_*(D, M) \approx H_*(\Gamma, D \otimes M),$$

whence 3.1.

Remarks. 1. Conversely, if Γ is a group such that there exist isomorphisms of the form 3.1 which are natural in M, (where D is a fixed Γ-module and n is a fixed integer), then Γ is a duality group. Indeed, Γ is then of type (FP) by [13] or [36], and condition (ii) above is easily derived from 3.1.

2. If the dualizing module D is $\mathbb{Z}$-free, which is the case in all known examples, then there are also isomorphisms

(3.2) $H_i(\Gamma, M) \approx H^{n-i}(\Gamma, \mathrm{Hom}(D, M))$,

where $\mathrm{Hom}(\ ,\ ) = \mathrm{Hom}_{\mathbb{Z}}(\ ,\ )$, with the diagonal Γ-action.

3. If Γ is an arbitrary group of type (FP), then one can still derive isomorphisms of the form 3.1 and 3.2, but with the dualizing module D replaced by a 'dualizing chain complex'. (The groups on the right-hand side of 3.1 and 3.2 must then be interpreted as in the appendix at the end of this section.) Conversely, the existence of such generalized duality isomorphisms, natural in M, implies that Γ is of type (FP).

A duality group is said to be a <u>Poincaré duality group</u> if D, as $\mathbb{Z}$-module, is infinite cyclic. In this case the duality isomorphisms take a form more familiar to topologists:

$$H^i(\Gamma, M) \approx H_{n-i}(\Gamma, \tilde{M}),$$

where $\tilde{M}$ denotes M with the Γ-action 'twisted' by the character $\Gamma \to \{\pm 1\}$ by which Γ acts on D. (For example, if there exists a $K(\Gamma, 1)$ which is a closed manifold, then Γ is a Poincaré duality group.) From the point of view of group theory, however, Poincaré duality is rather rare. Torsion-free arithmetic groups, for example, are always duality groups, but they are Poincaré duality groups only in the rank 0 case ([8], 11.4).

A group Γ is said to be a <u>virtual duality group</u> if it contains a subgroup of finite index which is a duality group. This is equivalent to saying that Γ is of type (VFP) and that Γ satisfies condition (ii) of the definition of 'duality group'. Again we set $D = H^n(\Gamma, \mathbb{Z}\Gamma)$ and we note that every torsion-free subgroup $\Gamma' \subseteq \Gamma$ of finite index is a duality group whose dualizing module is D, regarded as Γ'-module. [More generally, if Γ is an arbitrary group of type (VFP) then one can find a chain complex of Γ-modules which serves as dualizing complex in the sense of Remark 3 above for every torsion-free subgroup of finite index.]

We mention one example, which is a special case of the Borel-Serre results on arithmetic groups [8]: The group $GL_r(\mathbb{Z})$ is a virtual duality group of dimension $r(r-1)/2$, with

$$(3.3)\quad D = \begin{cases} St & \text{if } r \text{ is odd} \\ \tilde{St} & \text{if } r \text{ is even,} \end{cases}$$

where St is the 'Steinberg module' and $\tilde{St}$ denotes St with the $GL_r(\mathbb{Z})$-action twisted by $\det : GL_r(\mathbb{Z}) \to \{\pm 1\}$.

Appendix. Homology with coefficients in a chain complex

Let Γ be a group and $C = (C_i)_{i\geq 0}$ a chain complex of $\mathbb{Z}\Gamma$-modules. We then set

$$H_*(\Gamma, C) = H_*(P \otimes_{\mathbb{Z}\Gamma} C),$$

where P is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ and the tensor product is the total tensor product, i.e. the total complex associated to the double complex $P_\cdot \otimes_{\mathbb{Z}\Gamma} C_\cdot$. Note that if C consists of a single module M concentrated in dimension 0, then $H_*(\Gamma, C) = H_*(\Gamma, M)$.

The definition immediately gives us two spectral sequences converging to $H_*(\Gamma, C)$. The first has

$$E^2_{pq} = H_p(\Gamma, H_qC);$$

the second has

$$E^1_{pq} = H_q(\Gamma, C_p),$$

with the differential d^1 induced by the differential in C. In particular, one obtains from these spectral sequences the following two properties of $H_*(\Gamma, C)$:

(3.4) If each C_p is projective over $\mathbb{Z}\Gamma$ then $H_*(\Gamma, C) \approx H_*(C_\Gamma)$. [Here $C_\Gamma = H_0(\Gamma, C) = \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} C$] More generally, the same conclusion holds if each C_p is H_*-acyclic, i.e. if $H_q(C_p) = 0$ for $q > 0$.

(3.5) If $f : C \to C'$ is a weak equivalence of chain complexes (i.e. $f_* : H_*C \to H_*C'$ is an isomorphism), then f induces an isomorphism

$H_*(\Gamma, C) \overset{\approx}{\to} H_*(\Gamma, C')$.

One can also define cohomology groups $H^*(\Gamma, C)$, where $C = (C^i)_{i \geq 0}$ is a cochain complex, as the cohomology of the total complex associated to $\mathrm{Hom}_{\mathbb{Z}\Gamma}(P_\cdot, C^\cdot)$. Again there are two spectral sequences and properties analogous to those above.

PART II. EULER CHARACTERISTICS

Main references: [11], [30]; see also [4].

We wish to define the Euler characteristic of a group of type (FP) as the alternating sum of the 'ranks' of the projective modules P_i which occur in a finite projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$. We begin, therefore, by defining a suitable notion of rank.

§4. Ranks of projective modules

If Γ is a group and P a Γ-module, we denote by P_Γ the abelian group $H_0(\Gamma, P) = \mathbb{Z} \otimes_{\mathbb{Z}\Gamma} P$. If P is finitely generated and projective over $\mathbb{Z}\Gamma$ then P_Γ is a finitely generated free $\mathbb{Z}$-module, and we set

$$\varepsilon(P) = \mathrm{rank}_{\mathbb{Z}}(P_\Gamma).$$

We will sometimes write $\varepsilon_\Gamma(P)$ instead of $\varepsilon(P)$ when this is necessary for clarity. The following proposition shows that ε has the multiplicative property which one expects of a reasonable 'rank':

(4.1) **Proposition.** Let $\Gamma' \subset \Gamma$ be a subgroup of finite index. If P is a finitely generated projective $\mathbb{Z}\Gamma$-module, then P is also finitely generated and projective as $\mathbb{Z}\Gamma'$-module, and

$$\varepsilon_{\Gamma'}(P) = (\Gamma : \Gamma') \cdot \varepsilon_\Gamma(P).$$

Proof. Let $\Gamma'' \subseteq \Gamma'$ be a subgroup of finite index which is normal in Γ. Then we may replace Γ, Γ', and P by Γ/Γ'', Γ'/Γ'', and $P_{\Gamma''}$ to reduce to the case where Γ is finite. But in this case one knows by a theorem of Swan that $\mathbb{Q} \otimes_{\mathbb{Z}} P$ is free over $\mathbb{Q}\Gamma$. It is clear, then, that

$\varepsilon_\Gamma(P)$ is simply the rank of this free $\mathbb{Q}\Gamma$-module, and the proposition follows at once. (Proofs of Swan's theorem can be found in [38], [39], [2], and [3]; see also [4].)

Remark. There is another notion of rank, which we will denote $\rho(P)$, defined as the coefficient of the conjugacy class of 1 in the Hattori-Stallings rank of P. (Recall that the Hattori-Stallings rank of P, which we denote $r(P)$, is a finite linear combination of Γ-conjugacy classes, cf. [3], [4].) The rank ρ, like ε, has the multiplicative property

(4.2) $\rho_{\Gamma'}(P) = (\Gamma : \Gamma') \cdot \rho_\Gamma(P)$.

Bass's 'weak conjecture' ([3], p. 156) says that one always has $\varepsilon = \rho$, and, as Bass observed ([3], 6.10), this is easily proved if Γ is residually finite. To see this, note first that one can express $\varepsilon(P)$ as the sum of the coefficients of $r(P)$, hence $\varepsilon(P) = \rho(P)$ if $r(P)$ is concentrated at the conjugacy class of 1. Now if Γ is residually finite, then we can find a subgroup Γ' of finite index which does not contain the finitely many non-trivial conjugacy classes where $r(P)$ has a non-zero coefficient. We will then have $\varepsilon_{\Gamma'}(P) = \rho_{\Gamma'}(P)$, and hence $\varepsilon_\Gamma(P) = \rho_\Gamma(P)$ by 4.1 and 4.2.

§5. Euler characteristics for groups of type (FP)

One can use either of the ranks ε and ρ discussed in the previous section to define the Euler characteristic of a group of type (FP). For our purposes it will be more convenient to use ε. (Of course, the two definitions agree if Γ is residually finite by what we have just proved, and they agree for all Γ if Bass's weak conjecture is true. See [3], [4], and [17] for a discussion of the Euler characteristic based on ρ.)

Thus let Γ be of type (FP) and let $P = (P_i)$ be a finite projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$. We then set

$$\chi(\Gamma) = \sum (-1)^i \varepsilon(P_i) = \sum (-1)^i \operatorname{rank}_{\mathbb{Z}} (P_i)_\Gamma .$$

Note that the homology of the complex P_Γ is $H_*\Gamma$, so we can also write

$$\chi(\Gamma) = \sum(-1)^i \operatorname{rank}_{\mathbb{Z}}(H_i\Gamma) .$$

Thus $\chi(\Gamma)$ is simply the 'naive' Euler characteristic, which could have been defined a priori, without any discussion of ranks. The point of the definition in terms of ε, however, is that we immediately obtain from 4.1 the multiplicative property

(5.1) $\chi(\Gamma') = (\Gamma : \Gamma') \cdot \chi(\Gamma)$

if $\Gamma' \subset \Gamma$ is a subgroup of finite index. This property is by no means obvious from the naive definition, and some argument like that of §4 is needed in order to prove it. On the other hand, (5.1) is obvious if Γ is of type (FL). We will also need a multiplicative property of the Euler characteristic with respect to the coefficient module; again this is obvious if Γ is of type (FL) but requires some work in general.

(5.2) **Proposition.** *Suppose* Γ *is of type* (FP), k *is a field, and* V *is a* $k\Gamma$-*module of finite dimension over* k. *Then*

$$\sum(-1)^i \dim_k H_i(\Gamma, V) = \chi(\Gamma) \cdot \dim_k V = \sum(-1)^i \dim_k H^i(\Gamma, V).$$

A proof of the second equality can be found in [11], §4, and the first equality is proved similarly.

Before proceeding further, we mention a group theoretic application of the existence of an integer-valued Euler characteristic satisfying 5.1 for groups of type (FP):

(5.3) **Proposition.** *Let* Γ *be a group of type* (FP). *If* Γ *can be embedded as a subgroup of finite index in a torsion-free group* $\overline{\Gamma}$, *then* $\chi(\Gamma)$ *is divisible by* $(\overline{\Gamma} : \Gamma)$.

The proof is immediate, for $\overline{\Gamma}$ is of type (FP) by 1.3, hence

$$\frac{\chi(\Gamma)}{(\overline{\Gamma} : \Gamma)} = \chi(\overline{\Gamma}) \in \mathbb{Z} .$$

Thus $|\chi(\Gamma)|$, if non-zero, provides an obstruction to the existence of torsion-free enlargements of Γ.

(5.4) **Remark.** Even if one is only interested in the case where Γ is of type (FL), the proof requires a theory of Euler characteristics for groups of type (FP), since one does not know that $\overline{\Gamma}$ will be of type (FL).

(5.5) **Corollary.** Let $1 \to \Gamma \to E \to P \to 1$ be a group extension, where Γ is of type (FP) and P has prime order p. If $p \nmid \chi(\Gamma)$ then the extension splits.

In fact, E necessarily has torsion by 5.3; Γ being torsion-free, it follows that any non-trivial finite subgroup of E must map isomorphically to P, thus providing a splitting.

We will see later (Cor. 7.3) that 5.5 can be substantially improved.

§6. Extension to groups of type (VFP)

Let Γ be a group of type (VFP). Following the method of Wall [44], we then define $\chi(\Gamma)$ by choosing a subgroup Γ' of finite index which is of type (FP) and setting

$$\chi(\Gamma) = \frac{\chi(\Gamma')}{(\Gamma : \Gamma')},$$

the right-hand side being independent of the choice of Γ' by 5.1. Note that $\chi(\Gamma)$ is a rational number and is not, in general, an integer. For example, if Γ is finite then $\chi(\Gamma) = 1/|\Gamma|$. If Γ is torsion-free, on the other hand, then Γ is of type (FP) and hence $\chi(\Gamma) \in \mathbb{Z}$.

We list some useful properties of the Euler characteristic:

(6.1) If $\Gamma' \subset \Gamma$ is a subgroup of finite index, then

$$\chi(\Gamma') = (\Gamma : \Gamma') \cdot \chi(\Gamma).$$

This is immediate from the definition.

(6.2) Let $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ be a group extension, where Γ' and Γ'' are of type (VFP). If Γ is virtually torsion-free then Γ is of type (VFP) and

$$\chi(\Gamma) = \chi(\Gamma') \cdot \chi(\Gamma'').$$

The proof that Γ is of type (VFP) is straightforward. To prove the Euler characteristic formula, one reduces to the case where all groups are of type (FP), in which case the result follows from a spectral sequence argument together with 5.2.

(6.3) Let Γ be an amalgamation $\Gamma_1 *_A \Gamma_2$ where $A \hookrightarrow \Gamma_i$, and suppose Γ_1, Γ_2, and A are of type (VFP). If Γ is virtually torsion-free then Γ is of type (VFP) and

$$\chi(\Gamma) = \chi(\Gamma_1) + \chi(\Gamma_2) - \chi(A).$$

This can be proved exactly as in [30], where the (WFL) case is treated.

As an example of 6.3 we may take $\Gamma = SL_2(\mathbb{Z}) \approx \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ (where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$). We obtain

$$\chi(SL_2(\mathbb{Z})) = \frac{1}{4} + \frac{1}{6} - \frac{1}{2} = -\frac{1}{12}.$$

(Alternatively, one can derive this formula from the fact that $SL_2(\mathbb{Z})$ contains a subgroup of index 12 which is free on two generators, cf. [30], 1.8, Ex. 2).

The theory of Euler characteristics becomes especially interesting when applied to Chevalley groups over a ring of algebraic integers. In this case one has Harder's formula expressing $\chi(\Gamma)$ in terms of values of ζ-functions (see [20], [30], [32]). For future reference we record two special cases of this formula:

$$(6.4)\quad \chi(Sp_{2n}(\mathbb{Z})) = \prod_{i=1}^{n} \zeta(1-2i) = \prod_{i=1}^{n} -B_{2i}/2i,$$

where B_{2i} is the $2i^{th}$ Bernoulli number ($B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, ...).

$$(6.5)\quad \chi(E_7(\mathbb{Z})) = -\frac{691 \cdot 43867}{2^{21} \cdot 3^9 \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 19}.$$

Note that $Sp_2 = SL_2$, so we recover from 6.4 (with $n = 1$) the formula $\chi(SL_2(\mathbb{Z})) = -1/12$.

§7. Integrality properties of $\chi(\Gamma)$

Throughout this section Γ will denote an arbitrary group of type (VFP). We have seen that $\chi(\Gamma)$ need not be an integer if Γ has torsion.

The results of this section and the next resulted from an attempt to explain more precisely the relation between the torsion in Γ and the non-integrality of $\chi(\Gamma)$. The first result along these lines is the following observation due to Serre ([30], 1.8, Prop. 13):

(7.1) **Proposition.** If p is a prime such that Γ has no p-torsion, then $\chi(\Gamma)$ is p-integral, i.e. p does not occur in the denominator of $\chi(\Gamma)$.

[Taking $\Gamma = E_7(\mathbf{Z})$, for example, it follows from this proposition and 6.5 that $E_7(\mathbf{Z})$ must have p-torsion for $p = 2, 3, 5, 7, 11, 13, 19$.]

To prove the proposition choose a torsion-free normal subgroup $\Gamma' \subseteq \Gamma$ of finite index, and choose $\Gamma_p (\Gamma' \subseteq \Gamma_p \subseteq \Gamma)$ so that Γ_p/Γ' is a p-Sylow subgroup of Γ/Γ'. Then $(\Gamma : \Gamma_p)$ is relatively prime to p, and Γ_p is torsion-free (since any torsion would be p-torsion). Thus $\chi(\Gamma_p) \in \mathbf{Z}$ and $\chi(\Gamma) = \chi(\Gamma_p)/(\Gamma : \Gamma_p)$ is indeed p-integral.

Serre went on to conjecture the following more precise result, which was proved in [11]:

(7.2) **Theorem.** Let m be the least common multiple of the orders of the finite subgroups of Γ. Then $m \cdot \chi(\Gamma) \in \mathbf{Z}$.

Note that the p-part of m for a given prime p is simply the maximal order of a p-subgroup of Γ, so the theorem can be restated as follows: If a prime power p^k occurs in the denominator of $\chi(\Gamma)$, then Γ has a subgroup of order p^k.

For example, taking $\Gamma = E_7(\mathbf{Z})$ again, we see that not only must $E_7(\mathbf{Z})$ contain elements of order $2, 3, \ldots$, but it must contain subgroups of order $2^{21}, 3^9, \ldots$. This application of Theorem 7.2 to the study of torsion in the exceptional Chevalley groups is due to Serre. See [32] for a more detailed discussion.

Another application is the promised improvement of 5.5.

(7.3) **Corollary.** Let $1 \to \Gamma \to E \to P \to 1$ be a group extension such that Γ is of type (FP) and P is a p-group for some prime p. If $p \nmid \chi(\Gamma)$ then the extension splits.

Proof. One has $\chi(E) = \chi(\Gamma)/|P|$, and this fraction is in lowest terms. By the theorem, E must contain a subgroup of order equal to $|P|$, and any such subgroup provides a splitting of the extension.

As an example of the corollary, take $\Gamma = F_n$, the free group on n generators. Then $\chi(F_n) = 1 - n$, so an extension as above must split if $p \nmid n - 1$. This result is vacuous if $n = 1$ and easy to prove directly if $n = 2$, using the known structure of the group of outer automorphisms of F_2 ([24], §3.5, Cor. N4). If $n \geq 3$, however, I know of no proof other than that given here, based on the theory of Euler characteristics.

We now prove Theorem 7.2. Let X be a finite-dimensional contractible simplicial complex on which Γ acts properly (2.1). Let $\Gamma' \subseteq \Gamma$ be a torsion-free normal subgroup of finite index and let $Y = X/\Gamma'$. [Note: Replacing X by its barycentric subdivision, if necessary, we can assume that Y inherits a cell-decomposition from that of X. Taking another barycentric subdivision, we can even make Y simplicial, cf. [10].] Since Γ' is of type (FP) and acts freely on X, we have $\chi(\Gamma') = \chi(Y)$, the latter being, by definition, $\sum(-1)^i \mathrm{rk}(H_i Y)$. Hence $\chi(\Gamma) = \chi(Y)/(\Gamma : \Gamma')$ and what we are trying to prove, then, is that

$$\frac{m}{(\Gamma : \Gamma')} \cdot \chi(Y) \in \mathbb{Z},$$

or, in other words, that $\chi(Y)$ is divisible by the integer $d = (\Gamma : \Gamma')/m$.

To this end we note that the action of Γ on X induces a (simplicial) action of $G = \Gamma/\Gamma'$ on $Y = X/\Gamma'$. Moreover, the isotropy groups G_y $(y \in Y)$ are simply the images in G of the isotropy groups Γ_x $(x \in X)$, hence they all have order dividing m. Thus every orbit Gy has cardinality divisible by d, and one would like to conclude that $\chi(Y)$ is divisible by d. This is trivially true if Y is compact, since $\chi(Y)$ can then be computed by counting simplices, and the number of these in each dimension is divisible by d. If Y is not compact, one still knows that Y is finite dimensional and that $H_* Y$ is finitely generated, and it turns out that these finiteness conditions on Y are enough to yield the result that $d \mid \chi(Y)$. In fact, one can prove:

(7.4) Theorem. Let Y be a paracompact space of finite cohomological dimension in the sense of sheaf theory, and assume that

$H^*(Y, \mathbb{Z})$ is finitely generated. If a finite group G acts on Y and the cardinality of each orbit Gy is divisible by some integer d, then $d \mid \chi(Y)$.

[Here $H^*(Y, \mathbb{Z})$ denotes the sheaf-theoretic (or Čech) cohomology of Y, and $\chi(Y)$ is defined to be $\sum(-1)^i \mathrm{rk}(H^i(\Gamma, \mathbb{Z}))$.]

We will sketch the proof of this theorem; for further details see [11], §2. Note first that, by a Sylow argument, we may reduce to the case where G is a p-group for some prime p. Moreover, since $H^*(Y, \mathbb{Z})$ is finitely generated, $\chi(Y)$ is equal to the mod p Euler characteristic $\sum(-1)^i \dim_{\mathbb{Z}_p} H^i(Y, \mathbb{Z}_p)$. Throughout the remainder of this proof, then, $H^*(\ )$ will denote $H^*(\ , \mathbb{Z}_p)$ and χ will denote the mod p Euler characteristic.

(a) If G acts freely on Y, then the desired result that $|G| \,\big|\, \chi(Y)$ is a well-known consequence of Smith theory, cf. [7], ch. III. More generally, one has the following relative version of this result: If $Y' \subset Y$ is a G-invariant closed subspace such that $H^*(Y, Y')$ is finitely generated and G acts freely in $Y - Y'$, then $|G| \,\big|\, \chi(Y, Y')$.

(b) In the general case we use the technique of 'stratification by orbit type'. For any subgroup $H \subseteq G$ let $Y_H = \{y \in Y : G_y = H\}$ and let $Y_{\{H\}} = G \cdot Y_H$. (Thus $Y_{\{H\}}$ is the union of all orbits of type G/H.) Let $\mathcal{C}$ be a set of representatives for the conjugacy classes of subgroups of G which occur as isotropy groups in Y. It is easy to see that there is a filtration of Y by closed subspaces $\emptyset = Y_0 \subset \ldots \subset Y_n = Y$, such that the successive differences $Y_i - Y_{i-1}$ are the subspaces $Y_{\{H\}}$ $(H \in \mathcal{C})$. It follows that

$$\chi(Y) = \sum_{H \in \mathcal{C}} \chi'(Y_{\{H\}}),$$

where χ' is defined as follows: If A is a locally closed subspace of Y then we write $A = B - B'$, where B and B' are closed and $B' \subseteq B$; if $H^*(B, B')$ is finitely generated then we set $\chi'(A) = \chi(B, B')$, this being independent of the choice of (B, B'). (Alternatively, $\chi'(A)$ can be defined in terms of the cohomology of A with supports in the family of subsets of A which are closed in Y.) One must verify, of course, that $\chi'(Y_{\{H\}})$ is defined, but this follows easily from the fact (known from Smith theory) that each fixed point set X^H has finitely generated mod p cohomology.

It suffices, therefore, to prove that $\chi'(Y_{\{H\}})$ is divisible by $(G : H)$. Now clearly

$$Y_{\{H\}} = \coprod_{g \in G/N(H)} g \cdot Y_H,$$

where $N(H)$ is the normalizer of H in G, so

$$\chi'(Y_{\{H\}}) = (G : N(H)) \cdot \chi'(Y_H).$$

On the other hand, the group $N(H)/H$ acts freely in Y_H, so the relative version of (a) implies that $\chi'(Y_H)$ is divisible by $(N(H) : H)$. Thus $\chi'(Y_{\{H\}})$ is indeed divisible by $(G : N(H)) \cdot (N(H) : H) = (G : H)$.

§8. Formulas for $\chi(\Gamma)$

A careful examination of the proof of Theorem 7.2 yields more precise information than what was stated. For example, suppose Γ satisfies the following condition:

(8.1) Γ has only finitely many conjugacy classes of finite subgroups, and for each finite subgroup H the normalizer $N(H)$ is of type (VFP).

One can then derive ([11], §6) a formula of the form

(8.2) $\chi(\Gamma) = \tilde{\chi}(\Gamma) + \sum_{H \in \mathcal{C}} c_H / |H|$,

where $\tilde{\chi}(\Gamma)$ is the 'naive' Euler characteristic $\sum(-1)^i \mathrm{rk}_{\mathbb{Z}}(H_i \Gamma)$, $\mathcal{C}$ is a set of representatives for the conjugacy classes of non-trivial finite subgroups of Γ, and c_H is an integer which is defined in terms of the conjugation action of $N(H)$ on the ordered set of finite subgroups of Γ containing H. This formula then 'explains', in terms of the torsion in Γ, the failure of $\chi(\Gamma)$ to equal the integer $\tilde{\chi}(\Gamma)$. We will not prove 8.2 here, but we will instead give some results which are less precise but easier to use in practice.

We will need the notion of 'equivariant Euler characteristic' for a pair (Γ, K), where Γ is a group and K a CW-complex on which Γ acts. For simplicity we will assume that the following two conditions are satisfied:

(i) The Γ-action permutes the cells of K.

(ii) For each cell σ of K, the isotropy group Γ_σ fixes σ pointwise.

We then say that K is an admissible Γ-complex. [Note: Condition (ii) is harmless in practice; in the case of a simplicial action, for example, it can always be achieved by passing to the barycentric subdivision.] If, in addition, K/Γ is compact and each isotropy group Γ_σ is of type (VFP), then we define the equivariant Euler characteristic $\chi_\Gamma(K)$ by

$$\chi_\Gamma(K) = \sum (-1)^{\dim \sigma} \chi(\Gamma_\sigma),$$

where σ ranges over a set of representatives for the cells of K mod Γ. It is easy to verify (cf. [30], 1.8, proof of Prop. 14(b)), that

$$\chi_{\Gamma'}(K) = (\Gamma : \Gamma') \cdot \chi_\Gamma(K)$$

if $\Gamma' \subset \Gamma$ is a subgroup of finite index.

We will be particularly interested in the case where K arises from a partially ordered set S on which Γ operates, i.e. K is the simplicial complex $K(S)$ (sometimes called the nerve of S) whose vertices are the elements of S and whose n-simplices correspond to the chains $s_0 < s_1 < \ldots < s_n$ in S. In this case we set

$$\chi_\Gamma(S) = \chi_\Gamma(K(S)),$$

if the right-hand side is defined.

We can now state (cf. [11], §6):

(8.3) Theorem. *Let Γ be a group which satisfies condition 8.1 and let $\mathfrak{F}$ be the set of non-trivial finite subgroups of Γ. Regard $\mathfrak{F}$ as an ordered set under inclusion, with Γ-action by conjugation. Then $\chi_\Gamma(\mathfrak{F})$ is defined and*

$$\chi(\Gamma) \equiv \chi_\Gamma(\mathfrak{F}) \pmod{\mathbb{Z}}.$$

This theorem can be regarded as a formula for the 'fractional part' of $\chi(\Gamma)$ in terms of the Euler characteristics of groups of the form $N(H_0) \cap \ldots \cap N(H_n)$, where $H_0 \subset \ldots \subset H_n$ is a chain of non-trivial finite subgroups of Γ. There is also a 'local' version of the theorem, proved in [12], which says that if we just want the 'p-fractional part' of $\chi(\Gamma)$ for a fixed prime p, then it suffices to consider those finite subgroups H which are p-groups, i.e.

$$\chi(\Gamma) \equiv \chi_\Gamma(\mathfrak{F}_p) \pmod{\mathbb{Z}_{(p)}},$$

where $\mathfrak{F}_p$ is the set of non-trivial finite p-subgroups of Γ and $\mathbb{Z}_{(p)}$ is $\mathbb{Z}$ localized at p. Quillen [26] improved this result by showing that $\mathfrak{F}_p$ can be replaced by the smaller set $\mathfrak{A}_p$ consisting of the non-trivial elementary abelian p-subgroups of Γ. (Recall that an elementary abelian p-group is a group isomorphic to $(\mathbb{Z}/p)^r$ for some integer r, called the rank of the group.) The precise statement of this improved result is:

(8.4) Theorem. Let Γ be a group and p a prime such that N(H) is of type (VFP) for every elementary abelian p-subgroup $H \subseteq \Gamma$. Then $\chi_\Gamma(\mathfrak{A}_p)$ is defined and

$$\chi(\Gamma) \equiv \chi_\Gamma(\mathfrak{A}_p) \pmod{\mathbb{Z}_{(p)}}.$$

We will give the proofs of Theorems 8.3 and 8.4 in the next section.

Remark. Theorem 8.4 (unlike Theorem 8.3) is non-vacuous even if Γ is finite. In this case the congruence above can be unscrambled to yield

$$\chi(\mathfrak{A}_p) \equiv 1 \pmod{p^k},$$

where p^k is the highest power of p dividing $|\Gamma|$ and $\chi(\mathfrak{A}_p)$ is the Euler characteristic of the finite complex $K(\mathfrak{A}_p)$. See Quillen [26] for further results about the homotopy type of $K(\mathfrak{A}_p)$.

The simplest case of Theorem 8.4 is that where every elementary abelian p-subgroup of Γ has rank ≤ 1, i.e. Γ contains no subgroup isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$. In this case $K(\mathfrak{A}_p)$ is discrete and one has

$$\chi_\Gamma(\mathfrak{a}_p) = \sum \chi(N(P)) ,$$

where P ranges over the subgroups of Γ of order p, up to conjugacy. Using the fact that each P contains exactly p - 1 elements of order p, one can easily rewrite the right-hand side of this equation in terms of the elements of Γ of order p and their centralizers, and one obtains:

(8.5) **Corollary.** *Let Γ be a group of type (VFP) and p a prime such that Γ contains no subgroup isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$. For each element α of Γ of order p, assume that the centralizer $Z(\alpha)$ is of type (VFP). Then Γ has only finitely many conjugacy classes of elements of order p, and*

$$\chi(\Gamma) \equiv \frac{1}{p-1} \sum \chi(Z(\alpha)) \qquad (\mathrm{mod}\ \mathbb{Z}_{(p)}) ,$$

where α ranges over the elements of order p, up to conjugacy.

As an application of this corollary, due to Serre, one can recover Kummer's criterion in terms of Bernoulli numbers for the irregularity of a prime p. This is done by taking $\Gamma = Sp_{p-1}(\mathbb{Z})$ and combining the above congruence with Harder's formula 6.4. See [11], §9.4, and [12], §4, for details and a generalization.

§9. Proofs of Theorems 8.3 and 8.4

The proofs will require the rudiments of equivariant homology theory. Specifically, we will need to know that there are groups $H_*^\Gamma(K)$, defined, say, if K is an admissible Γ-complex, and having the following three properties:

(9.1) If Γ acts freely on K then $H_*^\Gamma(K) \approx H_*(K/\Gamma)$.

(9.2) If $f : K \to K'$ is a Γ-equivariant cellular map which induces an isomorphism $H_*K \to H_*K'$, then f induces an isomorphism

$$H_*^\Gamma(K) \xrightarrow{\approx} H_*^\Gamma(K') .$$

(9.3) There is a spectral sequence converging to $H_*^{\Gamma}(K)$, with

$$E^1_{pq} = \bigoplus_{\sigma \in \Sigma_p} H_q(\Gamma_\sigma),$$

where Σ_p is a set of representatives for the p-cells of K mod Γ. Consequently, if K/Γ is compact and each Γ_σ is of type (FP), then

$$\sum (-1)^i \mathrm{rk}_{\mathbb{Z}} H_i^{\Gamma}(K) = \chi_{\Gamma}(K) .$$

There are various ways to define the equivariant homology groups and prove the above properties. For example, one can set

$$H_*^{\Gamma}(K) = H_*(\Gamma, C(K)) ,$$

where $C(K)$ is the cellular chain complex of K and the right-hand side is to be interpreted in the sense of the appendix to §3. The properties 9.1-9.3 then follow from results stated in that appendix.

We can now prove Theorem 8.3. First, the fact that $\chi_{\Gamma}(\mathfrak{F})$ is defined is an easy consequence of 8.1, cf. [11], §5, Lemma. Now let X, as in the proof of Theorem 7.2, be a finite-dimensional contractible simplicial complex on which Γ acts properly, let $\Gamma' \subseteq \Gamma$ be a torsion-free normal subgroup of finite index, and let Y be the Γ/Γ'-complex X/Γ'. Assume further that X has been chosen so that X^H is contractible for $H \in \mathfrak{F}$, cf. 2.2. Let X_0 be the set of points of X with non-trivial isotropy group and let $Y_0 = X_0/\Gamma'$. I claim that Y_0 has finitely generated homology. Accepting this for the moment, and noting that Γ/Γ' acts freely in $Y - Y_0$, we obtain (cf. proof of Theorem 7.4)

$$\chi(Y) \equiv \chi(Y_0) \quad (\mathrm{mod}(\Gamma : \Gamma')).$$

Thus

$$(9.4)\quad \chi(\Gamma) = \frac{\chi(Y)}{(\Gamma : \Gamma')} \equiv \frac{\chi(Y_0)}{(\Gamma : \Gamma')} \qquad (\mathrm{mod}\ \mathbb{Z}) .$$

Observe now that $X_0 = \cup_{H \in \mathfrak{F}} X^H$. Since each X^H is contractible, one deduces that X_0 is homotopy equivalent to the 'nerve' of the covering $\{X^H\}$, and in the present context 'nerve' can be taken to mean the complex $K(\mathfrak{F})$:

(9.5) $X_0 \simeq K(\mathfrak{F})$.

(Cf. [11], Appendix B, and [26], proof of 4.1.) Moreover, this homotopy equivalence can be taken to be compatible with the Γ-action, in the sense that there is a third Γ-complex which maps to both X_0 and $K(\mathfrak{F})$ by Γ-equivariant maps which are homotopy equivalences. Using 9.1 and 9.2, we conclude that

$$H_*(Y_0) \approx H_*^{\Gamma'}(X_0) \approx H_*^{\Gamma'}(K(\mathfrak{F})) .$$

Thus $H_*(Y_0)$ is indeed finitely generated and, by 9.3, $\chi(Y_0) = \chi_{\Gamma'}(\mathfrak{F})$; the right-hand side of 9.4 is therefore equal to $\chi_\Gamma(\mathfrak{F})$, and the proof is complete.

Theorem 8.4 will be deduced from:

(9.6) **Proposition.** Let Γ be a group of type (VFP) and let K be an admissible Γ-complex such that $\chi_\Gamma(K)$ is defined. If p is a prime such that K^H is contractible for every non-trivial finite p-subgroup $H \subseteq \Gamma$, then

$$\chi_\Gamma(K) \equiv \chi(\Gamma) \qquad (\text{mod } \mathbb{Z}_{(p)}).$$

Proof. Let $\Gamma' \subset \Gamma$ be a torsion-free normal subgroup of finite index. Replacing Γ by a subgroup Γ_p such that Γ/Γ_p is a p-Sylow subgroup of Γ/Γ', we may reduce to the case where Γ/Γ' is a p-group, in which case we will prove

$$\chi_\Gamma(K) \equiv \chi(\Gamma) \qquad (\text{mod } \mathbb{Z}).$$

Note that every finite subgroup of Γ is now a p-group, so our hypothesis says that K^H is contractible for every $H \in \mathfrak{F}$. We may therefore argue as in the proof above to deduce

$$\chi(\Gamma) = \frac{\chi(Y)}{(\Gamma : \Gamma')} \equiv \frac{\tilde{\chi}_{\Gamma'}(\mathfrak{F})}{(\Gamma : \Gamma')} \qquad (\text{mod } \mathbb{Z}) ,$$

where $\tilde{\chi}_{\Gamma'}(\mathfrak{F}) = \sum(-1)^i \mathrm{rk}_{\mathbb{Z}_p} H_i^{\Gamma'}(K(\mathfrak{F}), \mathbb{Z}_p)$. (One needs to use here the fact that $H_*(Y_0, \mathbb{Z}_p)$ is finitely generated by Smith theory, cf. proof of Theorem 7.4.) On the other hand, we may apply the same argument with X replaced by $\overline{X} = X \times K$, since $\overline{X}^H = X^H \times K^H$ is still contractible for $H \in \mathfrak{F}$. Writing $\overline{Y} = \overline{X}/\Gamma'$, we find

$$\chi_\Gamma(K) = \frac{\chi_{\Gamma'}(K)}{(\Gamma : \Gamma')} = \frac{\chi(\overline{Y})}{(\Gamma : \Gamma')} \equiv \frac{\tilde{\chi}_{\Gamma'}(\mathfrak{F})}{(\Gamma : \Gamma')} \qquad (\mathrm{mod}\ \mathbb{Z}),$$

whence the proposition.

Proof of Theorem 8.4. We remark first that Γ has only finitely many conjugacy classes of p-subgroups. This follows from the fact that, with the notation we have been using, $H_*(Y^P, \mathbb{Z}_p)$ is finitely generated (and hence Y^P has only finitely many connected components) for every p-subgroup $P \subseteq \Gamma/\Gamma'$; see [15], proof of Lemma 4.11(a), or [25], proof of Prop. 14.5, for more details. In particular, Γ has only finitely many conjugacy classes of elementary abelian p-subgroups, and it follows easily that $\chi_\Gamma(\mathfrak{A}_p)$ is defined. The theorem will now follow from Proposition 9.6 applied with $K = K(\mathfrak{A}_p)$, if we verify that $K(\mathfrak{A}_p)^H$ is contractible for each non-trivial p-subgroup $H \subseteq \Gamma$. Fix a central subgroup C of H of order p. If $A \in \mathfrak{A}_p^H$, i.e. A is a non-trivial elementary abelian p-subgroup of Γ normalized by H, then A^H is non-trivial; hence we have a sequence of inclusions $A \supseteq A^H \subseteq C \cdot A^H \supseteq C$ in $\mathfrak{A}_p^H$ and this yields the required contracting homotopy of $K(\mathfrak{A}_p)^H = K(\mathfrak{A}_p^H)$, cf. [26], 4.4.

PART III. FARRELL COHOMOLOGY THEORY

References: [18], [14].

Let Γ be an arbitrary group of finite virtual cohomological dimension. If Γ is torsion-free then $\mathrm{cd}\,\Gamma < \infty$ and therefore $H^*(\Gamma) = 0$ in high dimensions. This suggests (by analogy with the results of §§7 and 8) that, in general, one might try to 'explain' the high-dimensional cohomology of Γ in terms of the torsion in Γ. For this purpose it is convenient to use a modified cohomology theory $\hat{H}$ introduced by Farrell [18]. There is a map $H^i(\Gamma) \to \hat{H}^i(\Gamma)$ which is an isomorphism for $i > \mathrm{vcd}\,\Gamma$, and one has $\hat{H}^*(\Gamma) = 0$ if Γ is torsion-free. Thus it is reasonable to expect that, in some sense, $\hat{H}^*(\Gamma)$ isolates the cohomological contribution of the finite subgroups of Γ. It is not yet clear to what extent the Farrell theory will be useful in the study of the low-dimensional cohomology of Γ (which is often more interesting than the high-dimensional cohomology, e.g. for applications to algebraic K-theory), but at the very least it allows one to break the study of $H^*(\Gamma)$ into two steps: (a) understand $\hat{H}^*(\Gamma)$; (b) understand the map $H^*(\Gamma) \to \hat{H}^*(\Gamma)$.

Farrell's cohomology theory is a generalization of the Tate cohomology theory for finite groups. We will therefore begin by reviewing the latter (§10); then in §§11 and 12 we discuss the foundations of Farrell's theory. Two of the well-known applications of Tate cohomology theory are the Nakayama-Rim theory of cohomologically trivial modules (cf. [29]) and the theory of groups with periodic cohomology (cf. [16]); in §§13 and 14 we give the generalizations of these theories to infinite groups, using Farrell cohomology. Finally, §15 contains the results alluded to above, relating $\hat{H}^*(\Gamma)$ to the finite subgroups of Γ; as an application, we obtain some results on $H^*(SL_3(\mathbb{Z}[\frac{1}{2}]))$.

§10. Review of Tate cohomology theory

Let G be a finite group. To define the Tate groups $\hat{H}^*(G)$, one begins with a projective resolution $P = (P_i)_{i \geq 0}$ of $\mathbb{Z}$ over $\mathbb{Z}G$ and 'completes' it to a complex of projectives $\hat{P}$ which is acyclic in <u>all</u> dimensions:

$$\cdots \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \cdots .$$

The existence of such a completion is easily proved as follows. To begin, one chooses an injection $i : \mathbb{Z} \hookrightarrow P_{-1}$ of $\mathbb{Z}G$-modules, such that P_{-1} is projective and i is $\mathbb{Z}$-split (e.g. take $P_{-1} = \mathbb{Z}G$ and $i(1) = N = \sum_{g \in G} g$). Let $C = \operatorname{coker} i$. Since C is $\mathbb{Z}$-free, one can find a $\mathbb{Z}$-split injection $j : C \hookrightarrow P_{-2}$, where P_{-2} is projective (e.g. take $P_{-2} = \mathbb{Z}G \otimes C$, with G acting on the first factor, and let $j(c) = \sum_{g \in G} g \otimes g^{-1}c$). Continuing in this way we obtain $\hat{P}$:

$$\begin{array}{ccccccccc} \cdots \to P_1 \to & P_0 & & \to & P_{-1} & & \to & P_{-2} & \cdots . \\ & & \searrow & & \nearrow \quad \searrow & & \nearrow & & \\ & & & \mathbb{Z} & & C & & & \end{array}$$

It is easy to see that any two such completions are canonically homotopy equivalent, and we can therefore define

$$\hat{H}^*(G, M) = H^*(\mathrm{Hom}_{\mathbb{Z}G}(\hat{P}, M))$$

for any G-module M. The Tate theory has the following properties:

(10.1) $\hat{H}^*(G, M) = 0$ if G is the trivial group.

(10.2) As in ordinary cohomology theory one has long exact cohomology sequences associated to short exact sequences of modules, Shapiro's lemma, restriction and transfer maps, and cup products.

(10.3) $\hat{H}^*(G, M) = 0$ if M is an induced module $\mathbb{Z}G \otimes A$ for some abelian group A; hence the functors $\hat{H}^i(G, -)$ are effaceable and coeffaceable.

(10.4) $\hat{H}^i = H^i$ for $i > 0$.

(10.5) $\hat{H}^0$ is a quotient of H^0, namely, the cokernel of the norm map $N : H_0 \to H^0$.

(10.6) $\hat{H}^{-1}$ is a subgroup of H_0, namely, the kernel of the norm map $N : H_0 \to H^0$.

(10.7) $\hat{H}^i = H_{-i-1}$ if $i < -1$.

Properties 10.1-10.5 are easy to verify directly from the definition, while 10.6 and 10.7 follow from the fact that a complete resolution can be constructed by splicing together a finite type resolution P of $\mathbb{Z}$ over $\mathbb{Z}G$ with its dual $P' = \mathrm{Hom}_{\mathbb{Z}G}(P, \mathbb{Z}G) \approx \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$:

(10.8) $$\dots \to P_1 \to P_0 \to P'_0 \to P'_1 \to \dots \quad (P_0 \to \mathbb{Z} \to P'_0).$$

Thus we have a cohomology theory $\{\hat{H}^i\}$ consisting of the functors H^i and H_i for $i > 0$, together with modified H^0 and H_0 functors:

$$
\begin{array}{ccccccccc}
 & & & & H^0 & H^1 & H^2 & \cdots \\
 & & & \nearrow & \downarrow & \| & \| & \\
\cdots & \hat{H}^{-3} & \hat{H}^{-2} & \hat{H}^{-1} & \hat{H}^0 & \hat{H}^1 & \hat{H}^2 & \cdots \\
 & \| & \| & \downarrow & & & & \\
\cdots & H_2 & H_1 & H_0 & & & &
\end{array}
$$

§11. Definition of $\hat{H}^*(\Gamma)$

Let Γ be a group such that $\operatorname{vcd}\Gamma = n < \infty$. (The previous section treated the case $n = 0$.) Let P be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$. By a <u>completion</u> of P we will mean an acyclic complex $\hat{P}$ of projectives which agrees with P in sufficiently high dimensions. A completion of P can be constructed as follows: Let $K = \operatorname{Im}\{P_n \to P_{n-1}\}$. If $\Gamma' \subseteq \Gamma$ is a torsion-free subgroup of finite index, then K is $\mathbb{Z}\Gamma'$-projective, hence we can find an embedding $i : K \hookrightarrow \hat{P}_{n-1}$ where $\hat{P}_{n-1}$ is $\mathbb{Z}\Gamma$-projective and i is $\mathbb{Z}\Gamma'$-split (e.g. take $\hat{P}_{n-1} = \mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma'} K$ and $i(x) = \sum \gamma \otimes \gamma^{-1}x$, where γ ranges over a set of representatives for the cosets Γ/Γ'). Applying the same process to $\operatorname{coker} i$ and continuing as in the previous section, we obtain a completion of P:

$$
\cdots \to P_{n+1} \to P_n \to \hat{P}_{n-1} \to \cdots .
\qquad (P_n \searrow K \nearrow \hat{P}_{n-1})
$$

In case Γ is of type (VFP), we can also use the following method for constructing complete resolutions, which generalizes the splicing construction (10.8) available if Γ is finite: Take the original resolution P to be of finite type and let P' be the dual complex $\operatorname{Hom}_{\mathbb{Z}\Gamma}(P, \mathbb{Z}\Gamma)$. One can show that there exists a chain complex $Q = (Q_i)_{i\geq 0}$ of finitely generated projectives which maps to P' by a weak equivalence of the form

$$
\begin{array}{ccccccccccccc}
\cdots \to & Q_{n+1} & \to & Q_n & \to & \cdots & \to & Q_0 & \to & 0 & \to & \cdots \\
& \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots \to & 0 & \to & P'_0 & \to & \cdots & \to & P'_n & \to & P'_{n+1} & \to & \cdots .
\end{array}
$$

(If Γ is a virtual duality group, for example, then Q is simply a finite type projective resolution of the module $D = H^n(\Gamma, \mathbb{Z}\Gamma)$.) The mapping cone of this weak equivalence is then an acyclic complex of projectives, whose dual is the desired completion $\hat{P}$.

Returning to the general case, now, one shows that any two completions are canonically homotopy equivalent, hence we can define the <u>Farrell cohomology groups</u> by

$$\hat{H}^*(\Gamma, M) = H^*(\mathrm{Hom}_{\mathbb{Z}\Gamma}(\hat{P}, M)).$$

One shows also that there is a chain map $\hat{P} \to P$, well-defined up to homotopy, whence a map

$$H^*(\Gamma, M) \to \hat{H}^*(\Gamma, M).$$

We will often suppress the coefficient module M from the notation and simply write $\hat{H}^*(\Gamma)$.

The Farrell theory has properties analogous to the properties of the Tate theory listed in §10.

(11.1) $\hat{H}^*(\Gamma) = 0$ if Γ is torsion-free.

(11.2) One has long exact cohomology sequences, Shapiro's lemma, restriction and transfer maps, and cup products. Moreover, there is a 'Hochschild-Serre' spectral sequence associated to a short exact sequence $1 \to \Gamma' \to \Gamma \to \Gamma'' \to 1$ of groups of finite virtual cohomological dimension, provided either Γ' or Γ'' is torsion-free. If Γ'' is torsion-free this takes the form

$$E_2^{pq} = H^p(\Gamma'', \hat{H}^q(\Gamma')) \Rightarrow \hat{H}^{p+q}(\Gamma),$$

and if Γ' is torsion-free then it takes the form

$$E_2^{pq} = \hat{H}^p(\Gamma'', H^q(\Gamma')) \Rightarrow \hat{H}^{p+q}(\Gamma).$$

(Note, in particular, that the edge homomorphism of the latter spectral sequence yields an <u>inflation map</u> $\hat{H}^*(\Gamma'') \to \hat{H}^*(\Gamma)$ for any Γ''-module of coefficients in the case where Γ' is torsion-free.)

(11.3) $\hat{H}(\Gamma, M) = 0$ if M is an induced module $\mathbb{Z}\Gamma \otimes_{\mathbb{Z}\Gamma'} M'$, where Γ' is a torsion-free subgroup of finite index and M' is a Γ'-module; hence the functors $\hat{H}^i(\Gamma, -)$ are effaceable and co-effaceable.

(11.4) $\hat{H}^i = H^i$ for $i > n = \mathrm{vcd}\,\Gamma$.

(11.5) $\hat{H}^n(\Gamma, M)$ is isomorphic to the cokernel of the transfer map $H^n(\Gamma', M) \to H^n(\Gamma, M)$, where Γ' is any torsion-free subgroup of finite index.

Assume now that Γ is a virtual duality group (§3), let $D = H^n(\Gamma, \mathbb{Z}\Gamma)$, and let $\tilde{H}_i(\Gamma, M) = H_i(\Gamma, D \otimes M)$. Then we have:

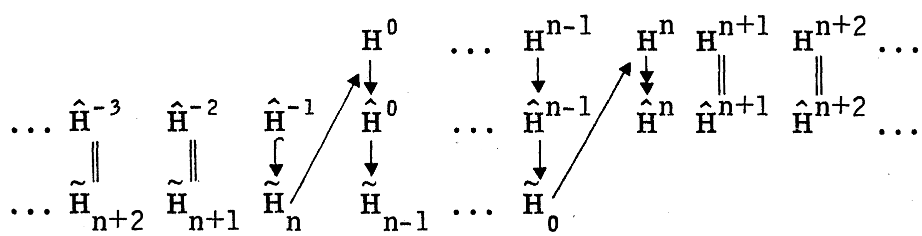
(11.6) $\hat{H}^{-1}(\Gamma)$ is isomorphic to the kernel of the transfer map $\tilde{H}_n(\Gamma) \to \tilde{H}_n(\Gamma')$, with Γ' as in 11.5.

(11.7) $\hat{H}^i = \tilde{H}_{n-i-1}$ for $i < -1$.

(11.8) There is an exact sequence

$$0 \to \hat{H}^{-1} \to \tilde{H}_n \to H^0 \to \hat{H}^0 \to \tilde{H}_{n-1} \to H^1 \to \hat{H}^1 \to \dots \to \tilde{H}_0 \to H^n \to \hat{H}^n \to 0.$$

To summarize, then, the Farrell cohomology theory $\{\hat{H}^i\}$ (at least if Γ is a virtual duality group) consists of the cohomology functors H^i for $i > n$; the homology functors $\tilde{H}_i$ for $i > n$; modified H^n and $\tilde{H}_n$ functors; and n additional functors $\hat{H}^0, \dots, \hat{H}^{n-1}$, which are some sort of mixture of the functors H^i and $\tilde{H}_i$ for $i \leq n$:



Remarks. 1. Properties 11.6-11.8 generalize to the case where Γ is an arbitrary group of type (VFP); the module D must then be replaced by a suitable complex, cf. §3.

2. There are also Farrell homology groups $\hat{H}_*(\Gamma, M)$, defined by

$$\hat{H}_*(\Gamma, M) = H_*(\hat{P} \otimes_{\mathbb{Z}\Gamma} M),$$

and having properties analogous to those above. If Γ is a virtual duality group then one has

$$\hat{H}^i(\Gamma, M) \approx \hat{H}_{n-i-1}(\Gamma, D \otimes M).$$

If, in addition, D is $\mathbb{Z}$-free, then

$$\hat{H}_i(\Gamma, M) \approx \hat{H}^{n-i-1}(\Gamma, \mathrm{Hom}(D, M)).$$

As usual, both of these isomorphisms can be generalized to the case where Γ is only assumed to be of type (VFP).

Finally, we mention that the groups $\hat{H}^*(\Gamma, M)$ are torsion groups; in fact, by transfer theory they are annihilated by the greatest common divisor d of the indices of the torsion-free subgroups Γ' of finite index. One might expect, by analogy with Theorem 7.2, that they are in fact annihilated by the least common multiple m of the orders of the finite subgroups of Γ, but it is not known whether or not this is true. [Note that m and d involve the same primes, and $m \mid d$.] In view of the theory of cup products, it would suffice to show that $1 \in \hat{H}^0(\Gamma, \mathbb{Z})$ is annihilated by m.

Example. Suppose $\Gamma = SL_3(\mathbb{Z})$. Then $d = 48$ and $m = 24$, and the calculations of Soulé [33] show that $\hat{H}^*(\Gamma, \mathbb{Z})$ is indeed annihilated by 24.

§12. Equivariant Farrell cohomology

It has been known for a long time that equivariant cohomology theory provides a machine for relating the cohomology of a discrete group to the cohomology of its finite subgroups. In this section we present the Farrell cohomology version of this equivariant theory. This generalizes a theory introduced by Swan [37] for finite groups. Throughout this section, Γ denotes an arbitrary group of virtually finite cohomological dimension.

For simplicity, we will define the equivariant cohomology groups $\hat{H}^*_\Gamma(K)$ only in the case where K is an <u>admissible</u> Γ-<u>complex</u> (§8). Moreover, we will assume that K is <u>finite-dimensional.</u> In this case we define, for any Γ-module M,

$$\hat{H}^*_\Gamma(K, M) = H^*(\mathrm{Hom}_{\mathbb{Z}\Gamma}(\hat{P}, C(K, M))).$$

Here $\hat{P}$ is a complete resolution for Γ; $C(K, M)$ is the cellular cochain complex of K with coefficients in the underlying abelian group of M, and $C(K, M)$ is given the diagonal Γ-action; and Hom denotes the total homomorphism complex, i. e. the total complex associated to the double complex $\mathrm{Hom}_{\mathbb{Z}\Gamma}(\hat{P}_\cdot, C^\cdot(K, M))$. As before we will often suppress M and simply write $\hat{H}^*_\Gamma(K)$. Note that $\hat{H}^*_\Gamma(\mathrm{pt.}) = \hat{H}^*(\Gamma)$, hence for any K there is a canonical map

$$\hat{H}^*(\Gamma) \to \hat{H}^*_\Gamma(K),$$

induced by the map $K \to \mathrm{pt.}$ of Γ-complexes.

We immediately obtain from the above definition two spectral sequences converging to $\hat{H}^*_\Gamma(K)$. The first has

$$E_2^{pq} = \hat{H}^p(\Gamma, H^q(K)),$$

and the second has

$$E_1^{pq} = \prod_{\sigma\in\Sigma_p} \hat{H}^q(\Gamma_\sigma),$$

where Σ_p is a set of representatives for the p-cells of K mod Γ. The E_2-term of the second spectral sequence is given by

$$E_2^{pq} = H^p(K/\Gamma, \{\hat{H}^q(\Gamma_\sigma)\}),$$

where the right-hand side is to be interpreted as follows. Fix $q \in \mathbb{Z}$. To each cell τ of K/Γ we may associate the group $A_\tau = \hat{H}^q(\Gamma_\sigma)$, where σ is any cell of K lying over τ; this group is independent of the choice of

σ, up to canonical isomorphism. Given a face relation $\tau' < \tau$, we may choose liftings σ' and σ with $\sigma' < \sigma$. We then have $\Gamma_{\sigma'} \supseteq \Gamma_{\sigma}$ by admissibility, hence there is a restriction map $\hat{H}^q(\Gamma_{\sigma'}) \to \hat{H}^q(\Gamma_{\sigma})$ which yields a well-defined map $A_{\tau'} \to A_{\tau}$. These maps satisfy the obvious compatibility condition whenever $\tau'' < \tau' < \tau$, and hence we have a 'coefficient system' on K/Γ. What occurs above, then, is the cohomology of K/Γ with coefficients in this system.

Remarks. 1. A coefficient system of this sort gives rise to a sheaf which is constant with stalk A_{τ} on the interior of τ, and the E_2-term above is isomorphic to the cohomology of K/Γ with coefficients in this sheaf.

2. The first spectral sequence above lives in the first and second quadrants, and the second one lives in the first and fourth quadrants. There is no problem with convergence, however, in view of the finite-dimensionality of K.

We record, now, two properties of equivariant Farrell cohomology which follow easily from the above spectral sequences:

(12.1) If $f : K \to L$ is a cellular Γ-map which induces an isomorphism $H_*K \to H_*L$, then f induces an isomorphism $\hat{H}^*_\Gamma(K) \xleftarrow{\approx} \hat{H}^*_\Gamma(L)$. In particular, if K is contractible, then $\hat{H}^*_\Gamma(K) \approx \hat{H}^*(\Gamma)$.

(12.2) If $K' \subseteq K$ is a Γ-invariant subcomplex such that Γ acts freely in $K - K'$, then $\hat{H}^*_\Gamma(K) \xrightarrow{\approx} \hat{H}^*_\Gamma(K')$.

Finally, we call attention to an important special case where we will apply the equivariant Farrell theory. Let X be, as in 2.1, a finite-dimensional contractible complex on which Γ acts properly. Then $\hat{H}^*_\Gamma(X) \approx \hat{H}^*(\Gamma)$, and the second spectral sequence therefore takes the form:

(12.3) $E_2^{pq} = H^p(X/\Gamma, \{\hat{H}^q(\Gamma_\sigma)\}) \Rightarrow \hat{H}^{p+q}(\Gamma)$.

This spectral sequence relates the Farrell cohomology of Γ to the Tate cohomology of its finite subgroups.

(12.4) **Exercise.** Suppose that X has been chosen so that X^G is connected and non-empty for every finite subgroup $G \subseteq \Gamma$; we know by 2.2 that this is possible. Show that the left-hand edge E_2^{0*} of the above spectral sequence can be identified with $\varprojlim \hat{H}^*(G)$, where G ranges over the finite subgroups of Γ and the limit is taken with respect to all maps between finite subgroups given by conjugation by an element of Γ. Explicitly, an element of this limit is a compatible family $\{u_G\}$, where G ranges over the finite subgroups of Γ, $u_G \in \hat{H}^*(G)$, and the compatibility condition is the following: If G and G' are finite subgroups and γ is an element of Γ such that $\gamma G \gamma^{-1} \subseteq G'$, then $u_{G'}$ maps to u_G under the map $\hat{H}^*(G') \to \hat{H}^*(G)$ induced by conjugation by γ.

§13. Cohomologically trivial modules

Γ continues to denote an arbitrary group of finite virtual cohomological dimension. As an immediate consequence of the equivariant cohomology spectral sequence 12.3, we have:

(13.1) **Lemma.** Let M be a Γ-module such that $\hat{H}^*(G, M) = 0$ for every finite subgroup $G \subseteq \Gamma$. Then $\hat{H}^*(\Gamma, M) = 0$.

We will say that M is cohomologically trivial if, as in the lemma, $\hat{H}^*(G, M) = 0$ for every finite $G \subseteq \Gamma$. It then follows from the lemma that $\hat{H}^*(\Gamma_0, M) = 0$ for every subgroup $\Gamma_0 \subseteq \Gamma$.

In case Γ is finite, we have the following characterization of cohomologically trivial modules, due to Rim [27] (see also [29]):

If Γ is finite then a Γ-module M is cohomologically trivial if and only if it has finite projective dimension over $\mathbb{Z}\Gamma$, and in this case $\operatorname{proj\,dim}_{\mathbb{Z}\Gamma} M \leq 1$. If M is $\mathbb{Z}$-free and cohomologically trivial, then $\operatorname{proj\,dim}_{\mathbb{Z}\Gamma} M = 0$, i.e. M is $\mathbb{Z}\Gamma$-projective.

We now extend this to the general case. Let $\operatorname{vcd} \Gamma = n$.

(13.2) **Theorem.** A Γ-module M is cohomologically trivial if and only if it has finite projective dimension, and in this case $\operatorname{proj\,dim}_{\mathbb{Z}\Gamma} M \leq n + 1$. If M is $\mathbb{Z}$-free and cohomologically trivial then $\operatorname{proj\,dim}_{\mathbb{Z}\Gamma} M \leq n$.

Proof. Clearly projective modules are cohomologically trivial, hence so is any module of finite projective dimension. Conversely, suppose M is cohomologically trivial, and assume first that M is $\mathbb{Z}$-free. I claim that Hom(M, N) is cohomologically trivial for any Γ-module N, where $\text{Hom}(\ ,\ ) = \text{Hom}_{\mathbb{Z}}(\ ,\ )$ with the diagonal Γ-action. Indeed, it suffices to verify the claim in case Γ is finite, in which case it follows at once from Rim's theorem. We therefore have $\hat{H}^*(\Gamma, \text{Hom}(M, N)) = 0$, hence

$$\text{Ext}^i_{\mathbb{Z}\Gamma}(M, N) = H^i(\Gamma, \text{Hom}(M, N)) = 0$$

for $i > n$, and $\text{proj dim}_{\mathbb{Z}\Gamma} M \leq n$. In case M is not $\mathbb{Z}$-free, choose a surjection $P \twoheadrightarrow M$ with P projective. The kernel M' of this map will have $\text{proj dim}_{\mathbb{Z}\Gamma} M' \leq n$ by what we have just proved, hence $\text{proj dim}_{\mathbb{Z}\Gamma} M \leq n + 1$.

§14. Groups with periodic cohomology

A group Γ of finite virtual cohomological dimension will be said to have periodic cohomology if for some integer $d > 0$ there is an element of $\hat{H}^d(\Gamma, \mathbb{Z})$ which is invertible in the ring $\hat{H}^*(\Gamma, \mathbb{Z})$. Cup product with such an element then defines periodicity isomorphisms

$$\hat{H}^i(\Gamma, M) \approx \hat{H}^{i+d}(\Gamma, M)$$

for any Γ-module M and any integer i. Similarly, one can define p-periodicity for a fixed prime p in terms of the existence of an invertible element of positive degree in the ring $\hat{H}^*(\Gamma, \mathbb{Z})_{(p)}$, the p-primary component of $\hat{H}^*(\Gamma, \mathbb{Z})$. [One can show by a Bockstein argument that this is equivalent to the existence of an invertible element of positive degree in $\hat{H}^*(\Gamma, \mathbb{Z}/p)$.] Clearly Γ has periodic cohomology if and only if it has p-periodic cohomology for every prime p.

One way to prove periodicity (or p-periodicity) is by exhibiting a finite quotient Γ/Γ' which has periodic (or p-periodic) cohomology, where Γ' is torsion-free. This follows from the fact that the inflation map (11.2) is a ring homomorphism $\hat{H}^*(\Gamma/\Gamma', \mathbb{Z}) \to \hat{H}^*(\Gamma, \mathbb{Z})$. Similarly, if Γ has periodic or p-periodic cohomology, then so does every subgroup (because the restriction maps are ring homomorphisms).

The main result on groups with periodic cohomology is the following theorem:

(14.1) **Theorem.** Let Γ be a group such that $\operatorname{vcd}\Gamma < \infty$ and let p be a prime. The following conditions are equivalent:

(i) Γ has p-periodic cohomology.

(ii) There exist integers i and d with $d > 0$, such that $\hat{H}^i(\Gamma, M)_{(p)} \approx \hat{H}^{i+d}(\Gamma, M)_{(p)}$ for all Γ-modules M.

(iii) Every finite subgroup of Γ has p-periodic cohomology.

(iv) Γ does not contain any subgroup isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$.

(v) Every finite p-subgroup of Γ is a cyclic or generalized quaternion group.

Proof. Trivially (i) $\Rightarrow$ (ii). To prove (ii) $\Rightarrow$ (iii), note first that if (ii) holds for some i then it holds for all i by a standard 'dimension shifting' argument. Also, (ii) holds for any subgroup of Γ by Shapiro's lemma. In particular, if $G \subseteq \Gamma$ is finite then $\hat{H}^d(G, \mathbb{Z})_{(p)} \approx \hat{H}^0(G, \mathbb{Z})_{(p)}$, and this is well-known to imply that G has p-periodic cohomology, as required (cf. [16]). The equivalences (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v) are well-known from the theory of finite groups with periodic cohomology [16]; so it remains to prove (iii) $\Rightarrow$ (i).

We recall first that a weaker version of this implication was proved by Venkov ([41], [42]), although he did not use the language of Farrell cohomology theory. (He spoke, rather, of periodicity in the ordinary cohomology of Γ in sufficiently high dimensions.) Restated in terms of Farrell cohomology, his result is the following: If there exists an element $u \in \hat{H}^d(\Gamma, \mathbb{Z})_{(p)}$ $(d > 0)$ whose restriction to $\hat{H}^*(G, \mathbb{Z})_{(p)}$ is invertible for every finite $G \subseteq \Gamma$, then u is invertible and hence Γ has p-periodic cohomology. This result of Venkov is easily deduced from the multiplicative structure in the equivariant cohomology spectral sequence 12.3, localized at p:

$$E_2^{st} = H^s(X/\Gamma, \{\hat{H}^t(\Gamma_\sigma)\})_{(p)} \Rightarrow \hat{H}^{s+t}(\Gamma)_{(p)}.$$

Indeed, one need only observe that multiplication by u induces an isomorphism on E_2 and hence also on the abutment.

To prove (iii) ⇒ (i), then, it suffices to prove that (iii) implies the existence of such an element u. This again follows from the multiplicative structure in the above spectral sequence. For let u_G be an invertible element of positive degree in $\hat{H}^*(G, \mathbb{Z})_{(p)}$, where G ranges over the finite subgroups of Γ. Raising u_G to a power, if necessary, we may assume that $\{u_G\}$ is a compatible family in the sense of 12.4, so that $\{u_G\}$ represents an element of the edge E_2^{0*}. It now follows by an argument of Quillen ([25], proof of Prop. 3.2) that some power of $\{u_G\}$ is a permanent cycle in the spectral sequence and hence is the image of some element $u \in \hat{H}^*(\Gamma, \mathbb{Z})$ under the edge homomorphism

(14.2) $$\hat{H}^*(\Gamma, \mathbb{Z})_{(p)} \to \varprojlim \hat{H}^*(G, \mathbb{Z})_{(p)} .$$

This completes the proof.

Remark. In the language of Quillen [25], the above proof is based on the fact that the map 14.2 is an 'F-isomorphism'. It should be noted that Quillen's methods yield the much stronger result (for any group Γ with $\operatorname{vcd} \Gamma < \infty$) that the map

$$\hat{H}^*(\Gamma, \mathbb{Z})_{(p)} \to \varprojlim \hat{H}^*(A, \mathbb{Z})$$

is an F-isomorphism, where A ranges over the elementary abelian p-subgroups of Γ.

§15. The ordered set of finite subgroups

Γ continues to denote an arbitrary group of finite virtual cohomological dimension. As in §8, if Γ operates on a partially ordered set S then we set

$$\hat{H}_\Gamma^*(S) = \hat{H}_\Gamma^*(K(S)) .$$

As usual it is understood here that there is an arbitrary Γ-module M of coefficients. In this section we will prove analogues in Farrell cohomology of Theorems 8.3 and 8.4.

Recall that, for any finite-dimensional admissible Γ-complex K, there is a canonical map $\hat{H}^*(\Gamma) \to \hat{H}^*_\Gamma(K)$.

(15.1) **Theorem.** _Let_ $\mathcal{F}$ _be the set of non-trivial finite subgroups of_ Γ. _Then the canonical map_

$$\hat{H}^*(\Gamma) \to \hat{H}^*_\Gamma(\mathcal{F})$$

is an isomorphism.

Proof. Let X and X_0 be as in §9, proof of Theorem 8.3. Then

$$\begin{aligned} \hat{H}^*(\Gamma) &\approx \hat{H}^*_\Gamma(X) && \text{by } 12.1 \\ &\approx \hat{H}^*_\Gamma(X_0) && \text{by } 12.2 \\ &\approx \hat{H}^*_\Gamma(\mathcal{F}) && \text{by } 12.1 \text{ and } 9.5. \end{aligned}$$

It is easy to check that this composite isomorphism is in fact given by the canonical map $\hat{H}^*(\Gamma) \to \hat{H}^*_\Gamma(\mathcal{F})$.

Next we prove the analogue of 9.6:

(15.2) **Proposition.** _Let_ K _be a finite-dimensional admissible_ Γ-_complex and let_ p _be a prime such that_ K^H _is contractible for every non-trivial_ p-_subgroup_ $H \subseteq \Gamma$. _Then_

$$\hat{H}^*(\Gamma)_{(p)} \xrightarrow{\approx} \hat{H}^*_\Gamma(K)_{(p)} .$$

Proof[2]**.** Let $\Gamma' \subseteq \Gamma_p \subseteq \Gamma$ be as in the proof of 7.1. Since $(\Gamma : \Gamma_p)$ is relatively prime to p, we may use restriction and transfer maps in the usual way to obtain, for any finite-dimensional admissible Γ-complex L, a natural embedding of $\hat{H}^*_\Gamma(L)_{(p)}$ as a direct summand of $\hat{H}^*_{\Gamma_p}(L)$. Applying this to $L = K$ and $L = \text{pt.}$, we see that the canonical map $\hat{H}^*(\Gamma)_{(p)} \to \hat{H}^*_\Gamma(K)_{(p)}$ is a direct summand of the canonical map $\hat{H}^*(\Gamma_p) \to \hat{H}^*_{\Gamma_p}(K)$. It therefore suffices to prove that the latter is an

[2] I am grateful to D. Quillen for a suggestion which simplified my original proof.

isomorphism. Just as in the proof of 9.6, we now apply the method of proof of Theorem 15.1 to $(\Gamma_p, X \times K)$ and (Γ_p, X) to deduce

$$\hat{H}^*_{\Gamma_p}(K) \approx \hat{H}^*_{\Gamma_p}(\mathfrak{F}) \approx \hat{H}^*(\Gamma_p),$$

where $\mathfrak{F}$ is now the set of non-trivial finite subgroups of Γ_p. It is easily checked that the composite isomorphism is given by the canonical map, whence the proposition.

Let $\mathfrak{a}_p$ be the set of non-trivial elementary abelian p-subgroups of Γ. As in §9 (proof of Theorem 8.4), we may apply 15.2 with $K = K(\mathfrak{a}_p)$ to obtain:

(15.3) **Theorem.** <u>For any prime</u> p,

$$\hat{H}^*(\Gamma)_{(p)} \xrightarrow{\approx} \hat{H}^*_{\Gamma}(\mathfrak{a}_p)_{(p)} .$$

This theorem gives information about $\hat{H}^*(\Gamma)_{(p)}$ in terms of the elementary abelian p-subgroups and their normalizers, cf. [14], Prop. 2. In case Γ contains no elementary abelian p-group of rank 2 (i.e. if Γ has p-periodic cohomology), this information takes the following simple form:

(15.4) **Corollary.** <u>If</u> Γ <u>contains no subgroup isomorphic to</u> $\mathbb{Z}/p \times \mathbb{Z}/p$, <u>then</u>

$$\hat{H}^*(\Gamma)_{(p)} \approx \Pi\, \hat{H}^*(N(P))_{(p)} ,$$

<u>where</u> P <u>ranges over the subgroups of</u> Γ <u>of order</u> p, <u>up to conjugacy.</u>

In [14], §6, we applied the corollary with $\Gamma = SL_3(\mathbb{Z})$ to calculate the 3-primary component of $\hat{H}^*(SL_3(\mathbb{Z}))$, from which we obtained $H^*(SL_3(\mathbb{Z}), \mathbb{Z})$ and $H_*(SL_3(\mathbb{Z}), St)$ modulo 2-torsion. Here St denotes the Steinberg module, cf. 3.3. (Of course, these calculations have been subsumed in the work of Soulé [33].) We now give another example in which Theorem 15.3 yields concrete information relating $\hat{H}^*(\Gamma)$ to the cohomology of the normalizers of the elementary abelian p-subgroups of Γ.

Let $\Gamma = SL_3(\mathbb{Z}[\frac{1}{2}])$. We will apply Theorem 15.3 with $p = 2$. Note first that every elementary abelian 2-subgroup of Γ is diagonalizable and hence has rank ≤ 2; thus $K(\mathfrak{a}_2)$ is a graph. Let P_0 be the group of order 2 generated by

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and let P_1 be the group of order 4 consisting of the matrices

$$\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$$

in SL_3. Then $P_0 \subset P_1$ so there is an edge of $K(\mathfrak{A}_2)$ with vertices P_0 and P_1, and it is easy to see that this edge is a fundamental domain for the action of Γ on $K(\mathfrak{A}_2)$. The isotropy groups of the vertices P_0 and P_1 are the normalizers $N(P_0)$ and $N(P_1)$. Explicitly, $N(P_0)$ is isomorphic to $GL_2 (= GL_2(\mathbb{Z}[\frac{1}{2}]))$, embedded in SL_3 in the usual way,

$$A \mapsto \left(\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 & 0 & \det A^{-1} \end{array}\right),$$

and $N(P_1)$ is the group SM_3 of monomial matrices in SL_3. The isotropy group of the edge (P_0, P_1) is $GL_2 \cap SM_3 = M_2$, the group of 2×2 monomial matrices. The second spectral sequence of equivariant cohomology theory (§12) therefore yields a 'Mayer-Vietoris' sequence relating $\hat{H}^*_\Gamma(\mathfrak{A}_2)$ to $\hat{H}^*(GL_2)$, $\hat{H}^*(SM_3)$, and $\hat{H}^*(M_2)$, so we have by Theorem 15.3 a Mayer-Vietoris sequence

$$\cdots \to \hat{H}^{i-1}(M_2)_{(2)} \to \hat{H}^i(SL_3)_{(2)} \to \hat{H}^i(GL_2)_{(2)} \oplus \hat{H}^i(SM_3)_{(2)} \to \hat{H}^i(M_2)_{(2)} \to \cdots$$

The result can be stated more precisely, as follows. Let $\tilde{\Gamma} = GL_2 *_{M_2} SM_3$ and consider the canonical map $\tilde{\Gamma} \to \Gamma$. One can show that $K(\mathfrak{A}_2)$ is connected, whence this map is surjective and its kernel is isomorphic to the free group $\pi_1(K(\mathfrak{A}_2))$, cf. [31], ch. I, 5.4, Ex. 3. There is therefore an inflation map (11.2)

$$\hat{H}^*(\Gamma) \to \hat{H}^*(\tilde{\Gamma}),$$

and our result is that this induces an isomorphism on 2-primary components, with any Γ-module of coefficients. In particular, since vcd $\Gamma = 5$ by Borel-Serre [9], we have

$$H^i(\Gamma)_{(2)} \xrightarrow{\approx} H^i(\tilde{\Gamma})_{(2)}$$

for $i > 5$.

REFERENCES

1. A. Ash. Deformation retracts with lowest possible dimension of arithmetic quotients of self-adjoint homogeneous cones, Math. Ann. 225 (1977), 69-76.
2. H. Bass. Algebraic K-theory, Benjamin, New York (1968).
3. H. Bass. Euler characteristics and characters of discrete groups, Invent. Math. 35 (1976), 155-96.
4. H. Bass. Traces and Euler characteristics, these proceedings, 1-26.
5. R. Bieri. Homological dimension of discrete groups, Queen Mary College Mathematics Notes, London (1976).
6. R. Bieri and B. Eckmann. Groups with homological duality generalizing Poincaré duality, Invent. Math. 20 (1973), 103-24.
7. A. Borel et al. Seminar on transformation groups, Ann. of Math. Studies 46, Princeton (1960).
8. A. Borel and J.-P. Serre. Corners and arithmetic groups, Comment. Math. Helv. 48 (1974), 244-97.
9. A. Borel and J.-P. Serre. Cohomologie d'immeubles et de groupes S-arithmétiques, Topology 15 (1976), 211-32.
10. G. E. Bredon. Introduction to compact transformation groups, Academic Press, New York (1972).
11. K. S. Brown. Euler characteristics of discrete groups and G-spaces, Invent. Math. 27 (1974), 229-64.
12. K. S. Brown. Euler characteristics of groups: The p-fractional part, Invent. Math. 29 (1975), 1-5.
13. K. S. Brown. Homological criteria for finiteness, Comment. Math. Helv. 50 (1975), 129-35.

14. K. S. Brown. High dimensional cohomology of discrete groups, Proc. Nat. Acad. Sci. USA 73 (1976), 1795-7.

15. K. S. Brown. Complete Euler characteristics and fixed-point theory, preprint.

16. H. Cartan and S. Eilenberg. Homological algebra, Princeton University Press (1956).

17. I. M. Chiswell. Euler characteristics of groups, Math. Z. 147 (1976), 1-11

18. F. T. Farrell. An extension of Tate cohomology to a class of infinite groups, J. Pure Appl. Algebra 10 (1977), 153-61.

19. J. Fischer, A. Karrass, and D. Solitar. On one-relator groups having elements of finite order, Proc. Amer. Math. Soc. 33 (1972), 297-301.

20. G. Harder. A Gauss-Bonnet formula for discrete arithmetically defined groups, Ann. Sci. École Norm. Sup. (4) 4 (1971), 409-55.

21. S. Illman. Smooth equivariant triangulations of G-manifolds for G a finite group, Math. Ann. 233 (1978), 199-220.

22. R. Lee and R. H. Szczarba. On the torsion in $K_4(\mathbb{Z})$ and $K_5(\mathbb{Z})$, Duke Math. J. 45 (1978), 101-29.

23. R. C. Lyndon. Cohomology theory of groups with a single defining relation, Ann. of Math. 52 (1950), 650-65.

24. W. Magnus, A. Karrass, and D. Solitar. Combinatorial group theory, Wiley, New York (1966).

25. D. Quillen. The spectrum of an equivariant cohomology ring, I, II, Ann. of Math. 94 (1971), 549-72 and 573-602.

26. D. Quillen. Homotopy properties of the poset of non-trivial p-subgroups of a group, Advances in Math. 28 (1978), 101-28.

27. D. S. Rim. Modules over finite groups, Ann. of Math. 69 (1959), 700-12.

28. G. P. Scott and C. T. C. Wall. Topological methods in group theory, these proceedings, 137-203.

29. J.-P. Serre. Corps locaux, Hermann, Paris (1968).

30. J.-P. Serre. Cohomologie des groupes discrets, Ann. of Math. Studies 70 (1971), 77-169.

31. J.-P. Serre. Arbres, amalgames, SL_2, Astérisque 46 (1977).

32. J.-P. Serre. Arithmetic groups, these proceedings.

33. C. Soulé. The cohomology of $SL_3(\mathbb{Z})$, Topology 17 (1978), 1-22.

34. E. H. Spanier. Algebraic Topology, McGraw-Hill, New York (1966).

35. J. Stallings. On torsion-free groups with infinitely many ends, Ann. of Math. 88 (1968), 312-34.

36. R. Strebel. A homological finiteness criterion, Math. Z. 151 (1976), 263-75.

37. R. G. Swan. A new method in fixed point theory, Comment. Math. Helv. 34 (1960), 1-16.

38. R. G. Swan. Induced representations and projective modules, Ann. of Math. 71 (1960), 552-78.

39. R. G. Swan. K-theory of finite groups and orders, Lecture Notes in Mathematics 149, Springer, Berlin (1970).

40. R. G. Swan. Groups of cohomological dimension one, J. Algebra 12 (1969), 585-601.

41. B. B. Venkov. Cohomology of groups of units in algebras with division, Dokl. Akad. Nauk SSSR 137 (1961), 1019-21. [English translation: Soviet Math. Dokl. 2 (1961), 379-81.]

42. B. B. Venkov. On homologies of groups of units in division algebras, Trudy Mat. Inst. Steklov 80 (1965), 66-89. [English translation: Proc. Steklov Inst. Math. 80 (1965), 73-100.]

43. G. Voronoi. Nouvelles applications des paramètres continus à la théorie des formes quadratiques, I, J. Reine Angew. Math. 133 (1907), 97-178.

44. C. T. C. Wall. Rational Euler characteristics, Proc. Cambridge Philos. Soc. 57 (1961), 182-3.