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Advances in Mathematics 226 (2011) 992–1055
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

The heat semigroup and Brownian motion on strip complexes ☆

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Received 7 July 2009; accepted 26 July 2010

Available online 17 September 2010

Communicated by D.W. Stroock

Abstract

We introduce the notion of strip complex. A strip complex is a special type of complex obtained by gluing "strips" along their natural boundaries according to a given graph structure. The most familiar example is the one-dimensional complex classically associated with a graph, in which case the strips are simply copies of the unit interval (our setup actually allows for variable edge length). A leading key example is treebolic space, a geometric object studied in a number of recent articles, which arises as a horocyclic product of a metric tree with the hyperbolic plane. In this case, the graph is a regular tree, the strips are $[0,1] \times \mathbb{R}$, and each strip is equipped with the hyperbolic geometry of a specific strip in upper half plane. We consider natural families of Dirichlet forms on a general strip complex and show that the associated heat kernels and harmonic functions have very strong smoothness properties. We study questions such as essential self-adjointness of the underlying differential operator acting on a suitable space of smooth functions satisfying a Kirchhoff type condition at points where the strip complex bifurcates. Compatibility with projections that arise from proper group actions is also considered.

☆ S. Bendikov was partially supported by SFB 701 at the University of Bielefeld and by the Polish Goverment Scientific Research Fund, Grant No. N201 371736. L. Saloff-Coste was partially supported by NSF grant DMS–063886. W. Woess was partially supported by the Autrian Science Fund (FWF) project No. P19115-N18.
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0001-8708/$ – see front matter
doi:10.1016/j.aim.2010.07.014

Keywords: Strip complex; Dirichlet form; Laplacian; Brownian motion; Heat semigroup; Essential self-adjointness

Contents

1. Introduction

A. The treebolic spaces $\mathsf{HT}(\mathsf{p},\mathsf{q})$. Let $\mathbb{H} = \{x + \mathrm{i}y\colon\ x \in \mathbb{R},\ y > 0\}$ be the hyperbolic upper half space, and $\mathbb{T} = \mathbb{T}_\mathsf{p}$ be the homogeneous tree with degree $\mathsf{p}+1$, where $\mathsf{p} \in \mathbb{N}$. The *treebolic space* is a Riemannian 2-complex which can be viewed as a *horocyclic product* of $\mathbb{H}$ and $\mathbb{T}$. Let us start with a picture and an informal description.

Let $1 < \mathsf{q} \in \mathbb{R}$. Subdivide $\mathbb{H}$ into the strips $S_k = \{x + \mathrm{i}y\colon\ x \in \mathbb{R},\ \mathsf{q}^{k-1} \leqslant y \leqslant \mathsf{q}^k\}$, where $k \in \mathbb{Z}$. Each strip is bounded by two horizontal lines of the form $L_k = \{x + \mathrm{i}\mathsf{q}^k\colon\ x \in \mathbb{R}\}$, which, in hyperbolic geometry, are horospheres with respect to the boundary point at ∞ (or rather $\mathrm{i}\infty$). In the treebolic space $\mathsf{HT}(\mathsf{p},\mathsf{q})$, infinitely many copies of those strips are glued together in a tree-like fashion: for each $k \in \mathbb{Z}$, the bottom lines of p copies of S_k are identified among each other and with the top line of S_{k-1}. Each strip is equipped with the standard hyperbolic length element and, in this way, one obtains a natural metric on $\mathsf{HT}(\mathsf{p},\mathsf{q})$ as well as a natural measure.

This space admits interesting isometric group actions. On the one hand, when $\mathsf{q} = \mathsf{p}$, the amenable Baumslag–Solitar group $\mathsf{BS}(\mathsf{p}) = \langle a, b \mid ab = b^\mathsf{p} a\rangle$ acts on $\mathsf{HT}(\mathsf{p},\mathsf{p})$ by isometries and with compact quotient. This fact has been exploited by Farb and Mosher [19] in order to classify the Baumslag–Solitar groups up to quasi-isometry. See also the nice picture in Meier [25, p. 118]. On the other hand, for $\mathsf{p} \neq \mathsf{q}$, no discrete group can act in such a way on $\mathsf{HT}(\mathsf{p},\mathsf{q})$ and its isometry group is a non-unimodular locally compact group. This isometry group admits various subgroups that act with compact quotients, see our forthcoming paper [6].

This article is motivated by the following questions. What is Brownian motion on the treebolic space $\mathsf{HT}(\mathsf{p},\mathsf{q})$? What is the concrete description of the Laplacian, i.e., the generator of Brownian motion? Can one prove some essential self-adjointness results for this Laplacian? How smooth is the associated heat kernel? Can one describe explicitly the cone of positive harmonic functions? The last question, which is at the origin of this work, will be discussed in detail in [6]. Answers to the other questions are described in Theorems 2.13–2.17.

B. General strip complexes. The treebolic spaces $\mathsf{HT}(\mathsf{p},\mathsf{q})$ form one family of examples of what we call a *strip complex*, and this work is devoted to the study of the heat equation and heat

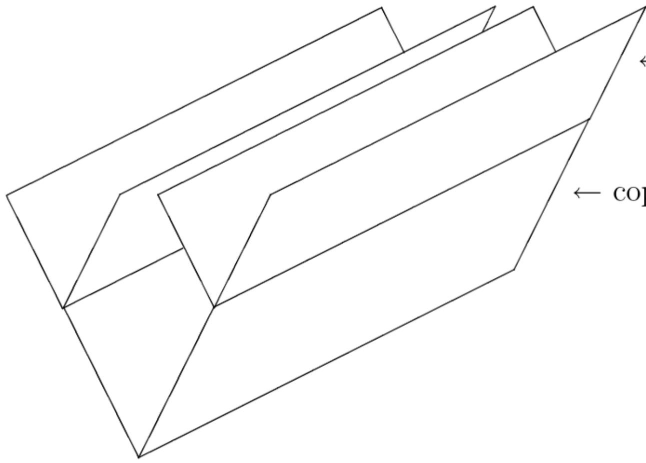

Fig. 1. A finite portion of treebolic space, with $\mathsf{p} = 2$.

kernel on strip complexes. The simplest family of strip complexes are metric graphs ("quantum graphs"). In fact, as a topological space, a strip complex is simply the direct product of a (connected) metric graph and a topological space M, e.g., $\{0\}$, $\mathbb{R}$, or a fixed manifold. In particular, strip complexes are typically not smooth as they bifurcate along the *bifurcation manifolds* at the vertices of the underlying graph structure. See, e.g., Fig. 1. We will equip those spaces with certain adapted geometries and adapted measures which will give rise to specific Laplacians and heat semigroups. Our aim is to show that, because of the specific structure of strip complexes, harmonic functions and solutions of the heat equation on such spaces have very strong global smoothness properties. Namely, these solutions have locally bounded derivatives of all orders *up to* the bifurcation manifolds even though these derivatives are typically not continuous *across* the bifurcation manifolds.

In order to carry this out in spite of the singularities of the underlying strip complex structure, we build the theory "from scratch", using the theory of strictly local regular Dirichlet forms. See, e.g., Fukushima, Oshima and Takeda [20, Cor. 1.3.1] and Sturm [33–35]. The Laplace operators constructed by this approach are somewhat esoteric objects and one of our goals is to describe them in a more concrete way as the closure of operators that are classical second order elliptic differential operators in the smooth part of the complex and whose domains of definition involve Kirchhoff type laws along bifurcation manifolds.

Our material and results should be compared with some previous work. First, the theory of the Laplacian, heat kernel, etc., on metric graphs is quite well understood. See, e.g., Baxter and Chacon [4], Cattaneo [12], Enriquez and Kifer [18] and Kuchment [23,24]. Note however that, even in this simple setting, the exact smoothness of the heat kernel is not entirely understood. See Bendikov and Saloff-Coste [5].

Second, Brin and Kifer [11] introduced Brownian motion on 2-dimensional Euclidean complexes (strongly connected simplicial complexes, where each simplex carries the Euclidean structure) via a local probabilistic construction. The Dirichlet form approach on more general Riemannian complexes is discussed by Eells and Fuglede [17] and Pivarski and Saloff-Coste [27]. None of these references provide the type of regularity results proved below for strip complexes.

It is worth emphasizing that, despite the existence of very many different approaches to the definition of Brownian motion on complexes such as $\mathsf{HT}(\mathsf{p},\mathsf{q})$, the basic problem of uniqueness is not adequately discussed in the literature. From this perspective, we view Theorem 2.17 (and its much more general version Theorem 7.11) as an important result.

Many of our results are local in nature. We note that, locally, the simplest strip complex structure (a star of finitely many Euclidean half spaces, glued along their boundaries) is the model for the neighbourhood of any generic singular point in a general n-dimensional Euclidean polytopal complex, that is, any point ξ where the n-dimensional closed faces containing ξ meet along an $(n-1)$-face. The strong regularity results that we obtain thus apply to small neighbourhoods of such points in any Euclidean polytopal complex.

This paper is organized as follows. In Section 2, we exhibit our main results in the key example of the treebolic space. We describe a two-parameter family of Dirichlet forms on HT whose associated Laplacians and heat semigroups satisfy all regularity and smoothness properties that one would wish to have (Theorem 2.13). In each case, the Laplacian is the unique self-adjoint extension of a naturally defined, essentially self-adjoint operator that is elliptic inside the strips of HT and acts on a space of smooth functions which satisfy a Kirchhoff condition along the bifurcation lines in HT (Theorem 2.17). To the best of our knowledge, this is the first time that essential self-adjointness is discussed is such a setting. This construction gives rise to a Hunt process ("Brownian motion") on HT with natural projections from HT onto the underlying (metric) tree and onto the hyperbolic plane ("sliced" into a strip complex by the lines L_v). On each of those objects, there is a corresponding Dirichlet form and associated Laplacian which is the infinitesimal generator of the respective projection of the process on HT (Theorem 2.23). Uniqueness properties are used here to identify the projections with the natural processes intrinsically defined on the quotient spaces.

In Section 3, we introduce the notion of strip complex as the product of a metric graph with a manifold. In a series of definitions, we introduce several function spaces that are needed to do analysis on such a complex. The geometry of a strip complex is obtained through the following data: a length function describing the length of the edges of the graph, a Riemannian structure on the manifold M, and a positive function ϕ on the metric graph that serves as a conformal factor to define the metric on each strip. We also introduce a second positive function ψ on the metric graph that serves as a weight function to define the underlying measure. These data turn the strip complex from a topological space into a geodesic metric measure space. This structure is used to define a Dirichlet form whose basic properties are discussed (Theorems 3.27–3.29). This Dirichlet form gives rise to the associated Laplacian, harmonic functions and heat equation.

Basic properties of the heat semigroup are derived in Section 4. Crucial geometric-analytic ingredients are the local doubling property and local Poincaré inequality (Theorem 4.1). Via the work of Sturm [33–35] and Saloff-Coste [28], this has far reaching consequences for weak solutions of the heat equation and for the heat diffusion semigroup (Theorems 4.2–4.4 plus corollaries).

In Section 5, we consider weak solutions of the Laplace and heat equations. We show that these weak solutions are smooth up to (but not across) the bifurcation manifolds and satisfy Kirchhoff type *bifurcation conditions* (Theorems 5.9, 5.19 and 5.23). These results are the most significant technical results contained in the present paper.

Section 6 studies how Dirichlet forms and the associated heat semigroups are compatible with natural projections of one strip complex onto another induced by a proper, continuous group action (Theorem 6.1).

Uniqueness of the heat semigroup is studied in Section 7. First, this question is dealt with on the space of continuous functions that vanish at infinity, where besides completeness, a uniform local doubling property plus uniform local Poincaré inequality is needed (Theorem 7.6). Second, a very precise essential self-adjointness result is obtained provided completeness and the existence of a *strip-adapted sequence of functions approximating* **1** (Theorem 7.11). The proof of this uses in an essential way the heat kernel regularity results proved earlier. Since we require the existence of an adapted approximation of **1**, this question is briefly dealt with in Section 8.

Finally, Appendix A contains a hypoellipticity result for the operator $\sqrt{-\Delta_M}$ on an arbitrary Riemannian manifold which is a key element for the proof of the regularity results in Section 5.

2. More on $\mathsf{HT}(\mathsf{p},\mathsf{q})$

A. First construction. We start with a rapid review of some relevant features of the homogeneous tree $\mathbb{T}=\mathbb{T}_{\mathsf{p}}$. Consider $\mathbb{T}$ as a one-complex, where each edge is a copy of the unit interval $[0,1]$. Let $\mathbb{T}^0$ be the vertex set (0-skeleton) of $\mathbb{T}$. This space is equipped with its natural metric. A geodesic in $\mathbb{T}$ is the image of an isometric embedding $t \to w_t \in \mathbb{T}$ of an interval $I \subset \mathbb{R}$.

An *end* of $\mathbb{T}$ is an equivalence class of geodesic rays (parametrized by $[0,\infty)$), where two rays (w_t) and $(\bar{w}_t)$ are equivalent if they coincide except perhaps on bounded initial pieces, i.e., there are $s_0, t_0 \geqslant 0$ such that $w_{s_0+t} = \bar{w}_{t_0+t}$ for all $t \geqslant 0$. We write $\partial\mathbb{T}$ for the space of ends, and $\widehat{\mathbb{T}} = \mathbb{T} \cup \partial\mathbb{T}$. For all $\mathfrak{u}, \mathfrak{v} \in \widehat{\mathbb{T}}$ there is a unique geodesic $\overline{\mathfrak{u}\,\mathfrak{v}}$ that connects the two. We choose and fix a reference vertex $o \in \mathbb{T}^0$ and a reference end $\varpi \in \partial\mathbb{T}$. For $v_1, v_2 \in \widehat{\mathbb{T}} \setminus \{\varpi\}$, their confluent $b = v_1 \curlywedge v_2$ with respect to ϖ is defined by $\overline{v_1\varpi} \cap \overline{v_2\varpi} = \overline{b\varpi}$. The *Busemann function* $\mathfrak{h} : \mathbb{T} \to \mathbb{R}$ and the *horocycles* H_t with respect to ϖ are defined as $\mathfrak{h}(w) = d(w, w \curlywedge o) - d(o, w \curlywedge o)$ and $H_t = \{w \in \mathbb{T}\colon\ \mathfrak{h}(w) = t\}$. Every horocycle is infinite and denumerable. The vertex set $\mathbb{T}^0$ is the union of all H_k with $k \in \mathbb{Z}$. Every vertex v in H_k has one neighbour v^- (its predecessor) in H_{k-1} and p neighbours (its successors) in H_{k+1}. We set $\partial^*\mathbb{T} = \partial\mathbb{T} \setminus \{\varpi\}$.

Fix $\mathsf{q} > 1$ and consider the hyperbolic plane $\mathbb{H}$ in its upper-half space representation. The horocycles (with respect to $i\infty$) are horizontal lines. Recall that $\mathbb{T}$ is subdivided horizontally by the horocycles H_k, $k \in \mathbb{Z}$. Similarly, subdivide $\mathbb{H}$ in the horizontal strips S_k delimited by the lines $y = \mathsf{q}^k$, $k \in \mathbb{Z}$, see Fig. 3. Note that all S_k are hyperbolically isometric.

As outlined in the Introduction, the treebolic space with parameters q and p is

$$\mathsf{HT}(\mathsf{p},\mathsf{q}) = \bigl\{(z,w) \in \mathbb{H} \times \mathbb{T}_{\mathsf{p}}\colon\ \mathfrak{h}(w) = \log_{\mathsf{q}}(\mathit{Im}\, z)\bigr\}, \tag{2.1}$$

where $\mathit{Im}\, z$ is the imaginary part of z. Thus, Figs. 2 and 3 are the "side" and "front" views of HT, that is, the images of HT under the projections $\pi_{\mathbb{T}} : (z,w) \mapsto w$ and $\pi_{\mathbb{H}} : (z,w) \mapsto z$, respectively.

For each end $\mathfrak{u} \in \partial^*\mathbb{T}$, treebolic space contains the isometric copy

$$\mathbb{H}_{\mathfrak{u}} = \bigl\{(z,w) \in \mathbb{H} \times \mathbb{T}_{\mathsf{p}}\colon\ \mathfrak{h}(w) = \log_{\mathsf{q}}(\mathit{Im}\, z),\ \ w \in \overline{\mathfrak{u}\varpi}\bigr\}$$

of $\mathbb{H}$, and if $\mathfrak{u}, \mathfrak{v} \in \partial^*\mathbb{T}$ are distinct and $v = \mathfrak{u} \curlywedge \mathfrak{v}$ (a vertex), then $\mathbb{H}_{\mathfrak{u}}$ and $\mathbb{H}_{\mathfrak{v}}$ bifurcate along the line

$$\mathsf{L}_v = \bigl\{(z,v) \in \mathbb{H} \times \mathbb{T}_{\mathsf{p}}\colon\ \mathit{Im}\, z = \mathsf{q}^{\mathfrak{h}(v)}\bigr\} = \mathbb{R} \times \{v\},$$

that is, $\mathbb{H}_{\mathfrak{u}} \cap \mathbb{H}_{\mathfrak{v}} = \{(z,w) \in \mathsf{HT}\colon\ w \in \overline{v\varpi}\}$. The metric of HT is induced by the hyperbolic length element in the interior of each $\mathbb{H}_{\mathfrak{u}}$.

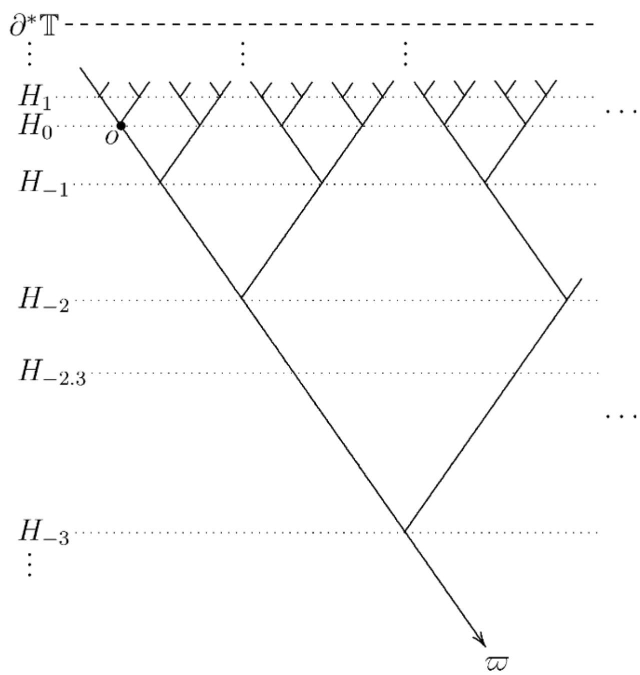

Fig. 2. The "upper half plane" drawing of $\mathbb{T}_2$ (top down, edge lengths are not meaningful in this picture).

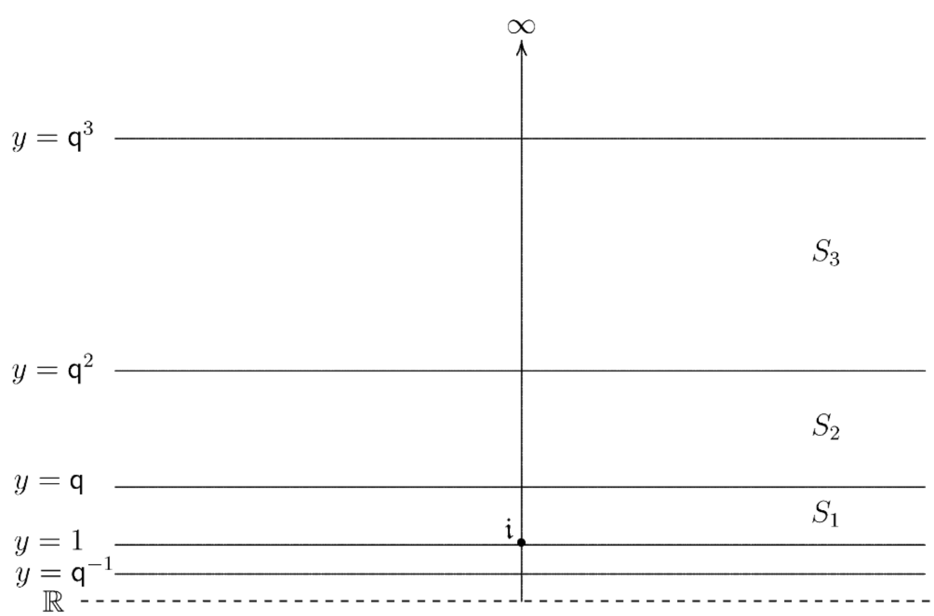

Fig. 3. Hyperbolic upper half plane $\mathbb{H}$ subdivided in isometric strips.

B. Second construction. We now present an alternative construction of $\mathsf{HT} = \mathsf{HT}(\mathsf{p}, \mathsf{q})$ which leads to further generalizations. It is clear that, as a topological space, HT is simply

$$\mathsf{HT} = \mathbb{T}_{\mathsf{p}} \times \mathbb{R}.$$

Note that topologically, q plays no role. Now, let us view $\mathbb{T}_{\mathsf{p}}$ as a metric tree $\mathbb{T}_{\mathsf{p},\mathsf{q}}$ by setting the length of all edges between the horocycles H_{k-1} and H_k to be $\mathsf{q}^{k-1}(\mathsf{q}-1)$. Hence, $\mathbb{T}_{\mathsf{p},\mathsf{q}} \times \mathbb{R}$ comes equipped with a natural geometry. Namely, given any edge $e = [v^-, v]$, parametrized by $s \in [q^{k-1}, q^k]$, $k = \mathfrak{h}(v)$, we can view $[v^-, v] \times \mathbb{R}$ as a manifold with global coordinates $(s, x) \in [q^{k-1}, q^k] \times \mathbb{R}$. We can equip this manifold with the length element $s^{-2}((ds)^2 + (dx)^2)$. Doing this for all edges yields a new metric structure on HT which is isometric to its treebolic structure described earlier. Indeed, any doubly infinite geodesic joining ϖ to another end of $\mathbb{T}$ determines an upper-half plane in $\mathbb{T}_{\mathsf{p},\mathsf{q}} \times \mathbb{R}$, and the construction outlined above yields the hyperbolic metric on any of these upper-half planes (with $s = y$, $z = x + \mathrm{i}y$). The natural measure on $\mathbb{T}_{\mathsf{p},\mathsf{q}} \times \mathbb{R}$ is given on a strip $[v^-, v] \times \mathbb{R}$, viewed as a manifold with global coordinates $(s, x) \in [q^{k-1}, q^k] \times \mathbb{R}$, by $s^{-2}\, ds\, dx$.

C. The two parameters family of Dirichlet forms $\mathcal{E}_{\alpha,\beta}$. Recall that the Riemannian metric and measure of the hyperbolic plane $\mathbb{H} = \mathbb{R}^2_+$ (upper half plane model) are given by $y^{-2}(dx^2 + dy^2)$ and $d\mu = y^{-2}\, dx\, dy$, respectively. The natural Dirichlet form on $\mathbb{H}$ is

$$\int_{\mathbb{H}} |\nabla f|^2\, d\mu = \int_{\mathbb{H}} \left(|\partial_x f|^2 + |\partial_y f|^2\right) dx\, dy.$$

The Laplacian is $y^2(\partial_x^2 + \partial_y^2)$. See, e.g., Chavel [13, pp. 263–265].

Any element ξ in HT is described uniquely by a pair (z, v) with $v \in \mathbb{T}^0$ and $z = x + \mathrm{i}y \in \mathbb{H}$ with $x \in \mathbb{R}$, $\mathsf{q}^{k-1} < y \leqslant \mathsf{q}^k$ and $k = \mathfrak{h}(v)$. In this case, we write $y = y(\xi)$ and $v = v(\xi)$.

Thus, for each $v \in \mathbb{T}^0$, we consider

$$S_v = \left\{(z, v):\ z = x + \mathrm{i}y \in \mathbb{H},\ x \in \mathbb{R},\ \mathsf{q}^{k-1} \leqslant y \leqslant \mathsf{q}^k\right\}$$
$$S_v^o = \left\{(z, v):\ z = x + \mathrm{i}y \in \mathbb{H},\ x \in \mathbb{R},\ \mathsf{q}^{k-1} < y < \mathsf{q}^k\right\}$$

where $k = \mathfrak{h}(v)$. The lines

$$L_v = \left\{(z, v):\ z = x + \mathrm{i}\mathsf{q}^{\mathfrak{h}(v)},\ x \in \mathbb{R}\right\}$$

are called bifurcation lines. With this notation, we have

$$\mathsf{HT} = \bigcup_{v \in \mathbb{T}^0} (S_v \setminus L_{v^-}) \quad \text{(a disjoint union)}.$$

Note that all the strips S_v^o are isometric and have hyperbolic width $\log \mathsf{q}$. However, above we have kept the Euclidean coordinates, taking into account the "height" of the strip S_v, i.e., $k = \mathfrak{h}(v)$.

As mentioned, the space HT carries a natural measure (again coming from $\mathbb{H}$) that we denote by $d\xi$. Namely,

$$\int_{\mathsf{HT}} f(\xi)\, d\xi = \sum_{v \in \mathbb{T}^0} \int_{S_v^o} f(x + \mathrm{i}y, v) y^{-2}\, dx\, dy. \tag{2.2}$$

For $\alpha \in \mathbb{R}$, $\beta > 0$, set

$$d\mu_{\alpha,\beta}(\xi) = \beta^{\mathfrak{h}(v)}\, y^\alpha \, d\xi = \beta^{\mathfrak{h}(v)} y^{\alpha-2}\, dx\, dy. \tag{2.3}$$

This means that

$$\int_{\mathsf{HT}} f(\xi)\, d\mu_{\alpha,\beta}(\xi) = \sum_{v\in\mathbb{T}^0} \beta^{\mathfrak{h}(v)} \int_{S_v^o} f(x+\mathrm{i}y, v) y^{-2+\alpha}\, dx\, dy. \tag{2.4}$$

For any open strip S_v^o equipped with the (x, y)-coordinates as above, let $\mathcal{W}^1(S_v^o)$ be the Sobolev space of those functions f in $\mathcal{L}^2(S_v^o)$ whose distributional first order partial derivatives $\partial_x f, \partial_y f$ can be represented by functions in $\mathcal{L}^2(S_v^o)$ (with respect to the measure $dx\, dy$, say). By a fundamental theorem concerning Sobolev spaces, such functions admit a trace $\mathrm{Tr}_L^{S_v^o}(f)$ on each of the lines bordering the strip. This trace is in fact in the fractional Sobolev space $\mathcal{W}^{1/2}(L)$ of the lines L. Namely, the trace theorem asserts that $\mathrm{Tr}_L^{S_v^o}$ defined on $\mathcal{C}^\infty(S_v)$ extends as a bounded operator

$$\mathrm{Tr}_L^{S_v^o} : \mathcal{W}^1(S_v^o) \to \mathcal{W}^{1/2}(L).$$

We can now describe a two parameters family of function spaces and Dirichlet forms on HT which all share the same underlying geometry.

2.5. Definition. Fix $\alpha \in \mathbb{R}$, $\beta > 0$. Let Ω be an open set in HT. We define $\mathcal{W}^1_{\alpha,\beta}(\Omega)$ as the space of all functions f in $\mathcal{L}^2(\Omega, \mu_{\alpha,\beta})$ such that the following two properties hold.

(1) For each $v \in \mathbb{T}^0$, the function f, restricted to $S_v^o \cap \Omega$, is in $\mathcal{W}^1(S_v^o \cap \Omega)$, and

$$\begin{aligned}\|f\|^2_{\mathcal{W}^1_{\alpha,\beta}(\Omega)} &= \sum_{v\in\mathbb{T}^0} \beta^{\mathfrak{h}(v)} \int_{S_v^o\cap\Omega} \left(\left|f(z,v)\right|^2 y^{-2} + \left|\partial_x f(z,v)\right|^2 + \left|\partial_y f(z,v)\right|^2\right) y^\alpha\, dx\, dy \\ &= \int_\Omega \left(\left|f(\xi)\right|^2 + \left|\nabla f(\xi)\right|^2\right) d\mu_{\alpha,\beta}(\xi) < \infty,\end{aligned}$$

where, for $\xi = (z, v)$, we have set $\nabla f(\xi) = (y^2\partial_x f(z,v), y^2\partial_y f(z,v))$ and

$$\left|\nabla f(\xi)\right|^2 = \left\langle \nabla f(\xi), \nabla f(\xi)\right\rangle_z = y^2\left(\left|\partial_x f(z,v)\right|^2 + \left|\partial_y f(z,v)\right|^2\right).$$

(The inner product is with respect to the hyperbolic metric in the z-variable.)

(2) For any pair of neighbours $u, v \in \mathbb{T}^0$ such that $S_v \cap S_u = L$, one has $\mathrm{Tr}_L^{S_v^o} f = \mathrm{Tr}_L^{S_u^o} f$ along $L \cap \Omega$.

Let $\mathcal{W}^1_{\alpha,\beta,0}(\Omega)$ be the completion of $\mathcal{W}^1_{\alpha,\beta}(\Omega) \cap \mathcal{C}_c(\Omega)$ with respect to the norm $\|\cdot\|_{W^1_{\alpha,\beta}(\Omega)}$.

2.6. Definition. Let $\mathcal{E}_{\alpha,\beta}$ be the bilinear form

$$\mathcal{E}_{\alpha,\beta}(f,g) = \sum_{v\in\mathbb{T}^0} \beta^{\mathfrak{h}(v)} \int_{S_v^o} \big(\partial_x f(z,v)\,\partial_x g(z,v) + \partial_y f(z,v)\partial_y g(z,v)\big) y^\alpha \,dx\,dy$$
$$= \int_{\mathsf{HT}} \big\langle \nabla f(\xi), \nabla g(\xi)\big\rangle_{z(\xi)}\, d\mu_{\alpha,\beta}(\xi), \tag{2.7}$$

with domain $\mathcal{D}(\mathcal{E}_{\alpha,\beta}) = \mathcal{W}^1_{\alpha,\beta}(\mathsf{HT}) \subset \mathcal{L}^2(\mathsf{HT}, \mu_{\alpha,\beta})$. Here, $z(\xi) = z$ if $\xi = (z,v) \in \mathsf{HT}$.

Note that for $f \in \mathcal{W}^1_{\alpha,\beta}(\mathsf{HT})$, the function $\xi \mapsto |\nabla f(\xi)|$ is well defined as an element of $\mathcal{L}^2(\mathsf{HT})$. In the present context, $|\nabla f|^2$ is the *carré du champ*, also often denoted by

$$|\nabla f|^2 = \Gamma(f,f) = \frac{d\Gamma_{\alpha,\beta}(f,f)}{d\mu_{\alpha,\beta}},$$

where $d\Gamma_{\alpha,\beta}(f,f)$ is the *energy measure* associated to $f \in \mathcal{W}^1_{\alpha,\beta}(\mathsf{HT})$. Observe that the *carré du champ* does not depend on the parameters α, β. This explains why we say that these Dirichlet forms all share the same geometry.

2.8. Definition. We let $\mathcal{C}^\infty(\mathsf{HT})$ be the set of those continuous functions f on HT such that, for each $v \in \mathbb{T}^0$, the restriction $f_v = f(\cdot, v)$ of f to the closed strip S_v has continuous derivatives $\partial_x^m \partial_y^n f(z,v)$ of all orders in the interior S_v^o which satisfy, for all $R > 0$,

$$\sup\big\{\big|\partial_x^m \partial_y^n f(z,v)\big| :\ (z,v) \in S_v^o,\ |Re\, z| \leqslant R\big\} < \infty.$$

Given an open set $\Omega \subset \mathsf{HT}$, we let $\mathcal{C}_c^\infty(\Omega)$ be the space of those functions in $\mathcal{C}^\infty(\mathsf{HT})$ that have compact support in Ω.

2.9. Remark. The condition implies that each partial derivative $\partial_x^m \partial_y^n f(z,v)$ extends continuously to the boundary of S_v. We write $\partial_x^m \partial_y^n f_v$ for this extension.

Note however that only the function $f \in \mathcal{C}^\infty(\mathsf{HT})$ itself has to be continuous at the bifurcation lines, not its derivatives. That is, if $w^- = v$ then it is in general *not* true that $\partial_x^m \partial_y^n f_w = \partial_x^m \partial_y^n f_v$ on $L_v = S_v \cap S_w$, unless $m = n = 0$.

2.10. Proposition. *For each $\alpha \in \mathbb{R}$ and $\beta > 0$, the form $(\mathcal{E}_{\alpha,\beta}, \mathcal{W}^1_{\alpha,\beta}(\mathsf{HT}))$ is a strictly local regular Dirichlet form, and $\mathcal{C}_c^\infty(\mathsf{HT})$ is a core for this Dirichlet form.*

For any open set Ω, the space $\mathcal{C}_c^\infty(\Omega)$ is dense in $\mathcal{W}^1_{\alpha,\beta,0}(\Omega)$.

Note that the regularity of these Dirichlet forms is not obvious at all. We will prove this result in a more general setting below.

D. The heat semigroup and Brownian motion. For each $\alpha \in \mathbb{R}$, $\beta > 0$, the Dirichlet form $(\mathcal{E}_{\alpha,\beta}, \mathcal{W}^1_{\alpha,\beta}(\mathsf{HT}))$ induces a self-adjoint contraction semigroup $e^{t\Delta_{\alpha,\beta}}$ with infinitesimal generator ("Laplacian") $\Delta_{\alpha,\beta}$ on $\mathcal{L}^2(\mathsf{HT}, \mu_{\alpha,\beta})$. The domain $\mathrm{Dom}(\Delta_{\alpha,\beta})$ of $\Delta_{\alpha,\beta}$ is the set of functions $f \in \mathcal{W}^1_{\alpha,\beta}(\mathsf{HT})$ for which there exists a constant C_f such that

$$\mathcal{E}_{\alpha,\beta}(f,g) = \int_{\mathsf{HT}} \langle \nabla f(\xi), \nabla g(\xi) \rangle_{z(\xi)} \, d\mu_{\alpha,\beta}(\xi) \leqslant C_f \|g\|_{\mathcal{L}^2(\mathsf{HT},\mu_{\alpha,\beta})}$$

for all $g \in \mathcal{W}^1_{\alpha,\beta}(\mathsf{HT})$. As $\mathcal{W}^1_{\alpha,\beta}(\mathsf{HT})$ is dense in $\mathcal{L}^2(\mathsf{HT}, \mu_{\alpha,\beta})$, this condition and the Riesz representation theorem imply that there exists a (unique) function $h \in \mathcal{L}^2(\mathsf{HT}, \mu_{\alpha,\beta})$ such that $\mathcal{E}_{\alpha,\beta}(f,g) = -\int_{\mathsf{HT}} hg \, d\mu_{\alpha,\beta}$. By definition, $\Delta_{\alpha,\beta} f = h$ see, e.g., [20, Cor. 1.3.1]. If f is in $\mathrm{Dom}(\Delta_{\alpha,\beta}) \cap C^\infty(\mathsf{HT})$ then, in each open strip,

$$\Delta_{\alpha,\beta} f = \left[y^2 \left(\partial_x^2 + \partial_y^2 \right) + \alpha y \partial_y \right] f, \tag{2.11}$$

but f must also satisfy the *bifurcation* or *Kirchhoff condition*

$$\partial_y f_v = \beta \sum_{w:\, w^- = v} \partial_y f_w \quad \text{on } L_v \quad \text{for each } v \in \mathbb{T}^0. \tag{2.12}$$

Note that the parameter β comes into play only at the bifurcation lines where it appears in the bifurcation condition (2.12) relating the different vertical partial derivatives in the $\mathsf{p}+1$ strips meeting along any given bifurcation line. This will be discussed in detail later on.

2.13. Theorem. *The semigroup $e^{t\Delta_{\alpha,\beta}}$, $t > 0$, acting on $\mathcal{L}^2(\mathsf{HT}, \mu_{\alpha,\beta})$ has the following properties*:

(a) *It admits a continuous positive symmetric transition kernel*

$$(0,\infty) \times \mathsf{HT} \times \mathsf{HT} \ni (t, \xi, \zeta) \mapsto h_{\alpha,\beta}(t, \xi, \zeta)$$

such that for all $f \in \mathcal{C}_c(\mathsf{HT})$,

$$e^{t\Delta_{\alpha,\beta}} f(\xi) = \int_{\mathsf{HT}} h_{\alpha,\beta}(t,\xi,\zeta)\, f(\zeta)\, d\mu_{\alpha,\beta}(\zeta).$$

(b) *For each fixed (t,ξ), the function $\zeta \mapsto h_{\alpha,\beta}(t,\xi,\zeta)$ is in $\mathcal{C}^\infty(\mathsf{HT})$ and satisfies* (2.12).
(c) *For each $k \in \mathbb{N}$, the function $(0,\infty) \times \mathsf{HT} \times \mathsf{HT} \ni (t,\xi,\zeta) \mapsto \partial_t^k h_{\alpha,\beta}(t,\xi,\zeta)$ is Hölder continuous, and for each $\xi \in \mathsf{HT}$, the function $\zeta \mapsto \partial_t^k h_{\alpha,\beta}(t,\xi,\zeta)$ is in $\mathcal{C}^\infty(\mathsf{HT})$ and satisfies* (2.12).
(d) *For any fixed $\epsilon \in (0,1)$ and $k \in \mathbb{N}$, there is a constant $C = C(\alpha,\beta,\mathsf{p},\mathsf{q},k,\epsilon)$ such that for all $(t,\xi,\zeta) \in (0,\infty) \times \mathsf{HT} \times \mathsf{HT}$,*

$$\left|\partial_t^k h_{\alpha,\beta}(t,\xi,\zeta)\right| \leqslant \frac{C}{\beta^{\mathfrak{h}(v(\xi))}\, y(\xi)^\alpha \min\{1,t\} t^k} \exp\left(-\frac{d(\xi,\zeta)^2}{4(1+\epsilon)t}\right). \tag{2.14}$$

(e) *It is conservative, that is, $e^{t\Delta_{\alpha,\beta}}\mathbf{1} = \mathbf{1}$. Equivalently, $\int_{\mathsf{HT}} h_{\alpha,\beta}(t,\xi,\cdot)\, d\mu_{\alpha,\beta} = 1$.*
(f) *It sends $\mathcal{L}^\infty(\mathsf{HT})$ into $\mathcal{C}(\mathsf{HT}) \cap \mathcal{L}^\infty(\mathsf{HT})$.*

(g) *It sends* $\mathcal{C}_0(\mathsf{HT})$ *into itself.*
(h) *The associated Hunt process is transient, that is, for all pairs of distinct points* $\xi, \zeta \in \mathsf{HT}$,

$$G_{\alpha,\beta}(\xi,\zeta) = \int_0^\infty h_{\alpha,\beta}(t,\xi,\zeta)\,dt < \infty.$$

(i) *The bottom* $\lambda = \lambda(\alpha,\beta,\mathsf{p},\mathsf{q})$ *of the* $\mathcal{L}^2(\mathsf{HT},\mu_{\alpha,\beta})$*-spectrum of* $-\Delta_{\alpha,\beta}$ *is strictly positive if and only if* $\mathsf{q}^{1-\alpha} \neq \beta\mathsf{p}$.
In particular, in addition to (2.14) *the following holds.*
For any fixed $\epsilon \in (0,1)$ *and* $k \in \mathbb{N}$, *there is a constant* $C = C(\alpha,\beta,\mathsf{p},\mathsf{q},k,\epsilon)$ *such that for all* $(t,\xi,\zeta) \in (0,\infty) \times \mathsf{HT} \times \mathsf{HT}$,

$$\left|\partial_t^k h_{\alpha,\beta}(t,\xi,\zeta)\right| \leqslant \frac{C}{\beta^{\mathfrak{h}(v(\xi))} y(\xi)^\alpha (\min\{1,t\})^{1+k}} \exp\left(-\lambda t - \frac{d(\xi,\zeta)^2}{4(1+\epsilon)t}\right). \tag{2.15}$$

Proof. Statements (a) through (g) follow from more general results proved in this paper. That λ is positive if and only if $q^{1-\alpha}/(\beta\mathsf{p}) \neq 1$ can be obtained by the techniques and results of Saloff-Coste and Woess [29] which also provides an explicit formula for λ in terms of the parameters. Transience is explained below after Theorem 2.23. □

2.16. Definition. Let $\mathsf{HT}^o = \bigcup_v S_v^o$ be the treebolic space without the bifurcation lines. For $f \in \mathcal{C}^\infty(\mathsf{HT}^o)$, set

$$\mathfrak{A}_\alpha f(\xi) = y^2\left(\partial_x^2 + \partial_y^2\right) f(\xi) + \alpha y \partial_y f(\xi), \qquad \xi = (x + \mathrm{i}y, v) \in \mathsf{HT}^o.$$

Let $\mathcal{D}^\infty_{\alpha,\beta,c}$ be the space of those functions in $\mathcal{C}_c^\infty(\mathsf{HT})$ such that:

- For any k, the function $\mathfrak{A}_\alpha^k f$, originally defined on HT^o, admits a continuous extension to all of HT. (Here, $\mathfrak{A}_\alpha^k$ is the k-th iterate of $\mathfrak{A}_\alpha$.) This implies that $\mathfrak{A}_\alpha^k f \in \mathcal{C}_c^\infty(\mathsf{HT})$ for each k.
- Using the same notation as in Remark 2.9 and formula (2.12),

$$\partial_y \mathfrak{A}_\alpha^k f_v = \beta \sum_{w:\, w^- = v} \partial_y \mathfrak{A}_\alpha^k f_w \quad \text{on } L_v \text{ for each } v \in \mathbb{T}^0.$$

The following statement yields a clear and fundamental uniqueness result concerning the Laplacian $\Delta_{\alpha,\beta}$ introduced above. For the proof, see Theorem 7.11 and Proposition 8.3.

2.17. Theorem. *The operator* $(\mathfrak{A}_\alpha, \mathcal{D}^\infty_{\alpha,\beta,c})$ *is symmetric on* $\mathcal{L}^2(\mathsf{HT},\mu_{\alpha,\beta})$. *It is essentially self-adjoint and its unique self-adjoint extension is the infinitesimal generator* $(\Delta_{\alpha,\beta}, \mathrm{Dom}(\Delta_{\alpha,\beta}))$ *associated with the Dirichlet form* $(\mathcal{E}_{\alpha,\beta}, \mathcal{W}^1_{\alpha,\beta}(\mathsf{HT}))$ *on* $\mathcal{L}^2(\mathsf{HT},\mu_{\alpha,\beta})$.

2.18. Remark. Let X be a topological space equipped with a Borel measure μ with full support. A densely defined operator $(\mathfrak{A}, \mathrm{Dom}(\mathfrak{A}))$ on $\mathcal{L}^1(X,\mu)$ is called strongly Markov-unique if and only if there is at most one sub-Markovian $\mathcal{C}^0$-semigroup on $\mathcal{L}^1(X,\mu)$ whose infinitesimal generator extends $(\mathfrak{A}, \mathrm{Dom}(\mathfrak{A}))$. It is not hard to see that a symmetric essentially self-adjoint operator is strongly Markov-unique. See, e.g., Eberle [16].

E. The (α,β)-Markov process. By the general theory of Markov processes, there is a Hunt process associated with the conservative semigroup $H_t^{\alpha,\beta} = e^{t\Delta_{\alpha,\beta}} : \mathcal{C}_0(\mathsf{HT}) \mapsto \mathcal{C}_0(\mathsf{HT})$. It is defined for every starting point $\xi \in \mathsf{HT}$, has infinite life time and continuous sample paths. Its family of distributions $(\mathbb{P}_\xi^{\alpha,\beta})_{\xi\in\mathsf{HT}}$ on $\boldsymbol{\Omega} = \mathcal{C}([0,\infty] \to \mathsf{HT})$ is determined by the one-dimensional distributions

$$\mathbb{P}_\xi^{\alpha,\beta}[X_t \in U] = \int_U h_{\alpha,\beta}(t,\xi,\zeta)\, d\mu_{\alpha,\beta}(\zeta) = H_t^{\alpha,\beta}\mathbf{1}_U(\xi)$$

where U is any Borel subset of HT.

Setting $\tau_U = \inf\{t:\ X_t \notin U\}$, we can define the exit distribution from a bounded Borel set U by

$$\pi_U^{\alpha,\beta}(\xi, B) = \mathbb{P}_\xi^{\alpha,\beta}[X_{\tau_U} \in B]$$

for any Borel set $B \subset U$ and set

$$\pi_U^{\alpha,\beta}(\xi, f) = \mathbb{E}_\xi^{\alpha,\beta}\big(f(X_{\tau_U})\big)$$

for any bounded Borel measurable function f. Since the process has continuous sample paths, the exit distribution is supported by ∂U for any starting point $\xi \in U$.

As outlined at the beginning of this section, the treebolic space $\mathsf{HT}(\mathsf{p},\mathsf{q}) = \{(z,w) \in \mathbb{H} \times \mathbb{T}_\mathsf{p}:\ \mathfrak{h}(w) = \log_\mathsf{q}(Im\, z)\}$ (here written in terms of the first construction) admits natural projections, $\pi_\mathbb{H} : (z,w) \mapsto z$ and $\pi_\mathbb{T} : (z,w) \mapsto w$, corresponding respectively to the “side” and “front” views of HT depicted in Figs. 2 and 3.

By the general theory of transformations of the state space, it is plain that the images of the Hunt process $(X_t, \mathbb{P}_\xi^{\alpha,\beta}, t \geqslant 0, \xi \in \mathsf{HT})$ by the projections $\pi_\mathbb{H}$ and $\pi_\mathbb{T}$ are Markov processes. What is not entirely obvious, *a priori,* is to describe what these processes are in intrinsic terms in $\mathbb{H}$ and $\mathbb{T}$. One of the multiple motivations behind this work was indeed to obtain an intrinsic description of each of these processes.

Analogously to HT, we can describe the metric tree $\mathbb{T} = \mathbb{T}_{\mathsf{p},\mathsf{q}}$ as

$$\mathbb{T} = \big\{(s,v):\ v \in \mathbb{T}^0,\ s \in \big(\mathsf{q}^{\mathfrak{h}(v)-1}, \mathsf{q}^{\mathfrak{h}(v)}\big]\big\},$$

where $\{v\} \times (\mathsf{q}^{\mathfrak{h}(v)-1}, \mathsf{q}^{\mathfrak{h}(v)}]$ parametrizes the “metric edge” $(v^-, v]$ as a left-open interval. On $\mathbb{T}$ we consider the measure $\mu_{\alpha,\beta}^{\mathbb{T}}$ defined by $d\mu_{\alpha,\beta}^{\mathbb{T}}(s,v) = \beta^{\mathfrak{h}(v)} s^{-2+\alpha}\, ds$, that is, for all $f \in \mathcal{C}_c(\mathbb{T})$

$$\int_{\mathbb{T}} f\, d\mu_{\alpha,\beta}^{\mathbb{T}} = \sum_{v\in\mathbb{T}^0} \beta^{\mathfrak{h}(v)} \int_{q^{\mathfrak{h}(v)-1}}^{q^{\mathfrak{h}(v)}} f(s,v) s^{-2+\alpha}\, ds, \tag{2.19}$$

and the Dirichlet form

$$\mathcal{E}_{\alpha,\beta}^{\mathbb{T}}(f,f) = \int_{\mathbb{T}} s^2 |\partial_s f|^2\, d\mu_{\alpha,\beta}^{\mathbb{T}} = \sum_{v\in\mathbb{T}^0} \beta^{\mathfrak{h}(v)} \int_{q^{\mathfrak{h}(v)-1}}^{q^{\mathfrak{h}(v)}} \big|\partial_s f(s,v)\big|^2 s^\alpha\, ds, \tag{2.20}$$

with domain

$$\mathcal{W}^1_{\alpha,\beta}(\mathbb{T}) = \{ f \in \mathcal{C}(\mathbb{T}) \cap \mathcal{L}^2(\mathbb{T}, \mu^{\mathbb{T}}_{\alpha,\beta}) \colon\ s\partial_s f \in \mathcal{L}^2(\mathbb{T}, \mu^{\mathbb{T}}_{\alpha,\beta}) \}.$$

Here $\partial_s f$ denotes the distributional derivative of f along any open edge $(v^-, v) = \{v\} \times (q^{\mathfrak{h}(v)-1}, q^{\mathfrak{h}(v)})$ of $\mathbb{T}$. Let $h^{\mathbb{T}}_{\alpha,\beta}(t, \cdot, \cdot)$, $t > 0$, be the heat kernel associated with this Dirichlet form.

On the hyperbolic space $\mathbb{H}$, subdivided by the horocycle lines $L_k = \{z = x + \mathrm{i}y\colon\ y = q^k\}$, consider the measure $\mu^{\mathbb{H}}_{\alpha,\beta}$ which is defined for all $f \in \mathcal{C}_0(\mathbb{H})$ by

$$\int_{\mathbb{H}} f\, d\mu^{\mathbb{H}}_{\alpha,\beta} = \sum_{k\in\mathbb{Z}} \beta^k \int_{q^{k-1}}^{q^k} \int_{-\infty}^{\infty} f(x + \mathrm{i}y) y^{-2+\alpha}\, dx\, dy, \tag{2.21}$$

and the Dirichlet form

$$\begin{aligned} \mathcal{E}^{\mathbb{H}}_{\alpha,\beta}(f, f) &= \int_{\mathbb{H}} |\nabla f|^2\, d\mu^{\mathbb{H}}_{\alpha,\beta} \\ &= \sum_{k\in\mathbb{Z}} \beta^k \int_{q^{k-1}}^{q^k} \int_{-\infty}^{\infty} \left(\left|\partial_x f(x + \mathrm{i}y)\right|^2 + \left|\partial_y f(x + \mathrm{i}y)\right|^2 \right) y^{\alpha}\, dx\, dy, \end{aligned} \tag{2.22}$$

where $|\nabla f|$ denotes the hyperbolic gradient length of f. The domain of this form is the space $\mathcal{W}^1_{\alpha,\beta}(\mathbb{H})$ of those functions in $\mathcal{L}^2(\mathbb{H}, \mu^{\mathbb{H}}_{\alpha,\beta})$ which admit locally integrable first order partial derivatives in the sense of distributions and such that $|\nabla f|$ is in $\mathcal{L}^2(\mathbb{H}, \mu^{\mathbb{H}}_{\alpha,\beta})$. Let $h^{\mathbb{H}}_{\alpha,\beta}(t, \cdot, \cdot)$, $t > 0$, be the heat kernel associated with this Dirichlet form on $\mathbb{H}$. (All this coincides precisely with what we have considered in the previous subsections on $\mathsf{HT}(\mathsf{p}, \mathsf{q})$, but now we are in the "degenerate" case when $\mathsf{p} = 1$ and the tree is a two-way-infinite linear graph.)

2.23. Theorem. *Fix $\mathsf{p} \in \{2, 3, \ldots\}$, $\mathsf{q} > 1$ and $\alpha \in \mathbb{R}$, $\beta > 0$. Let (X_t) be the process on $\mathsf{HT}(\mathsf{p}, \mathsf{q})$ associated with the Dirichlet form $(\mathcal{E}_{\alpha,\beta}, \mathcal{W}^1_{\alpha,\beta}(\mathsf{HT}))$. Let $Y_t = \pi^{\mathbb{T}}(X_t)$, $Z_t = \pi^{\mathbb{H}}(X_t)$, $t > 0$, be the projections on $\mathbb{T}$ and $\mathbb{H}$, respectively.*

(a) *The process (Y_t) is a Markov process on $\mathbb{T}$ and, for any $t > 0$ and $y \in \mathbb{T}$, the law of Y_t given $Y_0 = y_0$ has probability density $h^{\mathbb{T}}_{\alpha,\beta}(t, y_0, \cdot)$ with respect to $\mu^{\mathbb{T}}_{\alpha,\beta}$.*
In other words, (Y_t) is a version of the Hunt process associated with the strictly local regular Dirichlet form $(\mathcal{E}^{\mathbb{T}}_{\alpha,\beta}, \mathcal{W}^1_{\alpha,\beta}(\mathbb{T}))$.
(b) *The process (Z_t) is a Markov process on $\mathbb{H}$ and, for any $t > 0$ and $z \in \mathbb{H}$, the law of Z_t given $Z_0 = z_0$ has probability density $h^{\mathbb{H}}_{\alpha,\beta\mathsf{p}}(z_0, \cdot)$ with respect to $\mu^{\mathbb{H}}_{\alpha,\beta\mathsf{p}}$.*
In other words, (Z_t) is a version of the Hunt process associated with the strictly local regular Dirichlet form $(\mathcal{E}^{\mathbb{H}}_{\alpha,\beta\mathsf{p}}, \mathcal{W}^1_{\alpha,\beta\mathsf{p}}(\mathbb{H}))$.

See Proposition 6.6 and Example 6.8(C) at the end of Section 6.

2.24. Proposition. *Each of the processes* (X_t), (Y_t) *and* (Z_t) *appearing in Theorem* 2.23 *is transient.*

Proof. Via the projections, transience of (X_t) will follow from transience of (Z_t).

This amounts to showing that for every choice of $\alpha \in \mathbb{R}$ and $\beta > 0$, the process on $\mathsf{HT}(1, \mathsf{q}) = \mathbb{H}$ associated with $(\mathcal{E}^{\mathbb{H}}_{\alpha,\beta}, \mathcal{W}^1_{\alpha,\beta}(\mathbb{H}))$ is transient. Now, the associated measure $\mu^{\mathbb{H}}_{\alpha,\beta}$ can be compared below and above, up to multiplying with positive constants, with the measure $\mu^{\mathbb{H}}_{\bar\alpha,\bar\beta}$, where $\bar\alpha = \alpha + \log\beta/\log q$ and $\bar\beta = 1$. Hence, the associated metric measure spaces are (measure) quasi-isometric (i.e., quasi-isometric with adapted measures, see Coulhon and Saloff-Coste [14]). This implies that the corresponding processes are either both transient or both recurrent. Hence, it thus suffices to study the transience of the process on $\mathbb{H}$ associated with $(\mathcal{E}^{\mathbb{H}}_{\bar\alpha,1}, \mathcal{W}^1_{\bar\alpha,1}(\mathbb{H}))$. This process does not "see" the separating lines bounding the strips. Indeed, the associated infinitesimal generator on the whole upper half plane is

$$\Delta_{\bar\alpha,1} = y^2\big(\partial_x^2 + \partial_y^2\big) + \bar\alpha y \partial_y.$$

The process is just standard hyperbolic Brownian motion on $\mathbb{H}$ with an additional vertical drift term. It is very well known to be transient. For example, one finds nonconstant positive harmonic functions that are expressed in terms of the Poisson kernel. Another way is to identify $\mathbb{H}$ with the *affine group* of all transformations $x \mapsto ax + b$, where $a > 0$ and $b \in \mathbb{R}$, via $(a, b) \leftrightarrow b + \mathrm{i}a \in \mathbb{H}$. Then the law of our process is invariant under the action of the affine group on itself, whence it must be transient, compare e.g. with Guivarc'h, Keane and Roynette [22]. Namely, when we consider the process at integer times, we obtain a random walk on the affine group, which must be transient since that group is non–unimodular.

Also transience of (Y_t) can be shown by constructing non-constant positive harmonic functions. More details are deferred to forthcoming work [6], where among other things we shall describe *all* positive harmonic functions associated with $(\mathcal{E}^{\mathbb{T}}_{\alpha\beta}, \mathcal{W}^1_{\alpha,\beta}(\mathbb{T}))$. □

2.25. Remark. Theorems 2.13 and 2.17, which describe some basic properties of the (α, β)-heat semigroup and Laplacian on HT have obvious versions that apply to the heat semigroups and Laplacians on $\mathbb{T}$ and $\mathbb{H}$ (respectively) that appear in the above result on projections. All these results illustrate the more general theory developed below in the setting of what we call strip complexes. In fact, the introduction of the notion of strip complex is motivated in part by the justification of the projections described above and the need to treat all these objects and their properties in a unified way.

3. Strip complexes

A. The basic structure of strip complexes. Let V, E be countable sets equipped with a map

$$E \to V \times V, \qquad e \mapsto \big(e^-, e^+\big).$$

This defines an oriented graph Γ with vertex set V and edge set E. We will assume throughout that $e^- \neq e^+$. Hence multiple edges are allowed, but there are no loops. The "no loops" convention will simplify our considerations. Moreover, this is no real lack of generality for our purpose: loops can be handled by adding a virtual vertex in the middle of any existing loop.

The vertices e^-, e^+ are the extremities of the edge e. We set $V_e = \{e^-, e^+\} \subset V$ and $E_v = \{e\colon\ v \in V_e\}$. We let Γ^1 be the associated 1-dimensional complex. In Γ^1, the edge e is realized by a subset I_e of Γ^1, homeomorphic to the closed interval $[0, 1]$. We will also use the notation $I_e = [e^-, e^+]$ and $I_e^o = (e^-, e^+)$ for the closed and open intervals corresponding to edge e, respectively. Similarly, we write $\Gamma^o = \Gamma^1 \setminus V$. We assume throughout that Γ^1 is connected and that each vertex has only finitely many neighbours, that is, E_v is a finite set. For reasons that will become clear later, we refer to $\deg(v) = |E_v|$ as the *bifurcation number* at v.

Although the edges are oriented, this orientation will not play an important role for us. In particular, the notion of neighbours introduced above does not take the orientation into account. Observe also that we can view Γ^1 as the union of all the edges I_e, $e \in E$, with the appropriate identification at the vertices where several edges meet.

Given a topological space M (we will be mostly interested here in the case where M is $\{o\}$, a line, a circle, or more generally, a Riemannian manifold), the *strip complex* (more precisely, the M-strip complex) associated to Γ and M is simply the direct product

$$\Gamma\mathsf{M} = \Gamma^1 \times M.$$

This is a topological space with a simple "coordinate system" $\Gamma\mathsf{M} \ni \xi = (\gamma, m)$. However, this viewpoint is not entirely well suited to capture the additional structure that these spaces have in the cases of interest to us.

Instead, it will be essential to view $\Gamma\mathsf{M}$ as the union of the *strips*

$$\bigcup_{e \in E} S_e, \quad \text{where } S_e = I_e \times M.$$

This is not a disjoint union, as the strips $S_e = I_e \times M$, $e \in E_v$, $v \in V$, all meet along $M_v = \{v\} \times M$. We call M_v the *bifurcation manifold* at v. This is simply the copy of M passing through v in $\Gamma\mathsf{M}$.

(In Section 2, $M = \mathbb{R}$, and the strips were labeled by the vertices of the tree, because there is a one-to-one correspondence between vertices v and edges $[v^-, v]$.)

We let

$$S_e^o = \left(e^-, e^+\right) \times M$$

be the interior of the strip S_e and set

$$\Gamma\mathsf{M}^o = \bigcup_{e \in E} S_e^o,$$

the union of all open strips in $\Gamma\mathsf{M}$ (this is an open dense set in $\Gamma\mathsf{M}$). For any function f defined on $\Gamma\mathsf{M}^o$, we let

$$f_e = f|_{S_e^o}$$

be the restriction of f to the open strip S_e^o. This notation plays an important role and will be used throughout. In addition, we make the following natural convention. Whenever f_e admits a continuous extension to the closed strip S_e, we (abusively) use the same notation f_e to denote this continuous extension. Note that if f_e and $f_{e'}$ are defined on S_e and $S_{e'}$ with $M_v = S_e \cap S_{e'}$ then it may well be that f_e and $f_{e'}$ take different values along M_v.

We also set

$$X_v = M_v \cup \left(\bigcup_{e \in E_v} S_e^o \right).$$

The set X_v is called the *star of strips* at v. It is an open set in $\Gamma\mathsf{M}$.

3.1. Remark. Note that the definition of a strip complex given above is of a global nature and corresponds to what could be called "untwisted" strip complex in the context of the following more general definition which yields the same local structure. In this more general definition, the graph Γ is decorated at each vertex by a collection $\{g_v^e : e \in E_v\}$ of homeomorphisms g_v^e: $M \to M$ (when M is equipped with a Riemannian structure, these maps are required to be isometries). Then, the boundaries M_v^e of different strips S_e^o, $e \in E_v$, meeting at a vertex v, are identified with a unique copy M_v of M through the homeomorphisms g_v^e. For instance, if $M = (0, 1)$, and the graph Γ has two vertices a, a' and two edges e, e' joining a and a', the strip complex $\Gamma\mathsf{M} = \Gamma^1 \times M$ is a cylinder with two marked lines corresponding to a, a'. However, we could identify the two intervals $(0, 1)$ at a through the identity map and at a' through the flip $x \mapsto 1 - x$. In this case, we get a "twisted strip complex" which is a Moebius band with two marked lines. Note that this "twisted strip complex" is not globally the direct product of Γ^1 and M although, locally, it has the same structure. We will not discuss twisted strip complexes in this paper. But we note that all of our results (properly interpreted) will hold as well for such more general structures. In particular, our local smoothness results will apply to these twisted structures in an obvious way.

3.2. Remark. The treebolic space (see Fig. 1) gives a good illustration of a strip complex structure, but it may be useful for the reader to think of the case when M is the unit circle and Γ is some finite graph. Although one can easily draw sketches of such examples, in most cases, these circle strip complexes cannot be embedded (without crossings) in three-space.

B. Smooth functions on strip complexes. Fix a graph Γ as defined above. Let M be an n-dimensional manifold and consider the associated strip complex $\Gamma\mathsf{M}$. Let $\mathcal{C}_c(\Gamma\mathsf{M})$, $\mathcal{C}_0(\Gamma\mathsf{M})$ and $\mathcal{C}_b(\Gamma\mathsf{M})$ be the spaces of continuous functions on $\Gamma\mathsf{M}$ that are, respectively, compactly supported, vanishing at infinity, bounded.

Without further comments, we will assume that M is equipped with a Radon measure which, in any coordinate chart on M, admits a smooth positive density with respect to the Riemannian measure. The strip complex $\Gamma\mathsf{M}$ is then equipped with the product measure of one-dimensional Lebesgue measure on Γ^1 and the given Radon measure on M. Later we will make a more precise choice of such a measure. For the time being, this measure is used only for the definition of negligible sets (sets of measure zero) and the particular choice made is irrelevant.

3.3. Definition. A *relatively compact coordinate chart* in $\Gamma\mathsf{M}$ is an open, relatively compact set of the form $I \times U \subset \Gamma\mathsf{M}$ where $I \subset (e^-, e^+) \subset \Gamma^1$ for some $e \in E$ is an open interval and $(U; x_1, \ldots, x_n)$ is a relatively compact coordinate chart in M. The associated *local coordinate system* on the open subset $I \times U$ is denoted by $\xi = (s, x_1, \ldots, x_n)$, $s \in I$, $(x_1, \ldots, x_n) \in U$. For any $(n+1)$-tuple $\kappa = (\kappa_0, \kappa_1, \dots, \kappa_n)$ of integers and any smooth enough function f defined over $I \times U$, we set

$$\partial_\xi^\kappa f(\xi) = \partial_s^{\kappa_0} \partial_{x_1}^{\kappa_1} \dots \partial_{x_n}^{\kappa_n} f(s, x_1, \dots, x_n).$$

If necessary, we can also consider $\partial_\xi^\kappa f$ to be defined in the sense of distributions in $I \times U$.

3.4. Remark. The above definition never involves the bifurcation manifolds, except possibly at the boundary of $I \times U$. Hence, smoothness of a function in a relatively compact chart $I \times U$ as defined above is a classical notion.

3.5. Definition. (a) The space of *strip-wise smooth functions* on $\Gamma\mathsf{M}^o$, denoted $\mathcal{S}^\infty(\Gamma\mathsf{M}^o)$, is the set of those locally bounded functions f on $\Gamma\mathsf{M}^o$ such that, for any open edge $I_e^o = (e^-, e^+)$, $e \in E$, and any precompact coordinate chart $(U; x_1, \dots, x_n)$ in M, the function $f|_{I_e^o \times U}$ is a bounded continuous function with bounded continuous derivatives of all orders with respect to the coordinates $(s, x_1, \dots, x_n)$ in $I_e^o \times U$. The vector space $\mathcal{S}^\infty(\Gamma\mathsf{M}^o)$ is equipped with the family of seminorms

$$N_{K,I\times U}^k(f) = \sup\{|f(\xi)|:\ \xi \in K \cap \Gamma\mathsf{M}^o\} \\ + \sup\left\{|\partial_\xi^\kappa f(\xi)|:\ \xi \in I \times U,\ \kappa = (\kappa_0, \kappa_1, \dots, \kappa_n),\ \sum_0^n \kappa_i \leqslant k\right\}, \tag{3.6}$$

where k is an integer, K a compact subset of $\Gamma\mathsf{M}$ and $I \times U$ a relatively compact coordinate chart in $\Gamma\mathsf{M}$.

Abusing notation, we will also consider any function f in $\mathcal{S}^\infty(\Gamma\mathsf{M}^o)$ as a function on $\Gamma\mathsf{M}$ that is defined almost everywhere (a representative of a class of functions under the usual equivalence of coinciding almost everywhere).

(b) The space of *continuous strip-wise smooth functions* on $\Gamma\mathsf{M}$, denoted $\mathcal{C}^\infty(\Gamma\mathsf{M})$ is defined as

$$\mathcal{C}(\Gamma\mathsf{M}) \cap \mathcal{S}^\infty(\Gamma\mathsf{M}^o) = \{f \in C(\Gamma\mathsf{M}):\ f|_{\Gamma\mathsf{M}^o} \in \mathcal{S}^\infty(\Gamma\mathsf{M}^o)\}.$$

We also let

$$\mathcal{C}_c^\infty(\Gamma\mathsf{M}) = \mathcal{C}^\infty(\Gamma\mathsf{M}) \cap \mathcal{C}_c(\Gamma\mathsf{M}).$$

The vector space $\mathcal{C}^\infty(\Gamma\mathsf{M})$ is equipped with the same family of seminorms $N_{K,I\times U}^k$ as $\mathcal{S}^\infty(\Gamma\mathsf{M}^o)$.

3.7. Remarks. (i) A function $f \in \mathcal{S}^\infty(\Gamma\mathsf{M}^o)$ is not necessarily continuous across bifurcation manifolds (it need not even be defined on the latter). However, the functions f_e are bounded continuous with bounded continuous derivatives on $I_e^o \times U$ for any relatively compact set $U \subset M$. This implies that each f_e can be extended as a smooth continuous function to the closed strip S_e. According to our earlier convention, we still denote this extension by f_e. In particular, for any vertex v, a function $f \in \mathcal{S}^\infty(\Gamma\mathsf{M})$, yields $\deg(v)$ smooth functions

$$M \ni x \mapsto f_e(v, x) \in \mathcal{C}^\infty(M).$$

(ii) Note that a function in $\mathcal{S}^\infty(\Gamma\mathsf{M}^o)$ may not have a continuous extension to $\Gamma\mathsf{M}$ but is always (essentially) bounded on compact sets.

(iii) The space $\mathcal{C}^\infty(\Gamma\mathsf{M})$ is a complete seminormed space. In view of (i), a function $f \in \mathcal{C}^\infty(\Gamma\mathsf{M})$ is a continuous function on $\Gamma\mathsf{M}$ such that its restriction f_e to any closed strip S_e is a smooth function in the usual sense on the manifold S_e.

Since f is continuous it follows that the partial derivatives $\partial_x^\kappa f$, $\kappa = (\kappa_1, \ldots, \kappa_n)$ in the direction of M have to be continuous across bifurcation manifolds. That is, for any fixed coordinate chart $(U; x)$ in M, with $x = (x_1, \ldots, x_n)$,

$$\partial_x^\kappa f_{e_1}(v, x) = \partial_x^\kappa f_{e_2}(v, x), \quad \text{if } e_1, e_2 \in E_v.$$

Note, however, that the partial derivatives $\partial_s^k \partial_x^\kappa f_e$ with $k \geqslant 1$ and computed in different strips meeting along a bifurcation manifold M_v do not have to match along M_v.

3.8. Remark. We will sometimes consider functions f of space and time variables, such as for example $(0, T) \times \Gamma\mathsf{M} \ni (t, \xi) \mapsto f(t, \xi)$. Since $(0, T) \times \Gamma\mathsf{M}$ is also a strip complex, with M replaced by $M \times (0, T)$, (3.5.b) also defines $\mathcal{C}^\infty((0, T) \times \Gamma\mathsf{M})$.

The following subspace of $\mathcal{C}_c^\infty(\Gamma\mathsf{M})$ will be useful for our purpose. It is the subspace of those functions in $\mathcal{C}_c^\infty(\Gamma\mathsf{M})$ which are locally constant along Γ^1 near each bifurcation manifold M_v.

3.9. Definition. Let $\mathcal{C}_{c,c}^\infty(\Gamma\mathsf{M})$ be the subspace of $\mathcal{C}_c^\infty(\Gamma\mathsf{M})$ of those functions whose partial derivative $\partial_s f_e$ in any strip $S_e = I_e \times M$, $s \in I_e$, has compact support in S_e^o.

3.10. Lemma. *The space $\mathcal{C}_{c,c}^\infty(\Gamma\mathsf{M})$ is dense in $\mathcal{C}_0(\Gamma\mathsf{M})$ for the uniform norm.*

Proof. Since $\mathcal{C}_c(\Gamma\mathsf{M})$ is dense in $\mathcal{C}_0(\Gamma\mathsf{M})$ for the uniform norm, it suffices to show that for any $f \in \mathcal{C}_c(\Gamma\mathsf{M})$ and $\epsilon > 0$ there is $f_\epsilon \in \mathcal{C}_{c,c}^\infty(\Gamma\mathsf{M})$ such that $\|f - f_\epsilon\|_\infty \leqslant \epsilon$.

Let K be the support of f and $\{U_n, n \leqslant N\}$ be a finite covering of K by open precompact subsets which are so small that for each n, $\sup\{|f(\xi) - f(\zeta)|: \xi, \zeta \in U_n\} < \epsilon$ (uniform continuity of f) and U_n is either of the form $J_n \times V_n$ where V_n is a small coordinate chart in M and J_n is relatively compact in (e^-, e^+) for some e, or $U_n = \bigcup_{e \in E_v} J_n^e \times V_n$ where V_n is a small coordinate chart in M and each J_n^e is a semi-open interval in I_e with closed extremity at v. By standard arguments adapted to the present situation, we can construct a family of functions $\omega_n \in \mathcal{C}_{c,c}^\infty(\Gamma\mathsf{M})$ such that ω_n is supported in U_n and $\sum_{n \leqslant N} \omega_n = 1$ on K. For each $n \leqslant N$, pick $\xi_n \in U_n$ and set

$$f_\epsilon = \sum_{n \leqslant N} f(\xi_n)\omega_n.$$

By construction, $f_\epsilon \in \mathcal{C}_{c,c}^\infty(\Gamma\mathsf{M})$ and, for any $\xi \in K$,

$$|f - f_\epsilon|(\xi) \leqslant \sum_{n \leqslant N} \bigl|f(\xi) - f(\xi_n)\bigr|\omega_n(\xi) \leqslant \epsilon.$$

This provides the desired approximation. $\square$

The next definition introduces smoothness (of various orders) in an open subset Ω of $\Gamma\mathsf{M}$.

3.11. Definition. Let Ω be an open subset of $\Gamma\mathsf{M}$.

(a) A function f is in $\mathcal{C}^k(\Omega)$, where $k \geqslant 1$, if it is continuous in Ω, and for any relatively compact coordinate chart $I \times U$ with $\overline{I \times U} \subset \Omega$, f has continuous partial derivatives of order up to k in $I \times U$. This space is equipped with the family of seminorms $N^k_{K,I\times U}$ defined as in (3.6), where K runs over compact subsets of Ω and $I \times U$ over all relatively compact coordinate charts with $\overline{I \times U} \subset \Omega$.

(b) A function f is in $\mathcal{C}^\infty(\Omega)$ if it is continuous in Ω and for any relatively compact coordinate chart $I \times U$ with $\overline{I \times U} \subset \Omega$, f has continuous partial derivatives of all orders in $I \times U$. This space is equipped with the family of seminorms $N^k_{K,I\times U}$ defined as in (3.6), where k runs over the positive integers, K runs over compact subsets of Ω and $I \times U$ over all relatively compact coordinate charts with $\overline{I \times U} \subset \Omega$.

The spaces $\mathcal{C}^k(\Omega)$ and $\mathcal{C}^\infty(\Omega)$ are complete seminormed spaces.

C. Diffeomorphisms. Let $\Gamma_1\mathsf{M}_1$, $\Gamma_2\mathsf{M}_2$ be two strip complexes. Since these spaces are equipped with a natural topology, the notion of homeomorphism is well defined. Observe that bifurcation manifolds M_v with bifurcation number $\deg(v) = 2$ may be ignored by a homeomorphism. Otherwise, by definition, a homeomorphism must send strips to strips and send any bifurcation manifold with bifurcation number $\deg(v) > 2$ to a bifurcation manifold with the same bifurcation number.

3.12. Definition. Let $\Gamma_1\mathsf{M}_1$, $\Gamma_2\mathsf{M}_2$ be two strip complexes. A homeomorphism $\mathrm{j} : \Gamma_1\mathsf{M}_1 \to \Gamma_2\mathsf{M}_2$ is called a *diffeomorphism* if j and j^{-1} send any bifurcation manifold to a bifurcation manifold and, for any pair of closed strips $S_1 \subset \Gamma_1\mathsf{M}_1$, $S_2 \subset \Gamma_2\mathsf{M}_2$ such that $\mathrm{j}(S_1) = S_2$, the restriction $\mathrm{j}|_{S_1} : S_1 \mapsto S_2$ is a diffeomorphism.

A *local diffeomorphism between* open sets Ω_1, Ω_2 is a map $\mathrm{j} : \Omega_1 \to \Omega_2$ which is a homeomorphism, sends any trace of a bifurcation manifold to a trace of a bifurcation manifold and is a diffeomorphism between traces of closed strips.

3.13. Remarks. (1) Diffeomorphisms must respect the bifurcation structure, even for bifurcation manifolds with bifurcation number $\deg(v) = 2$.

(2) If $\mathrm{j} : \Gamma_1\mathsf{M}_1 \to \Gamma_2\mathsf{M}_2$ is a diffeomorphism then for any $f \in \mathcal{C}^\infty(\Gamma_2\mathsf{M}_2)$, resp. $\mathcal{S}^\infty(\Gamma_2\mathsf{M}_2^o)$, the function $f \circ \mathrm{j}$ is in $\mathcal{C}^\infty(\Gamma_1\mathsf{M}_1)$, resp. $\mathcal{S}^\infty(\Gamma_1\mathsf{M}_1^o)$. If $\mathrm{j} : \Omega_1 \to \Omega_2$ is a local diffeomorphism then for any $f \in \mathcal{C}^\infty(\Omega_2)$, the function $f \circ \mathrm{j}$ is in $\mathcal{C}^\infty(\Omega_1)$. The same holds for functions that are smooth up to order k.

D. Geometric structures on strip complexes. We now introduce a rather specific class of geometric structures on the strip complex $\Gamma\mathsf{M}$. This is done in two stages. The special features of these structures will play a central role in our analysis.

In the first stage, we introduce a product geometric structure on $\Gamma\mathsf{M}$ associated with given geometric structures on Γ and M as follows.

First, we assume that the edge map contains an additional information, namely, the *length* of the edge e. More precisely, we have a map

$$E \to V \times V \times (0,\infty), \quad e \mapsto \bigl(e^-, e^+, l_e\bigr).$$

Thus, with this additional information, the edge $I_e = [e^-, e^+]$ is isometric to the real interval $[0, l_e]$. We can view $\Gamma^1 = (\Gamma^1, l)$ as a metric space in the obvious way. We will always use the letter s to refer to an arc length parameter on Γ^1 or connected pieces of Γ^1. From now on, we always assume that Γ comes equipped with a specific edge length map l.

Second, we assume that (M, g) is a Riemannian manifold with gradient ∇_M. Given these two geometric inputs (length of edges, Riemannian metric on M), we immediately obtain a natural metric on $\Gamma\mathsf{M}$ by equipping each strip $S_e = I_e \times M$ with the Riemannian metric $(ds)^2 + g_x(\cdot, \cdot)$, where $(s, x) \in I_e \times M$.

Here and elsewhere, the subscript x in g_x indicates that g is considered with respect to the x-variable of (s, x).

The second stage of our construction depends on the choice of a function ϕ, positive and strip-wise smooth on $\Gamma^o = \Gamma^o$, that is, $\phi \in \mathcal{S}^\infty(\Gamma^o)$. On each strip $S_e = I_e \times M$, we consider the smooth Riemannian structure

$$\phi_e(s) \cdot \left[(ds)^2 + g_x(\cdot,\cdot)\right] \tag{3.14}$$

obtained from the product structure by multiplication by ϕ. The associated Riemannian measure is $\phi_e(s)^{(1+n)/2}\, ds\, dx$, where dx is the volume element of M (resp. area or length element, according to the dimension of M). This induces our reference measure on $\Gamma\mathsf{M}$ that reflects the underlying geome try, given by

$$\sum_{e \in E} \phi_e(s)^{(1+n)/2} \mathbf{1}_{S_e^o}\, ds\, dx. \tag{3.15}$$

Note that $\Gamma\mathsf{M} \setminus \Gamma\mathsf{M}^o$, the union of all the bifurcation manifolds, is a negligible set. (Below we shall consider a larger class of measures, associated forms and processes.) We are led to the following.

3.16. Definition. Let f, h be functions in $\mathcal{S}^\infty(\Gamma\mathsf{M}^o)$. The gradient ∇f and its length square are given at $(s, x) \in S_e^o$ by

$$\nabla f(s, x) = \frac{1}{\phi_e(s)} \big(\partial_s f_e(s, x), \nabla_M f_e(s, x)\big)$$

and

$$\left|\nabla f(s, x)\right|^2 = \frac{1}{\phi_e(s)} \Big(\left|\partial_s f_e(s, x)\right|^2 + g_x\big(\nabla_M f_e(s, x), \nabla_M f_e(s, x)\big)\Big),$$

that is, $|\nabla f|^2 = \sum_{e \in E} \frac{1}{\phi_e(s)} |\nabla f_e|^2 \mathbf{1}_{S_e^o}$. Correspondingly, the inner product of the gradients at $(s, x) \in \Gamma\mathsf{M}^o$ is

$$(\nabla f, \nabla h)(s, x) = \sum_{e \in E} \frac{1}{\phi_e(s)} \Big(\partial_s f_e(s, x) \partial_s h_e(s, x) + g_x\big(\nabla_M f_e(s, x), \nabla_M h_e(s, x)\big)\Big).$$

Note that these definitions involve the edge length function l, the metric g on M and the function ϕ, but these are omitted in our notation.

Now, if we have a continuous path in $\Gamma\mathsf{M}$ which is rectifiable (i.e., is rectifiable in each strip), then we can compute its length by adding the lengths of the parts of the path within each strip.

3.17. Definition. For any two points $\xi, \zeta \in \Gamma\mathsf{M}$, let $\rho(\xi, \zeta)$ be the infimum of the lengths of continuous rectifiable paths in $\Gamma\mathsf{M}$ joining ξ to ζ.

One easily checks that this defines a distance function on $\Gamma\mathsf{M}$ which defines the original topology of this space. We set

$$B(\xi, r) = \{\zeta \in \Gamma\mathsf{M}:\ \rho(\xi, \zeta) < r\},$$

the open ball with radius r around ξ.

The (easy) proof of the following lemma is left to the reader.

3.18. Lemma. *Assume that (M, g) is a complete Riemannian manifold. Then the metric space $(\Gamma\mathsf{M}, \rho)$ is complete if and only if the metric space (Γ^1, ρ) is complete. This is the case if and only if, for any infinite family $F \subset E$ of edges such that $\bigcup_{e\in F} I_e$ is connected in Γ^1, we have*

$$\sum_{e\in F} \int_{I_e} \sqrt{\phi_e(s)}\, ds = \infty. \tag{3.19}$$

3.20. Definition. Given two strip complexes $\Gamma_1\mathsf{M}_1, \Gamma_2\mathsf{M}_2$, each equipped with respective geometric structures (ϕ_1, ρ_1) and (ϕ_2, ρ_2) as above, we say that a diffeomorphism $\mathrm{j} : \Gamma_1\mathsf{M}_1 \to \Gamma_2\mathsf{M}_2$ is an *isometry* if it satisfies

$$\rho_2\big(\mathrm{j}(\xi), \mathrm{j}(\zeta)\big) = \rho_1(\xi, \zeta) \quad \text{for all } \xi, \zeta \in \Gamma_1\mathsf{M}_1.$$

A *local isometry* between two open sets Ω_1, Ω_2 is defined analogously.

3.21. Remark. If j is an isometry then for any $f \in \mathcal{C}^\infty(\Gamma_2\mathsf{M}_2)$ and any ξ in the interior of a strip in $\Gamma_1\mathsf{M}_1$, we have

$$(\nabla_1 f \circ \mathrm{j}, \nabla_1 h \circ \mathrm{j})_1(\xi) = (\nabla_2 f, \nabla_2 h)_2\big(\mathrm{j}(\xi)\big).$$

Indeed the differential map $d\mathrm{j}|_\xi$ is an isometry between the tangent spaces at ξ and $\mathrm{j}(\xi)$, when $\xi \in \Gamma_1\mathsf{M}_1^o$.

E. Dirichlet forms on $\Gamma\mathsf{M}$. We now equip $\Gamma\mathsf{M}$ with a measure $d\mu$ which will serve as our basic underlying measure to define $\mathcal{L}^p$ spaces on $\Gamma\mathsf{M}$, in particular, $\mathcal{L}^2(\Gamma\mathsf{M}, \mu)$. This measure μ is described by its density $\psi \in \mathcal{S}^\infty(\Gamma^o)$ with respect to the basic measure of (3.15).

3.22. Definition. (a) Given the positive function $\psi \in \mathcal{S}^\infty(\Gamma^o)$, let $\mu = \mu_\psi$ be the positive Radon measure on $\Gamma\mathsf{M}$ such that, for any $f \in \mathcal{C}_c(\Gamma\mathsf{M})$,

$$\begin{aligned}\int_{\Gamma\mathsf{M}} f\, d\mu &= \int_{\Gamma\mathsf{M}} f(s,x)\psi(s)\phi(s)^{(1+n)/2}\, ds\, dx \\ &= \sum_{e\in E} \int_{S_e^o} f_e(s,x)\psi_e(s)\phi_e(s)^{(1+n)/2}\, ds\, dx,\end{aligned}$$

where ds is Lebesgue measure on (Γ^1, l) and dx is the Riemannian measure on (M, g).

(b) For each $\xi \in \Gamma\mathsf{M}$ and $r > 0$ set

$$V(\xi, r) = \mu\big(B(\xi, r)\big) = \sum_{e \in E} \int\limits_{B(\xi,r)\cap S_e^o} \psi_e(s)\phi_e(s)^{(1+n)/2}\, ds\, dx.$$

Above, Lebesgue measure on (Γ^1, l) is of course the measure which restricted to each edge is Lebesgue measure assigning length l_e to I_e, while the vertex set has measure 0.

To construct Dirichlet forms on $\Gamma\mathsf{M}$, we need to recall a version of the classical trace theorem for Sobolev spaces. For any strip S_e, consider the set $\mathcal{W}^1(S_e, \mu) = \mathcal{W}^1(S_e^o, \mu)$ of all functions f in $\mathcal{L}^2(S_e^o, \mu)$ whose distributional first derivatives in S_e^o can be represented by functions in $\mathcal{L}^2(S_e^o, \mu)$. Note that, by definition, $\mathcal{W}^1(S_e, \mu) = \mathcal{W}^1(S_e^o, \mu)$. However choosing S_e or S_e^o makes a difference when considering the local versions of this space since compact subsets of S_e^o and S_e are different. We let $\mathcal{W}^1_{\mathrm{loc}}(S_e, \mu)$ be the space of all functions f in $\mathcal{L}^2_{\mathrm{loc}}(S_e, \mu)$ whose distributional first derivatives in S_e^o can be represented by functions in $\mathcal{L}^2_{\mathrm{loc}}(S_e, \mu)$.

For any f in $\mathcal{W}^1_{\mathrm{loc}}(S_e, \mu)$, using the global coordinates (s, x) on $S_e = I_e \times M$, we have that the derivative $\partial_s f$, the M gradient $\nabla_M f$ and the global gradient ∇f are well defined locally square integrable functions on S_e. In particular, for such functions, the length square and inner product of the gradient(s) are well defined as locally integrable functions in the sense of Definition 3.16.

By the classical trace theorem, those functions admit a trace on each of the copies M_{e^-} and M_{e^+} of M bounding the strip S_e. More precisely, there exist two continuous linear operators

$$\mathrm{Tr}^{S_e}_{M_{e^\pm}} : \mathcal{W}^1_{\mathrm{loc}}(S_e, \mu) \to \mathcal{L}^2_{\mathrm{loc}}(M_{e^\pm}, dx) \tag{3.23}$$

which extend the natural restriction operators defined from $\mathcal{C}^\infty(S_e)$ to $\mathcal{C}^\infty(M_{e^\pm})$.

3.24. Definition. Given Γ, (M, g) and $\phi, \psi \in \mathcal{S}^\infty(\Gamma^o)$, as above, let $\mathcal{W}^1(\Gamma\mathsf{M}, \mu)$ be the space of those functions f in $\mathcal{L}^2(\Gamma\mathsf{M}, \mu)$ whose restrictions f_e, $e \in E$, are all in $\mathcal{W}^1_{\mathrm{loc}}(S_e)$ and satisfy:

- $\int_{\Gamma\mathsf{M}} |\nabla f|^2\, d\mu < \infty$,
- For any vertex v and any two edges $e, e' \in E_v$, $\mathrm{Tr}^{S_e}_{M_v} f = \mathrm{Tr}^{S_{e'}}_{M_v} f$.

3.25. Definition. For $f, h \in \mathcal{W}^1(\Gamma\mathsf{M}, \mu)$, set

$$\mathcal{E}(f, h) = \int\limits_{\Gamma\mathsf{M}} (\nabla f, \nabla h)\, d\mu.$$

Let $\mathcal{W}^1_0(\Gamma\mathsf{M}, \mu)$ be the closure of $\mathcal{C}^\infty_c(\Gamma\mathsf{M})$ in $\mathcal{W}^1(\Gamma\mathsf{M}, \mu)$.

3.26. Example. Let $\Gamma = \mathbb{T} = \mathbb{T}_\mathsf{p}$ be a p-regular tree equipped with an origin o, a reference end ϖ, and the associated horocycle function $\mathfrak{h}$. Edges are oriented away from ϖ so that $\mathfrak{h}(e^+) = \mathfrak{h}(e^-) + 1$. See Fig. 2. Turn $\mathbb{T}$ into a metric tree by giving length $q^{k-1}(q-1)$ to all edges e with $\mathfrak{h}(e^-) = k - 1$. Define $\phi \in \mathcal{S}^\infty(\mathbb{T}^1 \setminus V)$ by $\phi_e(s) = s^{-2}$ on $I_e \cong [q^{k-1}, q^k]$ if $\mathfrak{h}(e^-) = k - 1$. Setting $M = \mathbb{R}$, the corresponding structure on $\Gamma\mathsf{M}$ is isometric to that of the treebolic space $\mathsf{HT}(\mathsf{p}, \mathsf{q})$. Next, for any fixed $\alpha \in \mathbb{R}$, define $\psi \in \mathcal{S}^\infty(\mathbb{T}^1 \setminus V)$ by $\psi_e(s) = s^\alpha$ on $I_e \cong [q^{k-1}, q^k]$ if $\mathfrak{h}(e^-) = k - 1$. Then the corresponding measure μ on $\Gamma\mathsf{M}$ is the measure $\mu_{\alpha,\beta}$ on $\mathsf{HT}(\mathsf{p}, \mathsf{q})$

(modulo the isometry mentioned earlier between these two spaces) and the associated Dirichlet form $(\mathcal{E}, \mathcal{W}^1(\Gamma\mathsf{M}, \mu))$ is the form $(\mathcal{E}_{\alpha,\beta}, \mathcal{W}^1_{\alpha,\beta}(\mathsf{HT}))$ from Definition 2.6.

3.27. Theorem. *The quadratic form $(\mathcal{E}, \mathcal{W}^1(\Gamma\mathsf{M}, \mu))$ is a strictly local Dirichlet form and the quadratic form $(\mathcal{E}, \mathcal{W}^1_0(\Gamma\mathsf{M}, \mu))$ is a strictly local regular Dirichlet form.*

Proof. The Markov character and strict locality of these forms are clear from the definitions. See [20]. The fact that the first form is closed follows from the fact that the corresponding forms on all strips are closed and from the continuity of the trace operators. The fact that the second form is closed and regular is obvious from the definition and the fact that $\mathcal{C}^\infty_c(\Gamma\mathsf{M})$ is dense in $\mathcal{C}_0(\Gamma\mathsf{M})$ for the uniform norm (see Lemma 3.10). □

3.28. Theorem. *Assume that $(\Gamma\mathsf{M}, \rho)$ is a complete metric space (see Lemma 3.18 for a necessary and sufficient condition). Then the forms $(\mathcal{E}, \mathcal{W}^1(\Gamma\mathsf{M}, \mu))$ and $(\mathcal{E}, \mathcal{W}^1_0(\Gamma\mathsf{M}, \mu))$ coincide. In particular, $(\mathcal{E}, \mathcal{W}^1(\Gamma\mathsf{M}, \mu))$ is a strictly local regular Dirichlet form.*

Proof. To prove this, we simply need to show that $\mathcal{C}^\infty_c(\Gamma\mathsf{M})$ is dense in $\mathcal{W}^1(\Gamma\mathsf{M}, \mu)$. First, we show that any $f \in \mathcal{W}^1(\Gamma\mathsf{M}, \mu)$ can be approximated in $\mathcal{W}^1(\Gamma\mathsf{M}, \mu)$ by functions with compact support. Consider the distance function ρ on $\Gamma\mathsf{M}$. Observe that, for any set U, the function $\xi \mapsto \rho(\xi, U)$ is a contraction in each strip S_e. Therefore this function is in $\mathcal{W}^1_{\mathrm{loc}}(\Gamma\mathsf{M})$ with $|\nabla \rho_U| \leqslant 1$. If follows that the functions

$$\theta_n = \max\{1 - \rho(\cdot, B(o,n))/n, 0\},$$

where o is a fixed point in $\Gamma\mathsf{M}$, are in $\mathcal{W}^1(\Gamma\mathsf{M}, \mu)$ and satisfy $|\nabla\theta_n| \leqslant 1/n$. The function θ_n is supported in $B(o, 2n)$, which is precompact since $(\Gamma\mathsf{M}, \rho)$ is a complete locally compact metric space. This yields that the compactly supported functions $\theta_n f$ converge to f in $\mathcal{W}^1(\Gamma\mathsf{M}, \mu)$.

Next, we show that any compactly supported function f in $\mathcal{W}^1(\Gamma\mathsf{M}, \mu)$ can be approximated in $\mathcal{W}^1(\Gamma\mathsf{M}, \mu)$ by compactly supported functions in $\mathcal{C}^\infty_c(\Gamma\mathsf{M})$. By compactness of the support of f, we can find a finite collection of functions ω_i in $\mathcal{C}^\infty_c(\Gamma\mathsf{M})$ such that $\sum \omega_i = 1$ on the support of f and each ω_i either has its compact support in an open strip $(e^-, e^+) \times M$ or ω_i has its compact support in a star of strips X_v at vertex v. At this point, it suffices to approximate each $f\omega_i$ by functions in $\mathcal{C}^\infty_c(\Gamma\mathsf{M})$. If ω_i has compact support within one open strip, this follows from a classical procedure.

The interesting case is when ω_i is compactly supported in a star X_v. In this case we can assume that the support of ω_i is so small that it is contained in an open set of the form $\bigcup_{e\in E_v} U_e$, where the U_e meet on M_v along an open set $U_v = \{v\} \times U \subset M_v$ and each U_e is of the form $J_e \times U$ where the $J_e \subset I_e$ are semi-open intervals of the same length all containing v.

Now pick one of the edges $\tilde{e} \in E_v$, and let $\tilde{f}$ be the function which, on each U_e, equals $f\omega_i|_{U_{\tilde{e}}}$, and is zero outside of $\bigcup_{e\in E_v} U_e$. That is, we copy the values of $f\omega_i$ from $U_{\tilde{e}}$ to all the other U_e, $e \in E_v$, via the obvious coordinate-wise correspondences between those sets, taking into account the identification between $U_{\tilde{e}}$ and the other sets U_e along U_v. On each strip S^o_e, the function $f\omega_i - \tilde{f}$ is in $\mathcal{W}^1_0(S^o_e, \mu)$ because, by construction, the functions $f\omega_i$ and $\tilde{f}$ coincide on $U_v \subset M_v$. Hence we can approximate $f\omega_i - \tilde{f}$ in $\mathcal{W}^1(\Gamma\mathsf{M}, \mu)$ by functions g_n whose restrictions to each S^o_e, $e \in E_v$, are smooth and compactly supported in the respective set U_e. Those g_n are in $\mathcal{C}^\infty_{c,c}(\Gamma\mathsf{M})$.

Next, the function $f\omega_i|_{U_{\tilde e}}$ is in $\mathcal{W}^1(S_{\tilde e},\mu)$ with compact support in $U_{\tilde e}$. Recall that $U_{\tilde e}$ contains U_v as part of its boundary. By classical constructions, $f\omega_i|_{U_{\tilde e}}$ can be approximated by functions $h_n \in \mathcal{C}_c^\infty(U_{\tilde e})$. We now use h_n to define $\tilde h_n$ on $\bigcup_{e\in E_v} U_e$ by setting, for each $e \in E_v$, $\tilde h_n|_{U_e} = h_n$ in the same way as above via the natural correspondence between U_e and $U_{\tilde e}$. Obviously, $\tilde h_n \in \mathcal{C}_c^\infty(\Gamma\mathsf{M})$ and it approximates $\tilde f$ in $\mathcal{W}^1(\Gamma\mathsf{M},\mu)$. This implies that $g_n + \tilde h_n$, which is in $\mathcal{C}_c^\infty(\Gamma\mathsf{M})$, approximates $f\omega_i$ in $\mathcal{W}^1(\Gamma\mathsf{M},\mu)$. □

In fact, the smaller space $\mathcal{C}_{c,c}^\infty(\Gamma\mathsf{M})$ is already dense in $\mathcal{W}_0^1(\Gamma\mathsf{M},\mu)$, and thus in $\mathcal{W}^1(\Gamma\mathsf{M},\mu)$ when $(\Gamma\mathsf{M},\rho)$ is complete. Recall that $\mathcal{C}_{c,c}^\infty(\Gamma\mathsf{M})$ is the set of those functions in $\mathcal{C}_c^\infty(\Gamma\mathsf{M})$ such that in any strip $S_e = I_e \times M$, the partial derivative $\partial_s f|_{S_e}$ has compact support contained in the open strip S_e^o (as usual, s is the variable in the interval I_e).

3.29. Theorem. *The subspace $\mathcal{C}_{c,c}^\infty(\Gamma\mathsf{M})$ of $\mathcal{W}_0^1(\Gamma\mathsf{M},\mu)$ is dense in $\mathcal{W}_0^1(\Gamma\mathsf{M},\mu)$, and thus in $\mathcal{W}^1(\Gamma\mathsf{M},\mu)$, when $(\Gamma\mathsf{M},\rho)$ is complete.*

Proof. To see that this is the case, we return to the end of the argument in the proof of Theorem 3.28. We claim that we can approximate $f\omega_i|_{U_{\tilde e}} \in \mathcal{W}^1(S_{\tilde e},\mu)$ by functions $h_n \in \mathcal{C}_{c,c}^\infty(U_{\tilde e})$. If that is the case, we use h_n to define $\tilde h_n$ on $\bigcup_{e\in E_v} U_e$ by copying the values of $\tilde h_n$ from $U_{\tilde e}$ to U_e for each $e \in E_v$. Obviously, $\tilde h_n \in \mathcal{C}_{c,c}^\infty(\Gamma\mathsf{M})$ and it approximates $\tilde f$ in $\mathcal{W}^1(\Gamma\mathsf{M},\mu)$. Then, as before, $g_n + \tilde h_n$, approximates $f\omega_i$ in $\mathcal{W}^1(\Gamma\mathsf{M},\mu)$ as desired. The function $g_n + \tilde h_n$ is in $\mathcal{C}_{c,c}^\infty(\Gamma\mathsf{M})$ because $\tilde h_n$ is in that space by construction and g_n has compact support in the union $\bigcup_{e\in E_v} S_e^o$ of the open strips surrounding M_v and thus is also in $\mathcal{C}_{c,c}^\infty(\Gamma\mathsf{M})$.

Thus, the only thing left to prove is that a function $f \in \mathcal{W}^1(R)$, $R = [e^-, e^+) \times U$, with compact support in R can be approximated in $\mathcal{W}^1(R)$ by a sequence of functions $h_n \in \mathcal{C}^\infty(R)$ with compact support in R and such that $\partial_s h_n$ has compact support in $I_e^o \times U$. Note that, by definition, R contains the bottom $\{e^-\} \times U$.

Since this is a local problem, we can regard U as a small open set in $\mathbb{R}^n$ that contains the origin, and ignore completely the role of the functions ϕ, ψ. Modifying notation in this sense, we use coordinates $(s,x) \in R = [0,l) \times U$ (instead of $s \in [e^-, e^+)$, with $l = l_e$) and write $d\mu = ds\,dx$. For $n = 1, 2, \ldots$, set

$$f_n(s,x) = n \int_s^{s+1/n} f(\tau,x)\,d\tau \quad \text{and} \quad \tilde f_n(s,x) = \begin{cases} f_n(s,x), & \text{if } s \in (1/n, l) \\ f_n(1/n,x), & \text{if } s \in [0,1/n]. \end{cases}$$

We can assume that the support of f in R is small enough so that f_n and $\tilde f_n$ are still supported in R. It is plain that f_n tends to f in $\mathcal{W}^1(R)$, and we claim that the same is true for $\tilde f_n$. It is clear that $\tilde f_n$ tends to f in $\mathcal{L}^2(R)$ and we only need to check that $|\nabla(f_n - \tilde f_n)|$ tends to 0 in $\mathcal{L}^2(R)$. Setting $R_n = [0,1/n] \times U$, we write

$$\begin{aligned}\int_R \left|\nabla(\tilde f_n - f_n)\right|^2 d\mu &= \int_{R_n} \left(|\partial_s f_n|^2 + \left|\nabla_M(\tilde f_n - f_n)\right|^2\right) d\mu \\ &\leqslant C \int_{R_n} \left(|\partial_s f_n|^2 + |\nabla_M f_n|^2\right) d\mu + \frac{C}{n} \int_U \left|\nabla_M \tilde f_n(1/n,x)\right|^2 dx.\end{aligned}$$

It is plain that

$$\int_{R_n} \left(|\partial_s f_n|^2 + |\nabla_M f_n|^2\right) d\mu \to 0.$$

Moreover

$$\frac{1}{n}\int_U \left|\nabla_M \tilde{f}_n(1/n, x)\right|^2 dx \leqslant \frac{1}{n}\int_U \left| n \int_{1/n}^{2/n} \left|\nabla_M f(s,x)\right| ds \right|^2 dx$$
$$\leqslant \int_{1/n}^{2/n} \int_U \left|\nabla_M f(s,x)\right|^2 ds\, dx \to 0.$$

The functions $\tilde{f}_n$ satisfy $\partial_s \tilde{f}_n = 0$ in $[0, 1/n) \times U$ but are not smooth. To obtain smooth functions approximating f with the desired property, extend f and $\tilde{f}_n$ by symmetry to $R^* = (-l, l) \times U$, that is, $f(-s, x) = f(s, x)$ and $\tilde{f}_n(-s, x) = \tilde{f}_n(s, x)$. Obviously, $\|\tilde{f}_n - f\|_{\mathcal{W}^1(R^*)} \to 0$. For each n, let θ_n be a smooth non-negative function with integral 1 and support in the ball of radius less than $1/(5n)$ around $(0, 0)$ in $(-l, l) \times U$. Consider $h_n = \theta_n * \tilde{f}_n$ ($*$ denoting convolution). Now, the restriction of h_n to $[0, l) \times U$ is a smooth function which satisfies $\partial_s h_n = 0$ in a neighbourhood of $\{0\} \times U$ and approximates f in $\mathcal{W}^1(R)$. Indeed, $\theta_n * f \to f$ in $\mathcal{W}^1(R^*)$, and $\|\theta_n * (\tilde{f}_n - f)\|_{\mathcal{W}^1(R^*)} \leqslant \|\tilde{f}_n - f\|_{\mathcal{W}^1(R^*)} \to 0$. □

The Dirichlet form structure on a strip complex $\Gamma\mathsf{M}$ is based on the choice of

(a) the geometry determined by l, ϕ, and
(b) the measure μ determined by ψ.

The following definition takes this into account to introduce isometries that are compatible with this additional structure.

3.30. Definition. Let $\Gamma_1\mathsf{M}_1$ and $\Gamma_2\mathsf{M}_2$ be two strip complexes equipped respectively with ϕ_i, ψ_i and the associated measures μ_i, $i = 1, 2$ as above. An isometry (or local isometry, with obvious modifications) $\mathrm{j} : \Gamma_1\mathsf{M}_1 \to \Gamma_2\mathsf{M}_2$ is called *measure-adapted* if there is a positive constant $c(\mathrm{j})$ such that, for any compact set $A \subset \Gamma_2\mathsf{M}_2$,

$$\mu_1\left(\mathrm{j}^{-1}(A)\right) = c(\mathrm{j})\mu_2(A).$$

3.31. Remark. If j is a measure-adapted isometry and $f_1 = f_2 \circ \mathrm{j}$, where $f_2 \in \mathcal{W}^1(\Gamma_2\mathsf{M}_2)$, then $f_1 \in \mathcal{W}^1(\Gamma_1\mathsf{M}_1)$ and

$$\mathcal{E}_1(f_1, f_1) = c(\mathrm{j})\mathcal{E}_2(f_2, f_2).$$

3.32. Example. For any $\mathsf{p} \in \{1, 2 \ldots\}$ and $\mathsf{q} > 1$, the treebolic space $\mathsf{HT}(\mathsf{p}, \mathsf{q})$ (equipped with its stripwise hyperbolic geometry, as described in Section 2) admits a large group of isometries (see the introduction). These isometries are all measure-adapted whenever $\mathsf{HT}(\mathsf{p},\mathsf{q})$ is equipped with any of the measures $\mu_{\alpha,\beta}$ ($\alpha \in \mathbb{R}$, $\beta > 0$) defined in (2.3).

F. The Laplacian and the heat equation on strip complexes. Consider a strip complex $\Gamma\mathsf{M}$ where (M,g) is a Riemannian manifold, and equip $\Gamma\mathsf{M}$ with the data (l,ϕ,ψ), where $\phi,\psi \in \mathcal{S}^\infty(\Gamma^o)$ as in §3.D and §3.E. Let μ be the associated measure. For simplicity, we write $\mathcal{L}^p(\Gamma\mathsf{M}) = \mathcal{L}^p(\Gamma\mathsf{M},\mu)$, $\mathcal{W}_0^1(\Gamma\mathsf{M}) = \mathcal{W}_0^1(\Gamma\mathsf{M},\mu)$ and $\mathcal{W}^1(\Gamma\mathsf{M}) = \mathcal{W}^1(\Gamma\mathsf{M},\mu)$.

By the general theory of Dirichlet forms, there is a self-adjoint operator

$$\big(\Delta, \mathrm{Dom}(\Delta)\big)$$

on $\mathcal{L}^2(\Gamma\mathsf{M})$ which we call the Laplacian on $\Gamma\mathsf{M}$ and which is defined as follows.

3.33. Definition. Set

$$\mathrm{Dom}(\Delta) = \big\{ f \in \mathcal{W}_0^1(\Gamma\mathsf{M})\text{: there is } C_f \text{ such that } \mathcal{E}(f,h) \leqslant C_f\|h\|_2 \text{ for all } h \in \mathcal{W}_0^1(\Gamma\mathsf{M})\big\}.$$

For $f \in \mathrm{Dom}(\Delta)$, there exists a unique $u \in \mathcal{L}^2(\Gamma\mathsf{M})$ such that $\mathcal{E}(f,h) = -\int uh\,d\mu$ for all $h \in \mathcal{L}^2(\Gamma\mathsf{M})$ and we set

$$\Delta f = u.$$

Since the measure μ will be fixed most of the time, we will often omit it in our notation. The operator Δ with domain $\mathrm{Dom}(\Delta)$ is the infinitesimal generator of a strongly continuous semigroup of self-adjoint contractions $\{H_t = e^{t\Delta}\colon t \geqslant 0\}$, on $\mathcal{L}^2(\Gamma\mathsf{M})$ which has the Markov property:

$$f \in \mathcal{L}^2(\Gamma\mathsf{M}), \quad 0 \leqslant f \leqslant 1 \quad \Longrightarrow \quad 0 \leqslant H_t f \leqslant 1.$$

It follows that H_t extends to a contraction on each space $\mathcal{L}^p(\Gamma\mathsf{M})$, $1 \leqslant p \leqslant \infty$. For $1 \leqslant p < \infty$, the family $\{H_t\colon t \geqslant 0\}$ is a strongly continuous semigroup on $\mathcal{L}^p(\Gamma\mathsf{M})$. We call $\{H_t\colon t > 0\}$ the *heat semigroup* on $\Gamma\mathsf{M}$ (more precisely, on $(\Gamma\mathsf{M}; l, \phi, \psi)$).

The following is immediate by inspection.

3.34. Proposition. *Let $\Gamma_1\mathsf{M}_1$ and $\Gamma_2\mathsf{M}_2$ be two strip complexes, each equipped with data l_i, ϕ_i, ψ_i $(i = 1,2)$ as above. Let μ_i and $(\Delta_i, \mathrm{Dom}(\Delta_i))$, $i = 1,2$, be the associated measures and Laplacians. If $\mathsf{j} : \Gamma_1\mathsf{M}_1 \to \Gamma_2\mathsf{M}_2$ is a measure-adapted isometry then*

$$\textit{for all } f_2 \in \mathrm{Dom}(\Delta_2), \quad f_1 = f_2 \circ \mathsf{j} \in \mathrm{Dom}(\Delta_1) \ \textit{ and } \ \Delta_1 f_1 = (\Delta_2 f_2) \circ \mathsf{j}.$$

Also,

$$\textit{for all } t > 0 \ \textit{ and } \ f_2 \in \mathcal{L}^2(\Gamma_2\mathsf{M}_2), \qquad f_1 = f_2 \circ \mathsf{j} \in \mathcal{L}^2(\Gamma_1\mathsf{M}_1) \ \textit{ and } \ H_{1,t} f_1 = (H_{2,t} f_2) \circ \mathsf{j}.$$

In the general theory of regular strictly local Dirichlet forms $(\mathcal{E}, \mathrm{Dom}(\mathcal{E}))$, one introduces a notion of intrinsic distance. In the present setting, this definition reads

$$\tilde{\rho}(x,y) = \sup\big\{\big|f(x) - f(y)\big|\colon\ f \in \mathcal{C}(\Gamma\mathsf{M}) \cap \mathcal{W}^1(\Gamma\mathsf{M}),\ |\nabla f| \leqslant 1\big\}.$$

It is not hard to see that this intrinsic distance coincides with the distance ρ introduced earlier.

3.35. Definition. Let Ω be an open set in $\Gamma\mathsf{M}$. Set

$$\mathcal{W}_c^1(\Omega) = \{ f \in \mathcal{W}^1(\Gamma\mathsf{M}): \ f \text{ is compactly supported in } \Omega \}$$

and

$$\mathcal{W}_{\mathrm{loc}}^1(\Omega) = \left\{ f \in \mathcal{L}_{\mathrm{loc}}^2(\Omega): \ \begin{array}{l} \text{for every compact } K \subset \Omega \text{ there is} \\ \tilde{f} \in \mathcal{W}^1(\Gamma\mathsf{M}) \text{ such that } f|_K = \tilde{f}|_K \text{ a.e.} \end{array} \right\}.$$

Fix an open set Ω and consider the topological vector spaces $\mathcal{W}_c^1(\Omega) \subset \mathcal{W}_0^1(\Gamma\mathsf{M}) \subset \mathcal{L}^2(\Gamma\mathsf{M})$ and their duals $\mathcal{L}^2(\Gamma\mathsf{M}) \subset \mathcal{W}_0^1(\Gamma\mathsf{M})^* \subset \mathcal{W}_c^1(\Omega)^*$.

3.36. Definition. Let Ω be an open set in $\Gamma\mathsf{M}$. Let $f \in \mathcal{W}_c^1(\Omega)^*$. We say that a function u is a weak solution of the equation $\Delta u = f$ in Ω if

- $u \in \mathcal{W}_{\mathrm{loc}}^1(\Omega)$, and
- $\mathcal{E}(u, h) = -f(h)$ for all $h \in \mathcal{W}_c^1(\Omega)$.

Observe that $f(h)$ above is well defined since $f \in \mathcal{W}_c^1(\Omega)^*$ and $h \in \mathcal{W}_c^1(\Omega)$. Observe also that if f is represented by a locally integrable function on Ω (again called f) and if u is such that there exists $\tilde{u} \in \mathrm{Dom}(\Delta)$ satisfying $u = \tilde{u}|_\Omega$ then u is a weak solution of $\Delta u = f$ in Ω if and only if $(\Delta \tilde{u})|_\Omega = f$.

Given a Hilbert space H and an interval I, let $\mathcal{L}^2(I \to H)$ be the Hilbert space of those functions $f : I \to H$ such that

$$\|f\|_{\mathcal{L}^2(I\to H)} = \left(\int_I \|f(t)\|_H^2 \, dt \right)^{1/2} < \infty.$$

Let $\mathcal{W}^1(I \to H) \subset \mathcal{L}^2(I \to H)$ be the Hilbert space of those functions $f : I \to H$ in $\mathcal{L}^2(I \to H)$ whose distributional time derivative f' can be represented by functions in $\mathcal{L}^2(I \to H)$, equipped with the norm

$$\|f\|_{\mathcal{W}^1(I\to H)} = \left(\int_I \left(\|f(t)\|_H^2 + \|f'(t)\|_H^2 \right) dt \right)^{1/2} < \infty.$$

Given an open time interval I, set

$$\mathcal{F}(I \times \Gamma\mathsf{M}) = \mathcal{L}^2\big(I \to \mathcal{W}_0^1(\Gamma\mathsf{M})\big) \cap \mathcal{W}^1\big(I \to \mathcal{W}_0^1(\Gamma\mathsf{M})^*\big).$$

This notation is justified by the inclusions $\mathcal{W}^1(\Gamma\mathsf{M}) \subset \mathcal{L}^2(\Gamma\mathsf{M}) = \mathcal{L}^2(\Gamma\mathsf{M})^* \subset \mathcal{W}^1(\Gamma\mathsf{M})^*$, compare with [33–35]. While in these definitions it was convenient to consider $f(t)$ as a function on $\Gamma\mathsf{M}$ for each $t \in I$, we shall usually prefer the notation $f(t, \cdot)$, where we think of f as a function on $I \times \Gamma\mathsf{M}$.

Given an open time interval I and an open set $\Omega \subset \Gamma\mathsf{M}$ (both nonempty), let

$$\mathcal{F}_{\mathrm{loc}}(I \times \Omega) \tag{3.37}$$

be the set of all functions $f : I \times \Omega \to \mathbb{R}$ such that, for any open interval $I' \subset I$ relatively compact in I and any open subset Ω' relatively compact in U there exists a function $f^{\#} \in \mathcal{F}(I \times \Gamma\mathsf{M})$ satisfying $f = f^{\#}$ a.e. in $I' \times \Omega'$. Finally, let

$$\mathcal{F}_c(I \times \Omega) = \{f \in \mathcal{F}(I \times \Gamma\mathsf{M}):\ f(t, \cdot) \text{ has compact support in } \Omega \text{ for a.e. } t \in I\}.$$

3.38. Definition. Let I be an open time interval. Let Ω be an open subset in $\Gamma\mathsf{M}$ and set $Q = I \times \Omega$. A function $u : Q \mapsto \mathbb{R}$ is a *weak (local) solution of the heat equation* $(\partial_t - \Delta)u = 0$ in Q if

(1) $u \in \mathcal{F}_{\mathrm{loc}}(Q)$, and
(2) for any open interval J relatively compact in I and any $f \in \mathcal{F}_c(Q)$,

$$\int_J \int_U f\, \partial_t u\, d\mu\, dt + \int_J \mathcal{E}\big(f(t, \cdot), u(t, \cdot)\big)\, dt = 0.$$

The following proposition follows from the relevant definitions by inspection.

3.39. Proposition. *Let $\Gamma_1\mathsf{M}_1$ and $\Gamma_2\mathsf{M}_2$ be two strip complexes, each equipped with data ϕ_i, ψ_i, $i = 1, 2$, as above. Let μ_i and $(\Delta_i, \mathrm{Dom}(\Delta_i))$, $i = 1, 2$, be the associated measures and Laplacians. Let $\mathrm{j} : \Omega_1 \to \Omega_2$, where $\Omega_1 \subset \Gamma_1\mathsf{M}_1$ and $\Omega_2 \subset \Gamma_2\mathsf{M}_2$, be a measure-adapted local isometry between the open sets Ω_1 and Ω_2.*

- *If $f_2 \in \mathcal{W}_c^1(\Omega_2)^*$ and u_2 is a weak solution of $\Delta_2 u = f_2$ in Ω_2 then $f_1(h) = f_2(h \circ \mathrm{j}^{-1}) \in \mathcal{W}_c^1(\Omega_1)^*$ and $u_1 = u_2 \circ \mathrm{j}$ is a weak solution of $\Delta_1 u_1 = f_1$.*
- *For any time interval I, if u_2 is a weak solution of the heat equation on $\Gamma_2\mathsf{M}_2$ in $Q_2 = I \times \Omega_2$, then $u_1 = u_2 \circ \mathrm{j}$ is a weak solution of the heat equation on $\Gamma_1\mathsf{M}_1$ in $Q_1 = I \times \Omega_1$.*

4. Basic properties of the heat semigroup

In this and the next section, $\Gamma\mathsf{M}$ is a fixed strip complex based on a graph Γ and a Riemannian manifold (M, g). Furthermore, $\Gamma\mathsf{M}$ is equipped with data (l, ϕ, ψ), where $\phi, \psi \in \mathcal{S}^\infty(\Gamma^o)$, the associated distance ρ and measure μ, the Dirichlet form $(\mathcal{E}, \mathcal{W}_0^1(\Gamma\mathsf{M}))$ and the corresponding Laplacian Δ and heat semigroup $\{H_t = e^{t\Delta}:\ t \geqslant 0\}$. See §3.D–§3.F.

Because of the singular nature of strip complexes, the local regularity properties of weak solutions of the Laplace or heat equations are a non-trivial and crucial issue.

4.1. Theorem. *For any compact set $K \subset \Gamma\mathsf{M}$, there exist $r_K > 0$ and constants D_K, P_K such that for all $\xi \in K$, $r \in (0, r_K)$ the following properties hold.*

- *$V(\xi, r) \leqslant D_K V(\xi, 2r)$, and*

- *for every* $f \in \mathcal{W}^1(B)$,

$$\int_B |f - f_B|^2 \, d\mu \leqslant P_K r^2 \int_B |\nabla f|^2 \, d\mu,$$

where $B = B(\xi, r)$ *and* $f_B = \frac{1}{\mu(B)} \int_B f \, d\mu$.

Proof. The first property is clear by inspection because of the continuity of ϕ, ψ, the Riemannian nature of M, and the fact that the underlying graph (V, E) is locally finite. The second property, i.e., the Poincaré inequality, is more delicate to prove. First, such a (localized) Poincaré inequality holds on M (i.e., on any Riemannian manifold). See, e.g., [28, 5.6.3]. This applies to any strip S_e equipped with the ϕ-structure. By continuity and positivity of ψ, the desired local Poincaré inequality holds on balls that are contained in the interior of a strip.

The same is true when we have a vertex v with $\deg(v) = 1$, so that M_v sits at the boundary of a unique strip S_e, and B is contained in the half-open strip $S_e^o \cup M_v$.

By classical arguments, it thus suffices to prove the stated result assuming that the center ξ belongs to a bifurcation manifold M_v, where $\deg(v) \geqslant 2$. We can further assume that r is small enough so that the ball $B = B(\xi, r)$ is contained in X_v, the star of strips around v.

The crucial observation is that for any pair of edges $e, e' \in E_v$, the open set $X_v^{e,e'} = M_v \cup S_e^o \cup S_{e'}^o$ equipped with the ϕ-structure is locally bi-Lipschitz equivalent to a smooth Riemannian manifold $I \times M$, where the interval I corresponds to $\{v\} \cup I_e^o \cup I_{e'}^o$.

Therefore, setting $B = B(\xi, r)$ and $B_{e,e'} = B \cap X_v^{e,e'}$ the following Poincaré inequalities hold:

$$\int_{B_{e,e'}} |f - f_{B_{e,e'}}|^2 \, d\mu \leqslant C_K r^2 \int_{B_{e,e'}} |\nabla f|^2 \, d\mu \quad \text{for all } f \in \mathcal{W}^1(B),\ e, e' \in E_v,$$

$$\text{where } f_{B_{e,e'}} = \frac{1}{\mu(B_{e,e'})} \int_{B_{e,e'}} f \, d\mu.$$

Now choose and fix an edge $e \in E_v$ so that $\mu(B_e)$ is maximal among all edges in E_v, where $B_e = B \cap S_e^o$. We set $f_{B_e} = \frac{1}{\mu(B_e)} \int_{B_e} f \, d\mu$. Then

$$\max_{e' \in E_v} \mu(B_{e,e'}) \leqslant 2\mu(B_e).$$

Then we can estimate

$$\begin{aligned}
|f_{B_e} - f_{B_{e,e'}}| &= \left| \frac{1}{\mu(B_e)\mu(B_{e,e'})} \int_{B_e} \int_{B_{e,e'}} \big(f(\eta) - f(\zeta)\big) \, d\mu(\eta) \, d\mu(\zeta) \right| \\
&\leqslant \frac{2}{\mu(B_{e,e'})^2} \int_{B_{e,e'}} \int_{B_{e,e'}} \big|f(\eta) - f(\zeta)\big| \, d\mu(\eta) \, d\mu(\zeta) \\
&\leqslant \left(\frac{4}{\mu(B_{e,e'})^2} \int_{B_{e,e'}} \int_{B_{e,e'}} \big|\big(f(\eta) - f_{B_{e,e'}}\big) - \big(f(\zeta) - f_{B_{e,e'}}\big)\big|^2 \, d\mu(\eta) \, d\mu(\zeta) \right)^{1/2}
\end{aligned}$$

$$\leqslant \left(\frac{8}{\mu(B_{e,e'})} \int_{B_{e,e'}} |f - f_{B_{e,e'}}|^2 \, d\mu \right)^{1/2}$$
$$\leqslant \left(\frac{8C_K r^2}{\mu(B_{e,e'})} \int_{B_{e,e'}} |\nabla f|^2 \, d\mu \right)^{1/2}.$$

In the last inequality, we have used the Poincaré inequality on $B_{e,e'}$. Next,

$$\int_B |f - f_B|^2 \, d\mu = \min_{c \in \mathbb{R}} \int_B |f - c|^2 \, d\mu \leqslant \int_B |f - f_{B_e}|^2 \, d\mu,$$

and

$$\begin{aligned} \int_B |f - f_{B_e}|^2 \, d\mu &\leqslant \sum_{e' \in E_v \setminus \{e\}} \int_{B_{e,e'}} |f - f_{B_e}|^2 \, d\mu \\ &\leqslant 2 \sum_{e' \in E_v \setminus \{e\}} \left(\int_{B_{e,e'}} |f - f_{B_{e,e'}}|^2 \, d\mu + \mu(B_{e,e'}) |f_{B_{e,e'}} - f_{B_e}|^2 \right) \\ &\leqslant 18 C_K r^2 \sum_{e' \in E_v \setminus \{e\}} \int_{B_{e,e'}} |\nabla f|^2 \, d\mu \\ &\leqslant 18 C_K r^2 \big(\deg(v) - 1\big) \int_B |\nabla f|^2 \, d\mu. \end{aligned}$$

This is the desired Poincaré inequality when $\xi \in K \cap M_v$. From this, for all $\xi \in K$ and $r \in (0, r_K)$, elementary considerations give that for all $f \in \mathcal{W}^1(2B)$,

$$\int_B |f - f_B|^2 \, d\mu \leqslant P_K r^2 \int_{2B} |\nabla f|^2 \, d\mu,$$

where $2B = B(\xi, 2r)$. Now, it is well known (but not so elementary) that this suffices to obtain the desired Poincaré inequality where $f \in \mathcal{W}^1(2B)$ and $\int_{2B} |\nabla f|^2 d\,\mu$ are replaced by $f \in \mathcal{W}^1(B)$ and $\int_B |\nabla f|^2 \, d\mu$. See [28, 5.3]. Compare also with [17] and [27]. $\square$

Theorem 4.1 has far reaching consequences. The next three theorems follow from the arguments of [33–35], which are based on Moser iteration techniques and thus, in the present situation, rely heavily on Theorem 4.1. See also [28] and Biroli and Mosco [7].

4.2. Theorem. *Referring to the general setting of this section, the heat semigroup has the following properties.*

- *For any open interval I and compact intervals J, J' of I with $\max J < \min J'$ and for any connected open set $\Omega \subset \Gamma\mathsf{M}$ and compact $K \subset \Omega$, there are positive constants*

$\alpha_1 = \alpha_1(I, \Omega, J, K)$, $C_1 = C_1(I, \Omega, J, K)$ and $C_2 = C_2(I, \Omega, J, J', K)$ such that any weak solution u of the heat equation $(\partial_t - \Delta)u = 0$ in $I \times \Omega$ admits a continuous version which satisfies

$$\sup\left\{\frac{|u(t,\xi) - u(s,\zeta)|}{(|t-s|^{1/2} + \rho(\xi,\zeta))^{\alpha_1}}:\ (t,\xi), (s,\zeta) \in J \times K\right\} \leqslant C_1 \sup_{I\times\Omega} |u|,$$

and, if u is non-negative,

$$\sup_{J\times K} u \leqslant C_2 \inf_{J'\times K} u.$$

- *The heat diffusion semigroup $\{H_t = e^{t\Delta}:\ t > 0\}$ admits a continuous kernel $(t,\xi,\zeta) \mapsto h(t,\xi,\zeta)$—which we call the* heat kernel *of $\Gamma\mathsf{M}$—so that*

$$H_t f(\xi) = \int_{\Gamma\mathsf{M}} h(t,\xi,\zeta) f(\zeta)\, d\mu(\zeta).$$

  The heat kernel is symmetric in ξ, ζ.
- *For each $\xi, \zeta \in \Gamma\mathsf{M}$, the function $t \mapsto h(t,\xi,\zeta)$ is in $\mathcal{C}^\infty((0,\infty))$, and for each $\zeta \in \Gamma\mathsf{M}$, the function $(t,\xi) \mapsto \partial_t^k h(t,\xi,\zeta)$ is a weak solution of the heat equation in $(0,\infty) \times \Gamma\mathsf{M}$. Moreover, $(t,\xi,\zeta) \mapsto \partial_t^k h(t,\xi,\zeta)$ is a continuous function on $(0,\infty) \times \Gamma\mathsf{M} \times \Gamma\mathsf{M}$.*
- *For any fixed compact $K \subset \Gamma\mathsf{M}$, $\zeta_0 \in K$, compact time interval $I = [a,b] \subset (0,\infty)$ and integer k, there are positive constants $\alpha_2 = \alpha_2(I,K,k)$ and $C_3 = C_3(I,K,k)$ such that, for all $\xi \in \Gamma\mathsf{M}$, we have*

$$\sup\left\{\left|\partial_t^k h(t,\zeta,\xi):\ t \in I,\ \zeta \in K\right|\right\} \leqslant C_3 h(2b,\zeta_0,\xi)$$

  and

$$\sup\left\{\frac{|\partial_t^k h(t,\zeta,\xi) - \partial_t^k h(t,\zeta',\xi)|}{\rho(\zeta',\zeta)^{\alpha_2}}:\ t \in I,\ \zeta,\zeta' \in K\right\} \leqslant C_3 h(2b,\zeta_0,\xi).$$

- *Each operator H_t, $t > 0$, sends bounded measurable functions to continuous bounded functions, that is, $H_t\mathcal{L}^\infty(\Gamma\mathsf{M}) \subset \mathcal{C}_b(\Gamma\mathsf{M})$ for any $t > 0$.*

Note that no global results can be obtained under the present very general hypotheses. In particular, we have no bound on the volume of large balls, and stochastic completeness is not guaranteed. That is, it may very well occur that $\int h(t,\xi,\zeta)\, d\mu(\zeta) < 1$ for some t, ξ. Indeed, we have so far not even assumed the completeness of $(\Gamma\mathsf{M}, \rho)$, but will do so next.

4.3. Theorem. *Assume that $(\Gamma\mathsf{M}, \rho)$ is complete and that*

$$\int_1^\infty \frac{r\, dr}{\ln V(\xi_0, r)} = \infty.$$

Then unique solvability of the bounded Cauchy problem holds on $(0,T)\times\Gamma\mathsf{M}$ *for the heat equation. More precisely, if* $u:(0,T)\times\Gamma\mathsf{M}$ *is a weak solution of the heat equation on* $(0,T)\times\Gamma\mathsf{M}$ *which is bounded and satisfies* $\lim_{t\to 0}u(t,\xi)=0$ μ*-almost everywhere, then* $u=0$ *on* $(0,T)\times\Gamma\mathsf{M}$*. In particular, the semigroup* $\{H_t=e^{t\Delta}\colon t>0\}$*, is conservative, that is,* $e^{t\Delta}\mathbf{1}=\mathbf{1}$.

In the next theorem, we also assume that $(\Gamma\mathsf{M},\rho)$ is complete, and make uniform local assumptions on the geometry of $\Gamma\mathsf{M}$ that allow us to obtain more quantitative results.

4.4. Theorem. *Assume that* $(\Gamma\mathsf{M},\rho)$ *is complete and that there are constants* $D, P, r_0>0$ *such that*

(i) *for any* $\xi\in\Gamma\mathsf{M}$ *and* $r\in(0,r_0)$*, we have the doubling property* $V(\xi,r)\leqslant DV(\xi,2r)$*, and*
(ii) *for any* $\xi\in\Gamma\mathsf{M}$ *and* $r\in(0,r_0)$*, setting* $B=B(\xi,r)$*,*

$$\int_B |f-f_B|^2\,d\mu\leqslant Pr^2\int_B|\nabla f|^2\,d\mu\quad \text{for every } f\in\mathcal{W}^1(B),\quad \text{where } f_B=\frac{1}{\mu(B)}\int_B f\,d\mu.$$

Then the following properties hold.

(1) *For fixed* $R>0$ *there are positive constants* α*,* C_4 *and* C_5 *(depending only on* R*) such that for all* $\xi\in\Gamma\mathsf{M}$*,* $r\in(0,R)$*, any weak solution* u *of the heat equation* $(\partial_t-\Delta)u=0$ *in* $Q=(0,4r^2)\times B(\xi,2r)$ *satisfies*

$$\sup\left\{\frac{|u(t,\xi)-u(s,\zeta)|}{(|t-s|^{1/2}+\rho(\xi,\zeta))^{\alpha}}\colon (t,\xi),(s,\zeta)\in Q'\right\}\leqslant\frac{C_4}{r^{\alpha}}\sup_Q|u|$$

and, if u *is non-negative,*

$$\sup_{Q_-}u\leqslant C_5\inf_{Q_+}u,\quad \text{where}$$

$Q'=(r^2,3r^2)\times B(\xi,r)$*,* $Q_-=(r^2,2r^2)\times B(\xi,r)$ *and* $Q_+=(3r^2,4r^2)\times B(\xi,r)$.

(2) *For any fixed integer* $k\geqslant 0$ *and* $\epsilon\in(0,1)$ *there is a constant* $C_{\epsilon,k}$ *such that for all* $t>0$ *and all* $\xi,\zeta\in\Gamma\mathsf{M}$*, with* α *as above,*

$$\left|\partial_t^k h(t,\xi,\zeta)\right|\leqslant\frac{C_{\epsilon,k}}{t^kV(\xi,\min\{1,\sqrt{t}\})}\exp\left(-\frac{\rho(\xi,\zeta)^2}{4(1+\epsilon)t}\right).$$

Moreover,

$$\left|\partial_t^k h(t,\xi,\zeta)\right|\leqslant\frac{C_{\epsilon,k}}{\min\{1,t\}^k}h\big((1+\epsilon)t,\xi,\zeta\big)$$

and, for all ζ' *with* $\rho(\zeta,\zeta')\leqslant\min\{1,\sqrt{t}\}$*,*

$$\left|\partial_t^k h(t,\xi,\zeta)-\partial_t^k h\big(t,\xi,\zeta'\big)\right|\leqslant\frac{C_{\epsilon,k}\rho(\zeta,\zeta')^{\alpha}}{(\min\{1,\sqrt{t}\})^{\alpha+k/2}}h\big((1+\epsilon)t,\xi,\zeta\big).$$

Concerning the growth of the volume of large balls, we point out that the hypothesis that the volume doubling property holds locally uniformly as in Theorem 4.4 implies that

$$V(\xi, r) \leqslant e^{Cr/r_0} V(\xi, r_0) \quad \text{for all } r \geqslant r_0,$$

see [28, Lemma 5.2.7]. We collect three of the main features.

4.5. Corollary. *Under the hypotheses of Theorem* 4.4*, the following properties hold for the heat semigroup* $\{H_t = e^{t\Delta}\colon t > 0\}$.

(1) *It is conservative* (*stochastically complete*)*, that is,* $e^{t\Delta}\mathbf{1} = \mathbf{1}$.
(2) *It sends* $\mathcal{L}^\infty(\Gamma\mathsf{M})$ *into* $\mathcal{C}_b(\Gamma\mathsf{M})$.
(3) *It sends* $\mathcal{C}_0(\Gamma\mathsf{M})$ *into itself.*

The next corollary concerns global non-negative solutions of the heat equation.

4.6. Corollary. *Under the hypotheses of Theorem* 4.4*, there exists a constant* C *such that any non-negative weak solution* u *of the heat equation on* $(0, T) \times \Gamma\mathsf{M}$ *satisfies*

$$u(s, \xi) \leqslant u(t, \zeta) \exp\bigl(C\bigl(1 + t/s + \rho(\xi, \zeta)^2/(t - s)\bigr)\bigr) \quad \textit{for all } \xi, \zeta \in \Gamma\mathsf{M},\ 0 < s < t < T.$$

Moreover, unique solvability of the positive Cauchy problem holds on $(0, T) \times \Gamma\mathsf{M}$ *for the heat equation. More precisely, if* $u : (0, T) \times \Gamma\mathsf{M}$ *is non-negative and is a weak solution of the heat equation on* $(0, T) \times \Gamma\mathsf{M}$ *then there exist a non-negative Borel measure* σ *on* $\Gamma\mathsf{M}$ *and* $a > 0$ *such that*

$$\int_{\Gamma\mathsf{M}} e^{-a\rho(\xi_0, \xi)^2} \, d\sigma(\xi) < \infty$$

for some (*equivalently, any*) $\xi_0 \in \Gamma\mathsf{M}$*, and*

$$u(t, \xi) = \int_{\Gamma\mathsf{M}} h(t, \xi, \zeta) \, d\sigma(\zeta) \quad \textit{for all } (t, \xi) \in (0, T) \times \Gamma\mathsf{M}.$$

In particular, if u *is a non-negative weak solution of the heat equation in* $(0, T) \times \Gamma\mathsf{M}$ *and there is some* $u_0 \in \mathcal{L}^1_{\mathrm{loc}}(\Gamma\mathsf{M})$ *such that*

$$\lim_{t \to 0} \int_{\Gamma\mathsf{M}} u(t, \cdot) f \, d\mu = \int_{\Gamma\mathsf{M}} u_0 f \, d\mu \quad \textit{for all } f \in \mathcal{C}_c^\infty(\Gamma\mathsf{M}),$$

then $u(t, \xi) = \int_{\Gamma\mathsf{M}} h(t, \xi, \cdot) u_0 \, d\mu$.

Proof. See Ancona and Taylor [1], Aronson [2,3], Grigor'yan [21, Theorem 6.2], [33, Sec. 3] and [28, Sec. 5.5.2]. □

Next, we give some relatively simple sufficient conditions which imply that the hypotheses of Theorem 4.4 are satisfied.

4.7. Proposition. *Assume the following.*

- *The manifold (M, g) is complete and satisfies the doubling property and $\mathcal{L}^2$-Poincaré inequality at all scales, that is, there are positive constants D_M and P_M such that, for every $x_0 \in M$, $r > 0$,*

$$V_M(x_0, r) \leqslant D_M V_M(x_0, 2r),$$

where $V_M(x_0, r)$ is the Riemannian volume of the geodesic ball $B = B_M(x_0, r)$ of radius r around x_0 in M, and

$$\int_B |f - f_B|^2 \, dx \leqslant P_M r^2 \int_B |\nabla_M f|^2 \, dx \quad \text{for all } f \in \mathcal{W}^1(B),$$

where f_B is the average of f over B, and dx is the volume element of M.

- *There are finite positive constants c_0 and C_0 such that*

$$\int_{e^-}^{e^+} \sqrt{\phi_e(s)} \, ds \geqslant c_0 \quad \text{for every } e \in E, \quad \text{and} \quad \deg(v) \leqslant C_0 \quad \text{for every } v \in V.$$

Moreover, for any finite interval $I \subset \Gamma^1$ with $\int_I \sqrt{\phi(s)} \, ds \leqslant c_0$,

$$\frac{\max_I \phi}{\min_I \phi} \leqslant C_0 \quad \text{and} \quad \frac{\max_I \psi}{\min_I \psi} \leqslant C_0.$$

Under these hypotheses, $(\Gamma\mathsf{M}, \rho)$ *is complete, and there are constants D, P, r_0 such that the properties* (i) *and* (ii) *of Theorem* 4.4 *hold.*

Proof. Completeness follows clearly from Lemma 3.18. Moreover, under the above hypotheses on ϕ, ψ, for any fixed r_0, the functions ϕ and ψ behave like constant functions (that is, there is $c = c(r_0) > 0$ such that $c \leqslant \phi, \psi \leqslant 1/c$) on any ball of radius r_0 in $\Gamma\mathsf{M}$. This means that the geometry of $\Gamma\mathsf{M}$ in such a ball B is comparable to the product of a piece of Γ_1 scaled by a constant factor ϕ_B (corresponding to the size of ϕ in the ball in question) and $(M, \phi_B g)$. The uniform local doubling property thus follows from the global doubling property on M and the fact that ϕ and ψ are approximately constant in B. The uniform local Poincaré inequality follows by the argument used in the proof of Theorem 4.1 that can now be carried through up to a uniformly fixed scale. □

4.8. Examples. (a) Let (Γ^1, l) be a metric graph as above with $\min_{e \in E}\{l_e\} > 0$. Suppose that $\psi \in \mathcal{S}^\infty(\Gamma^1)$ has the property that for any interval $I \subset \Gamma^1$ of length 1, one has $\max_I \psi / \min_I \psi \leqslant C$ for some positive C, and that $\max_V \deg(v) < \infty$.

Then the weighted 1-complex $(\Gamma^1, l, \psi(s)\, ds)$ satisfies the hypotheses of Theorem 4.4. More generally, for any $k = 0, 1, 2, \ldots$, the strip complex $\Gamma\mathsf{M}$ with $M = \mathbb{R}^k$, $\phi \equiv 1$ and ψ as above, satisfies the hypotheses of Theorem 4.4.

(b) The treebolic space HT equipped with any one of the forms $(\mathcal{E}_{\alpha,\beta}, \mathcal{W}^1_{\alpha,\beta}(\mathsf{HT}))$ satisfies the hypotheses of Theorem 4.4. This follows from the local result and the fact that there is a group of measure preserving isometries acting with compact quotient for any one of these structures.

5. Smoothness of weak solutions

Throughout this section, we keep the setting and notation of Section 4.

A. Harmonic functions. By the general theory of Dirichlet forms, there is a Hunt process with continuous sample paths defined for every starting point $\xi \in \Gamma\mathsf{M}$ associated with the semigroup $H_t = e^{t\Delta} : \mathcal{L}^2(\Gamma\mathsf{M}) \to \mathcal{L}^2(\Gamma\mathsf{M})$. In general, since our semigroup is not always conservative, we must add an isolated point ∞ to $\Gamma\mathsf{M}$.

The distribution $(\mathbb{P}_\xi)_{\xi\in\Gamma\mathsf{M}}$ of this process on $\mathcal{C}([0,\infty] \mapsto \Gamma\mathsf{M} \cup \{\infty\})$ is determined by the one-dimensional distributions

$$\mathbb{P}_\xi(X_t \in U) = \int_U h(t,\xi,\zeta)\,d\mu(\zeta) = H_t \mathbf{1}_U(\xi)$$

for any open subset $U \subset \Gamma\mathsf{M}$, where ξ is the starting point. The life time of the process is

$$\tau_\infty = \sup\{t \geqslant 0\colon\ X_t \in \Gamma\mathsf{M}\},$$

and H_t is conservative if and only if $\mathbb{P}_\xi(\tau_\infty < \infty) = 0$ for some (equivalently, all) $\xi \in \Gamma\mathsf{M}$.

For any relatively compact open set U, define the exit time

$$\tau_U = \inf\bigl\{t > 0\colon\ X_t \in U^c\bigr\}$$

and, for $\xi \in U$, the exit distribution

$$\pi_U(\xi, B) = \mathbb{E}_\xi(X_{\tau_U} \in B).$$

Since the process has continuous paths, for $\xi \in U$, the measure $\pi_U(\xi,\cdot)$ is supported on the boundary ∂U of U. More generally, we set

$$\pi_U(\xi, f) = \mathbb{E}_\xi\bigl(f(X_{\tau_U})\bigr)$$

for any bounded Borel measurable function f defined everywhere on ∂U.

The *Green potential* of a continuous function $\varphi \geqslant 0$ with support in U can be written as

$$G_U\varphi(\xi) = \mathbb{E}_\xi\left(\int_0^{\tau_U} \varphi(X_t)\,dt\right) \leqslant +\infty.$$

5.1. Definition. A bounded Borel function u in an open set $\Omega \subset \Gamma\mathsf{M}$ is $\mathbb{P}$-*harmonic* (that is, harmonic with respect to the process $X = (X_t)_{t\geqslant 0}$ with law $\mathbb{P}$) if, for any open relatively compact set B with $\overline{B} \subset \Omega$, we have

$$\pi_B(\xi, u) = u(\xi) \quad \text{for all } \xi \in B.$$

Since the associated semigroup $\{H_t\colon t > 0\}$ sends bounded measurable functions to bounded continuous functions it follows that any harmonic function is continuous; see e.g. Dynkin [15, Vol. II]. The following result is important for our purpose.

5.2. Theorem. *Let $\Omega \subset \Gamma\mathsf{M}$ be an open set.*

(i) *If u is a weak solution of $\Delta u = 0$ in Ω then the continuous version of u is $\mathbb{P}$-harmonic in Ω.*
(ii) *If u is $\mathbb{P}$-harmonic in Ω then u is a weak solution of $\Delta u = 0$ in Ω.*

Proof. Part (i) is true in great generality, see [20, Theorem 4.3.2] (recall that, in our case, weak solutions are continuous).

We now prove Part (ii). Without loss of generality, we can assume that Ω is relatively compact and $u \geqslant \epsilon > 0$ in Ω.

Consider a fixed open set V with $\overline{V} \subset \Omega$ (i.e., V is relatively compact in Ω).

Let φ be a non-negative continuous function (not identically 0) with support in U, and let $w = G_\Omega \varphi \in \mathcal{W}_0^1(\Omega)$ be its Green potential in Ω.

Since u is bounded from above in U and the potential w is bounded from below in U, there exists $t > 0$ such that the excessive function $h = \min\{t \cdot w, u\}$ coincides with u in U, because $w|_{\partial\Omega} = 0$. Moreover, h coincides with $t \cdot w$ near the boundary of Ω. Since $h \leqslant t \cdot w$, the function $h = G_\Omega \nu$ is the Green potential of a measure ν with compact support in Ω and energy integral which is computed as

$$\mathcal{E}(h,h) = \int_\Omega h\, d\nu < \infty.$$

See Blumenthal and Getoor [8, Ch. VI, Theorem 2.10] and Silverstein [31, Ch. 1, Sec. 3]. In particular, $h \in \mathcal{W}_0^1(\Omega)$, and since u coincides with h in V, we see that u is in $\mathcal{W}_{\mathrm{loc}}^1(\Omega)$.

Next, u is represented inside any open set V with $\overline{V} \subset \Omega$ as $u = \pi_V(\cdot, u)$. Since u is in $\mathcal{W}^1(V)$, the function $\pi_V(\cdot, u)$ coincides with the Hilbert projection of u on the linear subspace of weakly harmonic functions in V. See [20, Theorem 4.3.2]. □

B. The bifurcation conditions. The aim of this section is to prove that weak solutions of $\Delta u = 0$ are actually very regular in each strip and up to the bifurcation manifolds although their various derivatives are typically not continuous across those bifurcation manifolds. This will allow us to see that weak solutions verify in a strong sense a particular bifurcation condition (or Kirchhoff's law) along each bifurcation manifold. This bifurcation law is a crucial ingredient in the analysis of our Dirichlet forms. It captures the influence of the jumps of the functions ϕ and ψ across bifurcation manifolds and is crucial for an understanding of the domain of the infinitesimal generator.

Let us start by observing that, in any open strip S_e^o, the infinitesimal generator Δ of our heat semigroup is simply the weighted Riemannian Laplacian

$$\Delta f = \frac{1}{\psi} \operatorname{div}\bigl(\psi \operatorname{grad}(f)\bigr),$$

where div and grad refer, respectively, to the divergence and gradient on the manifold

$$\big((e^-, e^+) \times M,\ \phi(s)\big((ds)^2 + g(\cdot,\cdot)\big)\big).$$

More concretely, this means that for any f in the domain of Δ and such that $f \in \mathcal{C}^\infty(S_e^o)$,

$$\Delta f = \frac{1}{\psi}\big[\partial_s^2 + \Delta_M + \eta \partial_s\big] f, \quad \text{where } \eta = \partial_s \ln\big(\phi^{(n-1)/2}\psi\big).$$

To be able to distinguish between the infinitesimal generator and its expression in the interior of a strip, we make the following definition.

5.3. Definition. For any $\xi \in \Gamma\mathsf{M}^o$ and any function f which coincides with a smooth function in a neighbourhood of ξ, set

$$\mathfrak{A} f(\xi) = \frac{1}{\psi(\xi)}\big[\partial_s^2 + \Delta_M + \eta(\xi)\partial_s\big] f(\xi), \quad \text{where } \eta = \partial_s \ln\big(\phi^{(n-1)/2}\psi\big).$$

In particular, $\mathfrak{A}$ (as well as any of its integer powers $\mathfrak{A}^k$) is a well defined continuous operator from $\mathcal{S}^\infty(\Gamma\mathsf{M}^o)$ to $\mathcal{L}^\infty_{\text{loc}}(\Gamma\mathsf{M})$.

In addition to the "differential operator" $\mathfrak{A}$, there is another crucial ingredient needed in order to describe harmonic functions on $\Gamma\mathsf{M}$ properly. Namely, harmonic functions must satisfy a bifurcation condition (or Kirchhoff law) along each bifurcation manifold M_v. To express this bifurcation condition, we introduce the following notation.

5.4. Definition. Given $v \in V$ and $e \in E_v$, let $\mathsf{n}_{v,e}$ be the outwards pointing normal unit vector relative to S_e^o along M_v.

We start by writing down Green's formulas for a domain Ω with piecewise smooth boundary contained in one strip S_e and for smooth functions f, h on Ω. Then Green's formulas read as follows.

$$\int_\Omega f \mathfrak{A} h \, d\mu + \int_\Omega (\nabla f, \nabla h)\, d\mu = \int_{\partial\Omega} (\mathsf{n}, \nabla h) f \, d\mu' \tag{5.5}$$

and

$$\int_\Omega (f\mathfrak{A}h - h\mathfrak{A}f)\, d\mu = \int_{\partial\Omega} \big((\mathsf{n}, \nabla h) f - (\mathsf{n}, \nabla f) h\big)\, d\mu', \tag{5.6}$$

where n is the outward unit normal vector to Ω and μ' is the induced measure on $\partial\Omega$. This measure has density $\psi_e(s)$ with respect to the Riemannian hypersurface measure on $(S_e, \phi \cdot ((ds)^2 + g(\cdot,\cdot)))$.

Let u be a weak solution of $\Delta u = 0$ in a general domain $\Omega \subset \Gamma\mathsf{M}$ and let U be a domain in a bifurcation manifold $M = M_v$ such that the closure of U is contained in Ω.

Fix a strip $S = S_e^o$ attached to M_v, and consider the outward unit normal derivative relative to S_e^o along M_v.

If $(U; x_1, \dots, x_n)$ is a local coordinate chart in M_v and $(s, x_1, \dots, x_n)$ denotes the corresponding coordinate chart in $S_e^o = (e^-, e^+) \times U$ then that derivative is given by

$$(\mathsf{n}_{v,e}, \nabla) = \pm\phi_e(v)^{-1/2}\partial_s, \quad \text{if } v = e^{\pm}. \tag{5.7}$$

(The two signs have to coincide.) Note that it is crucial here to use the notation $\phi_e(v)$ since ϕ is not necessarily defined at v and the values of the edge-wise extensions $\phi_e(v)$ of ϕ to the vertex v may be distinct for different $e \in E_v$.

Suppose for the sake of simplicity that $v = e^-$. Given u as above, we want to define

$$\delta = (\mathsf{n}_{v,e}, \nabla u)|_U = -\phi_e(v)^{-1/2}\partial_s u(v, \cdot)$$

as a distribution on U. For $\varepsilon > 0$ (small enough), let $L_\varepsilon = \{(s, x) \in S\colon s = s_\varepsilon\}$ be the "horizontal manifold" in S where s_ε is the point at distance ε from $e^- = v$ in the interval I_e. Let $U_\varepsilon = \{(s, x)\colon s = s_\varepsilon, \ x \in U\}$. We assume that ε is so small that the closure of U_ε is contained in Ω. For $0 < \varepsilon' < \varepsilon$ fixed small enough we let $R_{\varepsilon',\varepsilon}$ be the rectangle with $U_{\varepsilon'}$ and U_ε as horizontal sides.

Because u is smooth inside the strip S, for any sufficiently small $\varepsilon > 0$ and any smooth function θ on M with compact support in U,

$$\delta_\varepsilon(\theta) = -\phi(s_\varepsilon)^{(n-1)/2}\psi(s_\varepsilon)\int_U \partial_s u(s_\varepsilon, x)\theta(x)\,dx$$

is well defined, and $\theta \mapsto \delta_\varepsilon(\theta)$ is a distribution.

Now, we can compute $\delta_\epsilon(\phi) - \delta_{\epsilon'}(\phi)$ by setting $\Theta(s, x) = \theta(x)$ and writing

$$\int_{R_{\varepsilon',\varepsilon}} (\Theta\mathfrak{A}u - u\mathfrak{A}\Theta)\,d\mu = \int_{\partial R_{\varepsilon',\varepsilon}} \big((\mathsf{n}_{v,e}, \nabla u)\Theta - (\mathsf{n}_{v,e}, \nabla\Theta)u\big)\,d\mu'.$$

Recall that θ is a smooth function with compact support in U. It follows that Θ and $\nabla\Theta$ vanish on the vertical components of $\partial R_{\epsilon',\epsilon}$. In addition, since Θ is independent of s, $(\mathsf{n}_{v,e}, \nabla\Theta)$ vanishes on the horizontal components of $\partial R_{\varepsilon',\varepsilon}$. Furthermore $\mathfrak{A}u = 0$ in $R_{\varepsilon',\varepsilon}$. Hence

$$-\int_{R_{\varepsilon',\varepsilon}} u\mathfrak{A}\Theta\,d\mu = \int_{U_\varepsilon} (\mathsf{n}_{v,e}, \nabla u)\Theta\,d\mu' + \int_{U_{\varepsilon'}} (\mathsf{n}_{v,e}, \nabla u)\Theta\,d\mu',$$

whence

$$\left|\int_{R_{\varepsilon',\varepsilon}} u(s, x)\frac{1}{\phi(s)}\Delta_M\theta(x)\,d\mu\right| = \left|\delta_\varepsilon(\theta) - \delta_{\varepsilon'}(\theta)\right|.$$

Since u and $\Delta_M\theta$ are uniformly bounded in a domain containing all rectangles $R_{\varepsilon',\varepsilon}$ with sufficiently small $0<\varepsilon'<\varepsilon$, it follows that

$$\lim_{\varepsilon\to 0}\delta_\varepsilon(\phi)=\delta(\phi)$$

exists. If (as usually) u_e denotes the restriction of u to S_e^o, this defines

$$\delta=\big(\mathsf{n}_{v,e},\nabla u_e(v,\cdot)\big)$$

as a distribution on U.

In this way, we obtain $\deg(v)$ distributions $\delta_{v,e}$, one for each edge $e\in E_v$. Each $\delta_{v,e}$ corresponds to the unit outward normal derivative $(\mathsf{n}_{v,e},\nabla u_e(v,\cdot))$ in S_e^o along $U\subset M_v$. Now, the fact that u is a weak solution of $\Delta u=0$ in Ω implies that

$$\sum_{e\in E_v}\psi_e(v)\,\delta_{v,e}=\sum_{e\in E_v}\psi_e(v)\big(\mathsf{n}_{v,e},\nabla u_e(v,\cdot)\big)=0\quad\text{as distributions on } U\subset M_v. \tag{5.8}$$

We refer to this as the *bifurcation condition* along M_v or *Kirchhoff's law,* in the sense of distributions.

For later purpose, it is useful to observe that the argument developed above for weak solutions of $\Delta u=0$ also works for weak solutions of the Poisson equation

$$\Delta u=f$$

in an open set Ω with a function f that is Hölder continuous in Ω. To be precise, we require here that $u\in\mathcal{W}^1(\Omega)$ and that for any $h\in\mathcal{W}_0^1(\Omega)$,

$$\mathcal{E}(u,h)=-\int f h\,d\mu.$$

Note that by classical results, such a function u has continuous partial derivatives up to second order and satisfies $\mathfrak{A}u=f$ in the intersection of Ω with each open strip S_e^o. By an argument similar to the one used above for weak solutions, the function u must also satisfy the bifurcation condition (5.8) in the sense of distributions.

For instance, the function $u(\xi)=h(t,\zeta,\xi)$ is a weak solution of $\Delta u=f$ on HT with $f(\xi)=\partial_t h(t,\zeta,\xi)$. Hence it satisfies (5.8) in the sense of distributions along each of the bifurcation manifolds M_v in $\Gamma\mathsf{M}$.

C. Smoothness of harmonic functions. The aim of this section is to show that weak solutions of $\Delta u=0$ in an open set are smooth in the strip complex sense, that is, they belong locally to $\mathcal{C}^\infty(\Gamma\mathsf{M})$. Since Δ is a non-degenerate elliptic operator in each open strip, we know that harmonic functions are smooth there (in the usual sense of having continuous partial derivatives of all orders). The problem is to obtain smoothness up to the bifurcation manifolds in each strip separately. Recall here that smoothness on $\Gamma\mathsf{M}$ does not imply continuity of the derivatives across bifurcation manifolds.

5.9. Theorem. *Fix an open set $\Omega \subset \Gamma\mathsf{M}$. For each $e \in E$, set $\Omega_e = \Omega \cap S_e^o$ and, if $u \in \mathcal{C}(\Omega)$, $u_e = u|_{\Omega_e}$. A function u is a weak solution of $\Delta u = 0$ in Ω if and only if it has the following properties (more precisely, the continuous version of u has the following properties):*

- $u \in \mathcal{C}^\infty(\Omega)$.
- *For any $e \in E$, one has $\mathfrak{A} u_e = 0$ on Ω_e.*
- *For any $v \in V$, one has $\sum_{e \in E_v} \psi_e(v)(\mathsf{n}_{v,e}, \nabla u_e) = 0$ along $M_v \cap \Omega$.*

5.10. Remark. The first and third conditions are the crucial ones, since we already know that the second condition must hold by the local ellipticity of our Laplacian in each open strip. Concerning the first condition, we already know that weak solutions are continuous (more precisely, have a continuous representative) so the important part of the statement is that they belong locally to $\mathcal{S}^\infty(\Gamma\mathsf{M})$. We already observed in (5.8) that the third condition must hold in the sense of distributions but, if $u \in \mathcal{C}^\infty(\Omega)$, this is equivalent to a classical pointwise statement as given by the theorem.

Proof of Theorem 5.9. The proof goes through four steps and needs two auxiliary propositions.

Step 1: change of function. It will be useful to consider the functions

$$w_e(\xi) = \beta_e(s)u(\xi), \quad \text{where } \beta_e = \sqrt{\phi_e^{(n-1)/2}\psi_e} \quad \text{and} \quad \xi = (s, x) \in I_e \times M.$$

Recall that u satisfies

$$\mathfrak{A} u = \phi^{-1}\left[\partial_s^2 + \Delta_M + \eta\partial_s\right]u = 0, \quad \text{where } \eta = \partial_s \ln\left(\phi^{(n-1)/2}\psi\right)$$

in each set $\Omega_e = \Omega \cap S_e^o$ and the bifurcation equation

$$\sum_{e \in E_v} \psi_e(v)(\mathsf{n}_{v,e}, \nabla u_e) = 0$$

on each bifurcation manifold M_v, where this is understood in the sense of distributions. Observe that

$$\frac{2\partial_s\beta_e}{\beta_e} = \partial_s \ln\left(\phi_e^{(n-1)/2}\psi_e\right) = \eta_e.$$

This implies that the functions w_e, $e \in E$, satisfy

$$\left(\partial_s^2 + \Delta_M\right)w_e = \phi_e\beta_e\mathfrak{A} u_e + \left(\partial_s^2\beta_e\right)u_e = \frac{\partial_s^2\beta_e}{\beta_e}w_e$$

in each open strip S_e^0 and the bifurcation equation

$$\sum_{e \in E_v} \psi_e(v)(\mathsf{n}_{v,e}, \nabla w_e) = -\left(\frac{1}{\phi_e(v)^{1/2}\beta_e(v)}\sum_{e \in E_v} \varepsilon_{v,e}\psi_e(v)\left|\partial_s\beta_e(v)\right|w_e\right) \quad \text{along } M_v,$$

where

$$\epsilon_{v,e} = \begin{cases} 1, & \text{if } v = e^+, \\ -1, & \text{if } v = e^-. \end{cases} \tag{5.11}$$

Step 2: folding. As smoothness is a local property, we can assume that Ω is a small neighbourhood of a point $\xi_0 = (v, x_0)$ on a fixed bifurcation manifold M_v and that $\Omega_e = \Omega \cap S_e^o$ is of the form $(v, r_e) \times U$ where $r_e \in I_e^o = (e^-, e^+)$, and all intervals (v, r_e) in Γ^1 have the same (small) length l. This provides us with an obvious way to identify all the different Ω_e with a fixed set

$$\Omega_+ = (0, l) \times U \subset (0, \infty) \times M.$$

Using this identification, we can consider each w_e as a function defined on Ω_+, namely,

$$\Omega_+ \ni (s, x) \mapsto w_e(s_{(v,e)}, x)$$

where $s_{(v,e)}$ is the point on I_e, $e \in E_v$, at distance s from v. Now Theorem 5.9 will be an immediate consequence of the next result.

In the following proposition, E_v can be viewed as an arbitrary finite set of parameters whose elements are denoted by e.

5.12. Proposition. *Let U be a relatively compact domain in M. Let*

$$\Omega_+ = (0, l) \times U \subset (0, \infty) \times M$$

and $I = \{0\} \times U$ be the bottom of Ω_+. For all $e, e' \in E_v$, let $\delta_e > 0$, $\tilde{\delta}_e \in \mathbb{R}$ and $c_{e,e'} > 0$ be fixed numbers. Let γ_e, $e \in E_v$, be functions in $\mathcal{C}^\infty([0, l])$.

Assume that w_e, $e \in E_v$, are functions defined on Ω_+ that belong to $\mathcal{C}^\infty(\Omega_+)$ and satisfy the following hypotheses.

- *For each $e \in E_v$, the function w_e is in $\mathcal{C}^\alpha(\overline{\Omega_+})$ for some $\alpha \in (0, 1)$, and*

$$w_e|_I = c_{e,e'} w_{e'}|_I \quad \text{for all } e, e' \in E_v,$$

- $[\partial_s^2 + \Delta_M] w_e = \gamma_e w_e$ *in* Ω_+.
- *The partial derivatives $\partial_s w_e(0, \cdot)$, $e \in E_v$, whose existence in the sense of distributions in U is guaranteed by the first two hypotheses, satisfy*

$$\sum_e \delta_e \partial_s w_e(0, \cdot) = \sum_e \tilde{\delta}_e w_e(0, \cdot)$$

in the sense of distributions in U.

Then $w_e \in \mathcal{C}^\infty([0, l) \times U)$ for each $e \in E_v$, i.e., it is smooth up to the bottom I of Ω_+.

In order to prove this proposition, set

$$W(s, x) = \sum_{e \in E_v} \delta_e w_e(s, x).$$

The function W is continuous in $\overline{\Omega_+} = [0, l] \times \overline{U}$. Moreover, it satisfies

$$\begin{cases} \left[\partial_s^2 + \Delta_M\right] W = W_1 \quad \text{in } \Omega_+, \\ \partial_s W(0, \cdot) = W_2 \quad \text{on } U, \end{cases} \tag{5.13}$$

where

$$\begin{cases} W_1 = \sum_e \delta_e \gamma_e w_e \in \mathcal{C}^\alpha(\overline{\Omega_+}) \quad \text{and} \\ W_2 = \frac{1}{\delta} \sum_e \tilde{\delta}_e w_e(0, \cdot) \in \mathcal{C}^\alpha(\overline{U}), \quad \text{with } \delta = \sum_{e \in E_v} \delta_e. \end{cases} \tag{5.14}$$

At this point, the proof of Proposition 5.12 requires another auxiliary result, as follows.

Step 3: improved regularity.

5.15. Proposition. *With notation as in Proposition* 5.12*, fix* $\alpha \in (0, 1)$ *and a nonnegative integer* k*. Also fix* $h_1 \in \mathcal{C}^{k+\alpha}(\overline{\Omega_+})$ *and* $h_2 \in \mathcal{C}^{k+\alpha}(\overline{U})$*. Let* f *be a smooth function in* Ω_+ *which belongs to* $\mathcal{C}^{k+\alpha}(\overline{\Omega_+})$ *and satisfies*

$$\begin{cases} \left[\partial_s^2 + \Delta_M\right] f = h_1 \quad \text{in } \Omega_+, \\ \partial_s f = h_2 \quad \text{in } I \text{ (in the sense of distributions when } k = 0\text{)}. \end{cases}$$

Then f *belongs to* $\mathcal{C}^{k+1+\alpha}(\overline{\Omega'_+})$ *for every set* $\Omega'_+ = (0, l') \times U'$*, where* $0 < l' < l$ *and* U' *is open and relatively compact in* U.

Proof. Without loss of generality (because of well-known basic extension theorems, see e.g. Seeley [30]), we can assume that $h_1 = h|_{\Omega_+}$ is the restriction to Ω_+ of a function $h \in \mathcal{C}^{k+\alpha}(\mathbb{R} \times M)$ with compact support. Let B be a ball in $\mathbb{R} \times M$ containing the support of h. Let $H = G_B h$ be the Green potential of h relative to this ball B and with respect to the operator $\partial_s^2 + \Delta_M$. Then $H \in \mathcal{C}^{k+2+\alpha}_{\text{loc}}(B)$, and within Ω_+ we have

$$\left[\partial_s^2 + \Delta_M\right](f + H) = 0.$$

Obviously, on the boundary I, the function $f + H$ satisfies

$$\partial_s(f + H)|_I = h_2 + \partial_s H|_I.$$

Note that $f + H \in \mathcal{C}^{k+\alpha}(\overline{\Omega_+})$ and $h_2 + \partial_s H|_I \in \mathcal{C}^{k+\alpha}(\bar{I})$. Thus, replacing f by $f + H$, we are led to study the solutions $f \in \mathcal{C}^{k+\alpha}(\overline{\Omega_+})$ of

$$\begin{cases} \left[\partial_s^2 + \Delta_M\right] f = 0 \quad \text{in } \Omega_+, \\ \partial_s f = h \quad \text{on } I, \end{cases}$$

where $h \in \mathcal{C}^{k+\alpha}(\overline{U})$. Indeed, to prove Proposition 5.15, it suffices to show that such f must be in $\mathcal{C}^{k+1+\alpha}(\overline{\Omega'_+})$.

Recall that I is the bottom of Ω_+. Let $\Omega'_0 = \{0\} \times U'$. Identifying $\{0\} \times M$ with M, there exists a function $f_1 \in \mathcal{C}_c^{k+\alpha}(M)$ and a set J open in I with $\overline{I'} \subset J \subset I$ such that $f|_J = f_1|_J$. Thus, if we decompose $f|_I = f_1 + f_2$ on I, then $f_2 = 0$ on J.

Let $(s, x) \mapsto F_1(s, x)$ be the harmonic function on $(0, \infty) \times M$ which coincides with f_1 on $\{0\} \times M$, that is, the Poisson integral given formally by $F_1(s, x) = e^{-s\sqrt{-\Delta_M}} f_1(x)$. Then, in Ω_+, we have $f = F_1 + F_2$ where F_2 is harmonic in Ω_+ with boundary values 0 on J. In particular, F_2 has bounded continuous derivatives of all orders up to J. Moreover, along J we have in the sense of distributions on J

$$h = \partial_s f(0, \cdot) = -\sqrt{-\Delta_M} f_1 + \partial_s F_2(0, \cdot).$$

Write this as

$$[Id + \sqrt{-\Delta_M}] f_1|_J = \big(-h + f_1 + \partial_s F_2(0, \cdot)\big)\big|_J,$$

again in the sense of distributions. (Here, *Id* is the identity operator.) By hypothesis, the right-hand side is in $\mathcal{C}_{\mathrm{loc}}^{k+\alpha}(J)$. Let $f_3 \in \mathcal{C}_c^{k+\alpha}(J)$ be a function which coincides with $(-h + f_1 + \partial_s F_2(0, \cdot))|_J$ in a neighbourhood J' of $\overline{I'}$ that is contained in J. Let $f_4 = [Id + \sqrt{-\Delta_M}]^{-1} f_3$. Then $f_4 \in \mathcal{C}_{\mathrm{loc}}^{k+1+\alpha}(M) \cap \mathcal{L}^2(M)$, and the function $f_1 - f_4 \in \mathcal{L}^2(M)$ satisfies

$$[Id + \sqrt{-\Delta_M}](f_1 - f_4) = 0 \quad \text{in } J'.$$

In addition, the distribution $[Id + \sqrt{-\Delta_M}](f_1 - f_4) = ([Id + \sqrt{-\Delta_M}] f_1) - f_3$ can be represented by a function in $\mathcal{L}^2(M)$ outside I because f_1 is continuous with compact support in I. By the hypoellipticity of $[Id + \sqrt{-\Delta_M}]$ (see Theorem A.4 in Appendix A) it follows that $f_1 - f_4$ is in $\mathcal{C}_{\mathrm{loc}}^{\infty}(J')$. Hence f_1 is $\mathcal{C}_{\mathrm{loc}}^{k+1+\alpha}(J')$: it has the same smoothness as f_4 in J'. This implies that the Poisson integral F_1 of f_1 is in $\mathcal{C}^{k+1+\alpha}(\overline{\Omega'_+})$. Hence $f = F_1 + F_2$ is in $\mathcal{C}^{k+1+\alpha}(\overline{\Omega'_+})$. This is the desired result. □

Step 4: final bootstrap. We now prove Proposition 5.12 by induction on the smoothness parameter k, using Proposition 5.15. Assume we have proved that the functions w_e in Proposition 5.12 are in $\mathcal{C}^{k+\alpha}(\overline{\Omega'_+})$ for some integer k and any $\Omega' = (0, l') \times U'$ relatively compact in Ω_+. This implies that the functions W_1, W_2 of (5.14) are respectively in $\mathcal{C}^{k+\alpha}(\overline{\Omega'_+})$ and $\mathcal{C}^{k+\alpha}(\overline{U})$. Hence we can apply Proposition 5.15 to the function W of (5.13). This gives that $W \in \mathcal{C}^{k+1+\alpha}(\overline{\Omega^*_+})$ where $\Omega^*_+ = (0, l^*) \times U^*$ with l^* an arbitrary real in $(0, l')$ and U^* an arbitrary open relatively compact set in U'. Because $l' \in (0, l)$ and U', relatively compact in U, are arbitrary, we conclude that $W \in \mathcal{C}^{k+1+\alpha}(\overline{\Omega'_+})$ for any $\Omega' = (0, l') \times U'$ relatively compact in Ω_+.

The functions w_e, $e \in E_v$, are related on $\{0\} \times U$ by

$$w_e(0, x) = c_{e,e'} w_{e'}(0, x)$$

and thus are all equal on $\{0\} \times U$ to a fixed multiple of $W(0, \cdot) \in \mathcal{C}^{k+1+\alpha}(U)$. Each of the functions w_e is solution of

$$\begin{cases} \big[\partial_s^2 + \Delta_M\big] f = h_{e,1} & \text{in } \Omega_+, \\ f(0, \cdot) = h_{e,2} & \text{on } U, \end{cases}$$

where $h_{e,1} = \gamma_e w_e \in \mathcal{C}^{k+\alpha}(\overline{\Omega_+})$ and $h_{e,2} = w_e(0,\cdot) \in \mathcal{C}^{k+1+\alpha}(U)$.

Let $H_{e,1}$ be the Green potential (in a large ball in $\mathbb{R} \times M$) of a compactly supported extension of $h_{e,1}$ that belongs to $\mathcal{C}^{k+\alpha}(\mathbb{R} \times M)$. The function $H_{e,1}$ is $\mathcal{C}^{k+2+\alpha}(\overline{\Omega_+})$, and $w_e - H_{e,1}$ is solution of

$$\begin{cases} [\partial_s^2 + \Delta_M] f = 0 & \text{in } \Omega_+, \\ f(0,\cdot) = h_{e,2} - H_{e,1}(0,\cdot) & \text{on } U, \end{cases}$$

where $h_{e,2} - H_{e,1}(0,\cdot) \in \mathcal{C}^{k+1+\alpha}(U)$. It follows that $w_e - H_{e,1}$ is in $\mathcal{C}^{k+1+\alpha}(\overline{\Omega'_+})$. This means that each of the functions w_e is in $\mathcal{C}^{k+1+\alpha}(\overline{\Omega'_+})$. □

Given an open connected set Ω, consider the linear space $H(\Omega)$ of all weak solutions of the Laplace equation $\Delta u = 0$ in Ω. By the local Hölder regularity result and the fact that weak solutions and $\mathbb{P}$-harmonic functions coincide, it follows that $H(\Omega)$ equipped with the seminorms of the uniform convergence on compact subsets of Ω is a complete seminormed vector space.

By Theorem 5.9, any element u of $H(\Omega)$ is in $\mathcal{C}^\infty(\Omega)$. The closed graph theorem then yields the following result.

5.16. Corollary. *Let Ω be an open connected set in $\Gamma\mathsf{M}$ and Ω_0 relatively compact in Ω. Let $I \times U$ be a relatively compact coordinate chart in $\Gamma\mathsf{M}$ such that $K = \overline{I \times U} \subset \Omega_0$. Fix $\kappa = (\kappa_0, \kappa_1, \ldots, \kappa_n)$. Then there exists a constant $C = C(\Omega_0, K, \kappa)$ such that*

$$\sup_{\xi \in K} \left|\partial_\xi^\kappa u(\xi)\right| \leqslant C \sup_{\Omega_0} |u| \quad \textit{for all } u \in H(\Omega).$$

D. Regularity of certain weak solutions of the heat equation. Let $(t,\xi) \mapsto u(t,\xi)$ be a weak solution of the heat equation in $(0,T) \times \Omega$, where Ω is an open set in $\Gamma\mathsf{M}$. We already know that we can regard u as a Hölder continuous function on $(0,T) \times \Omega$. Our aim is to show that in some cases, including the case of the heat kernel, that $u(t,\cdot) \in \mathcal{C}^\infty(\Omega)$ for each $t \in (0,T)$, and moreover, for any positive integer k, $\partial_t^k u(t,\cdot) \in \mathcal{C}^\infty(\Omega)$. (See Definition 3.11 for the definition of $\mathcal{C}^\infty(\Omega)$.) It is plausible that this result holds for any weak solution, but our proof below does not provide this stronger result.

5.17. Definition. Fix $k \in \{0, 1, \ldots, \infty\}$, $T > 0$, $I = (0,T)$ and an open set $\Omega \subset \Gamma\mathsf{M}$. See (3.37) for the definition of $\mathcal{F}_{\mathrm{loc}}(I \times \Omega)$. We say that a weak solution $u \in \mathcal{F}_{\mathrm{loc}}(I \times \Omega)$ of the heat equation in $I \times \Omega$ is time regular to order k if, for each $m \in \{0, 1, \ldots, k\}$, the distributional time derivative $\partial_t^m u$ exists and can be represented by a function $u_m \in \mathcal{F}_{\mathrm{loc}}(I \times \Omega)$ which is a weak solution of the heat equation in $I \times \Omega$. When u is time regular to infinite order we simply say that u is a time regular weak solution in $I \times \Omega$.

5.18. Example. Fix $f \in \mathcal{L}^2(\Gamma\mathsf{M})$. Then $u(t,\xi) = H_t f(\xi) = e^{t\Delta} f(\xi)$ is a time regular weak solution up to infinite order in $(0,\infty) \times \Gamma\mathsf{M}$. Fix $\zeta \in \Gamma\mathsf{M}$ and set $u(t,\xi) = h(t,\xi,\zeta)$. Then u is again a time regular solution up to infinite order in $(0,\infty) \times \Gamma\mathsf{M}$. Fix an open set $\Omega \subset \Gamma\mathsf{M}$ and consider the Dirichlet Laplacian Δ_Ω in Ω. This is the infinitesimal generator associated with the closure of the form $(\int_\Omega |\nabla f|^2 \, d\mu, \mathcal{C}_c^\infty(\Omega))$. Let $f \in \mathcal{L}^2(\Omega)$ and consider $u(t,\xi) = e^{t\Delta_\Omega} f(\xi)$, $(t,\xi) \in (0,\infty) \times \Omega$. This is a time regular weak solution up to infinite order in $(0,\infty) \times \Omega$ and so is the corresponding Dirichlet heat kernel in Ω.

5.19. Theorem. *Fix $T > 0$ and an open set $\Omega \subset \Gamma\mathsf{M}$. For each $e \in E$, set $\Omega_e = \Omega \cap S_e^o$ and, if $u \in \mathcal{C}((0,T) \times \Omega)$, set $u_e = u|_{(0,T)\times\Omega_e}$. Any function u which is a weak solution of $[\partial_t - \Delta]u = 0$ in $Q = (0,T) \times \Omega$ and is time regular to order k has the following properties*:

- *For any $m = 0, 1, 2, \ldots, k$, the derivative $\partial_t^m u$ is a continuous function on $(0,T) \times \Omega$. Moreover, there is $\alpha \in (0,1)$ such that $\partial_t^m u(t,\cdot) \in \mathcal{C}^{k-m+\alpha}(\Omega)$ for any $t \in (0,T)$.*
- *For any $e \in E$, one has $[\partial_t - \mathfrak{A}]u_e = 0$ on $(0,T) \times \Omega_e$. In particular, u_e is smooth (in the usual sense) in the open set Ω_e.*
- *For any $m \in \{0, 1, \ldots, k-1\}$ and $v \in V$,*

$$\sum_{e \in E_v} \psi_e(v)\big(\mathsf{n}_{v,e}, \nabla \partial_t^m u_e\big) = 0 \quad \textit{along } (0,T) \times (M_v \cap \Omega).$$

Proof. The proof goes through three steps and involves Proposition 5.20 below.

Step 1: change of function. As in the elliptic case, we consider the functions

$$w_e(t,\xi) = \beta_e(s)u_e(t,\xi), \quad \text{where } \beta = \sqrt{\phi^{(n-1)/2}\,\psi} \text{ and } \xi = (s,x) \in I_e \times M.$$

Recall that u satisfies

$$\mathfrak{A}u = \frac{1}{\phi}\big[\partial_s^2 + \Delta_M + \eta\partial_s\big]u = \partial_t u, \quad \text{where } \eta = \partial_s \ln\big(\phi^{(n-1)/2}\psi\big)$$

in each set $\Omega_e = \Omega \cap S_e^o$ and the bifurcation equation

$$\sum_{e \in E_v} \psi_e(v)(\mathsf{n}_{v,e}, \nabla u_e) = 0$$

on each bifurcation manifold M_v, where this is understood in the sense of distributions. As in the proof of Theorem 5.9, this implies that the functions w_e satisfy

$$\big[\partial_s^2 + \Delta_M\big]w_e = \frac{\partial_s^2\beta_e}{\beta_e}w_e + \phi_e\partial_t w_e$$

in each open strip S_e^0 and the bifurcation equation

$$\sum_{e \in E_v} \psi_e(v)(\mathsf{n}_{v,e}, \nabla w_e) = -\left(\frac{1}{\phi_e(v)^{1/2}\beta_e(v)} \sum_{e \in E_v} \epsilon_{v,e}\psi_e(v)\big|\partial_s\beta_e(v)\big|w_e\right) \quad \text{along } M_v,$$

where $\epsilon_{v,e}$ is as in (5.11).

Step 2: folding and improved regularity. The following is analogous to Proposition 5.12 except for the role played by the function $\tilde{w}_e$.

5.20. Proposition. *Let U be a relatively compact domain in M. Let*

$$\Omega_+ = (0,l) \times U \subset (0,\infty) \times M$$

and $I = \{0\} \times U$ be the bottom of Ω_+. For all $e, e' \in E_v$, let $\delta_e > 0$, $\tilde{\delta}_e \in \mathbb{R}$ and $c_{e,e'} > 0$ be fixed numbers. Assume that $w_e, \tilde{w}_e$, $e \in E_v$, are functions defined on Ω_+ that belong to $\mathcal{C}^\infty(\Omega_+)$ and satisfy the following hypotheses.

- *For each $e \in E_v$, the functions $w_e, \tilde{w}_e$ are in $\mathcal{C}^{k+\alpha}(\overline{\Omega_+})$ for some integer k and $\alpha \in (0,1)$, and*

$$w_e|_I = c_{e,e'} w_{e'}|_I \in \mathcal{C}^{k+\alpha}(\overline{U}) \quad \text{for all } e, e' \in E_v,$$

- $[\partial_s^2 + \Delta_M] w_e = \tilde{w}_e$ *in* Ω_+.
- *The partial derivatives $\partial_s w_e(0,\cdot)$, $e \in E_v$, whose existence in the sense of distributions in U is guaranteed by the first two hypotheses, satisfy*

$$\sum_e \delta_e \partial_s w_e(0,\cdot) = \sum_e \tilde{\delta}_e w_e(0,\cdot)$$

in the sense of distributions in U.

Then $w_e \in \mathcal{C}^{k+1+\alpha}([0,l) \times U)$ for each $e \in E_v$.

The proof of this result follows exactly the same line as the proof of Proposition 5.12, except for the very last step (bootstrap) that cannot be performed in the present case because of the presence of the functions $\tilde{w}_e$ on the right-hand side of the second condition. This is why we only obtain improved smoothness from $\mathcal{C}^{k+\alpha}$ to $\mathcal{C}^{k+1+\alpha}$.

Step 3: finite order bootstrap. When applying Proposition 5.20 to weak solutions of the heat equation, the function $\tilde{w}_e$ has the form

$$\tilde{w}_e = \frac{\partial_s^2 \beta_e}{\beta_e} w_e + \phi_e \partial_t w_e.$$

In order to apply Proposition 5.20 repeatedly, we need to improve not only the smoothness of w_e but also the smoothness of $\partial_t w_e$. For instance, in order to apply Proposition 5.20 and obtain $\mathcal{C}^{1+\alpha}$-regularity of w_e, we need first to prove that $\partial_t w_e$ is Hölder continuous. Observe that this property immediately follows if we know that the original weak solution u_e of the heat equation is such that $\partial_t u_e$ is also a weak solution of the heat equation.

Assume now that u and all its time derivatives ∂_t^m up to order k are weak solutions of the heat equation in $(0,T) \times \Omega$. Then all the partial derivatives $\partial_t^m u$, $m \in \{0,\dots,k\}$ are Hölder continuous and we can apply Proposition 5.20 simultaneously to all the functions $\partial_t^m w_e$, where $e \in E_v$ and $m \in \{0,1,2,k-1\}$, to conclude that these functions are in $\mathcal{C}^{1+\alpha}$. Using this conclusion, and applying Proposition 5.20 to $\partial_t^m w_e$, where $e \in E_v$ and $m \in \{0,1,2,k-2\}$, we conclude that these functions are in $\mathcal{C}^{2+\alpha}$. Proceeding by finite induction, Theorem 5.19 follows. □

5.21. Definition. Fix $T > 0$ and an open set $\Omega \subset \Gamma\mathsf{M}$ and set $Q = (0,T) \times \Omega$. Let $R_k(Q)$ be the vector space of all weak solutions in $(0,T) \times \Omega$ that are time regular to order k in $(0,T) \times \Omega$, equipped with the seminorms

$$N_{k,Q'}(u) = \sup_{Q'} \sup_{m\in\{0,\dots,k\}} \left|\partial_t^m u(t,\xi)\right| + \sup_{v\in\mathcal{F}_c(Q')} \left| \int_{Q'} v\partial_t^{k+1} u\, d\mu\, dt \right| + \sup_{v\in\mathcal{F}_c(Q')} \left| \int_{Q'} \left(\nabla v, \nabla \partial_t^k u\right) d\mu\, dt \right|,$$

where $Q' = I' \times \Omega'$ is relatively compact in $(0,T) \times \Omega$.

The first term in the seminorm $N_{k,Q'}$ controls the sup-norms (hence the $\mathcal{L}^2$-norms) in Q' of the time derivatives up to order k. Since these functions are weak solutions, this yields a control of the $\mathcal{L}^2$-norms of $|\nabla\partial_t^m u|$ for m up to $k-1$. The last two terms provide the additional control needed to insure that the seminormed space $R_k(Q)$ is complete (a limit in this topology of a sequence of weak solutions that are all time regular up to order k is, itself, such a solution).

5.22. Corollary. *Let $T > 0$, (a', b') a relatively compact interval in $(0,T)$ and $[a,b]$ be a compact interval in (a,b). Let Ω be an open connected set in $\Gamma\mathsf{M}$ and Ω' be a subset that is relatively compact in Ω. Set $Q = (0,T) \times \Omega$, $Q' = (a', b') \times \Omega'$. Let $I \times U$ be a relatively compact coordinate chart in $\Gamma\mathsf{M}$ such that $K = \overline{I \times U} \subset \Omega'$. Fix integers k, κ_*, $\kappa = (\kappa_0, \kappa_1, \dots, \kappa_n)$ with $\kappa_* + \sum_0^n \kappa_i \leqslant k$. Then there exists a constant $C = C(a, a', b, b', \Omega', K, k)$ such that if $u \in R_k(Q)$ is a weak solution of the heat equation in Q, time regular to order k, then we have*

$$\sup\left\{\left|\partial_t^{\kappa_*}\partial_\xi^\kappa u(t,\xi)\right|:\ (t,\xi) \in [a,b] \times K\right\} \leqslant C N_{k,Q'}(u).$$

Applying this to the heat kernel which is a time regular weak solution up to infinite order, we obtain the following important result.

5.23. Theorem. *For any fixed $\zeta \in \Gamma\mathsf{M}$, and integer k, the function $\xi \mapsto \partial_t^k h(t,\xi,\zeta)$ is in $\mathcal{C}^\infty(\Gamma\mathsf{M})$.*

- *Fix a relatively compact coordinate chart $I \times U$ and $\kappa = (\kappa_0, \kappa_1, \dots, \kappa_n)$. Then, for fixed $\xi \in I \times U$, the function*

$$(t,\zeta) \mapsto u(t,\zeta) = \partial_t^k \partial_\xi^\kappa h(t,\xi,\zeta)$$

is in $\mathcal{C}^\infty(\Gamma\mathsf{M})$. It is a weak solution of the heat equation, and it satisfies the bifurcation condition

$$\sum_{e\in E_v} \psi_e(v)(\mathsf{n}_{v,e}, \nabla u) = 0$$

(in the classical sense) along each bifurcation manifold M_v, $v \in V$.
- *Fix a compact time interval $[a,b] \subset (0,\infty)$ and a relatively compact coordinate chart $I \times U$ in $\Gamma\mathsf{M}$ with $\xi_0 \in I \times U$. Fix also integers k and $\kappa_0, \dots, \kappa_n$ and set $\kappa = (\kappa_0, \dots, \kappa_n)$. Then there exists a constant $C = C(a,b,I,U,k,\kappa)$ such that*

$$\sup\left\{\left|\partial_t^k \partial_\xi^\kappa h(t,\xi,\zeta)\right|:\ (t,\xi) \in [a,b] \times I \times U\right\} \leqslant C h(2b, \xi_0, \zeta) \quad \textit{for all } \zeta \in \Gamma\mathsf{M}.$$

6. Projections

Recall the following simple version of transformation of phase space. See [15, Vol. II, Theorem 10.13].

Let X be a separable metrisable space equipped with a Radon measure μ with full support and with a symmetric Markov semigroup $\{H_t\colon t>0\}$ of operators on $\mathcal{L}^2(X)=\mathcal{L}^2(X,\mu)$. Denote also by H_t the extension of that operator from $\mathcal{L}^2(X)\cap\mathcal{L}^\infty(X)$ to $\mathcal{L}^\infty(X)$. Assume that (H_t) admits a transition function $h(t,x,\cdot)$, that is, for any $f\in\mathcal{L}^\infty(X)$ and for all $t>0$ we have $H_tf(x)=\int_X f(y)h(t,x,dy)$ for μ-almost every x. Let $((X_t)_{t\geqslant 0},\mathbb{P}_x)$ be the associated Markov process. In the applications of interest to us here, $X=\Gamma\mathsf{M}$ and the process is the one associated with our Dirichlet form.

Let G be a locally compact group acting properly and continuously on X, and let $\underline{X}$ be the topological quotient space and $\pi: X\to\underline{X}$ the quotient map. Assume that H_t commutes with the action of G, that is, $[H_tf](gx)=H_tf_g(x)$ for all bounded measurable functions f on X, where $f_g(x)=f(gx)$. Then H_t induces a semigroup of contractions $\underline{H}_t:\mathcal{L}^\infty(\underline{X})\to\mathcal{L}^\infty(\underline{X})$ defined by

$$\underline{H}_tf(\underline{x})=[H_tf\circ\pi](x),\quad \text{where } \underline{x}=\pi(x).$$

Moreover, the formula $\underline{X}_t=\pi(X_t)$, $t>0$, defines a Markov process on $\underline{X}$ with law $\mathbb{P}_{\underline{x}}$ satisfying $\mathbb{P}_{\underline{x}}(\underline{X}_t\in A)=\underline{H}_t\mathbf{1}_A(\underline{x})=\mathbb{P}_x[X_t\in\pi^{-1}(A)]$, where $\pi(x)=\underline{x}$. Note that in general there is no obvious natural way to project the $\mathcal{L}^2$-structure onto $\underline{X}$. In particular, in this abstract setting and unless either X or G is compact, there is *a priori* no natural reference measure on $\underline{X}$.

For the purpose of the next theorem, we say that a semigroup $\{P_t\colon t>0\}$ defined on $\mathcal{L}^\infty(X)$ is a *Markov semigroup* if it admits a transition function $p_t(x,f)$ as defined in [15, Vol. I, Ch. 2]. By [15, Vol. I, Theorem 2.1], this is equivalent to say that $\{P_t\colon t>0\}$ can be viewed as a semigroup of contractions on the space $\mathcal{B}(X)$ of all bounded measurable functions on X (not classes of functions!) that preserves positivity and such that $P_0f(x_0)=0$ if $f(x_0)=0$. As for any $t>0$ and $x\in X$, $p_t(x,\cdot)$ is a Borel measure on X, the action of P_t on $\mathcal{L}^\infty(X)$ is determined by its action on $\mathcal{C}_c(X)$.

6.1. Theorem. *Let $\Gamma\mathsf{M}$ and $\Gamma_0\mathsf{M}_0$ be two strip complexes. Assume that there is a locally compact group G that acts continuously and properly on $\Gamma\mathsf{M}$ and such that the quotient of $\Gamma\mathsf{M}$ by G is $\Gamma_0\mathsf{M}_0$ (as topological spaces). Let π be the quotient map. Assume that $\Gamma\mathsf{M}$ is equipped with the data (l,ϕ,ψ) that induce a geometry, measure and a Dirichlet form as discussed in the preceding sections. Let $\{H_t=e^{t\Delta}\colon t>0\}$ be the heat semigroup on $\Gamma\mathsf{M}$ associated with (l,ϕ,ψ).*

Let a Markov semigroup $\{H_{0,t}\colon t>0\}$ acting on $\mathcal{L}^\infty(\Gamma_0\mathsf{M}_0)$ be given that satisfies $\lim_{t\to 0}H_{0,t}\phi=\phi$ for all $\phi\in\mathcal{C}_c(\Gamma_0\mathsf{M}_0)$. Assume the following hypotheses.

(1) *$(\Gamma\mathsf{M},\rho)$ is complete and satisfies the volume condition*

$$\int_1^\infty \frac{r\,dr}{\ln V(\xi_0,r)}=\infty.$$

(2) *H_t commutes with the action of G on $\Gamma\mathsf{M}$.*
(3) *For any bounded function $\phi_0\in\mathcal{C}_c(\Gamma_0\mathsf{M}_0)$, the function $u_0:(0,\infty)\times\Gamma_0\mathsf{M}_0\to\mathbb{R}$ defined by $u_0(t,\xi)=H_{0,t}\phi_0(\xi)$ is such that $u=u_0\circ\pi$ is a weak solution of the heat equation on $(0,T)\times\Gamma\mathsf{M}$.*

Then the semigroup $\{\underline{H}_t\colon\ t>0\}$*, defined on* $\mathcal{L}^\infty(\Gamma_0\mathsf{M}_0)$ *by*

$$\underline{H}_t f(\underline{\xi}) = H_t[f\circ\pi](\xi), \quad \text{where } \pi(\xi)=\underline{\xi},$$

coincides with $H_{0,t}$*. Consequently, if* $(X_t,\mathbb{P}_\xi)$ *and* $(X_{0,t},\mathbb{P}_{0,\xi_0})$ *are the Markov processes associated with* $\{H_t\colon\ t>0\}$ *and* $\{H_{0,t}\colon\ t>0\}$ *on* $\Gamma\mathsf{M}$ *and* $\Gamma_0\mathsf{M}_0$*, respectively, then these processes are related by*

$$\mathbb{P}_{\xi_0}[X_{0,t}\in B] = \mathbb{P}_\xi\big[\pi(X_t)\in B\big], \quad \text{where } \xi_0=\pi(\xi),$$

for any measurable set $B\subset\Gamma_0\mathsf{M}_0$*.*

Proof. Let $\phi_0\in\mathcal{C}_c(\Gamma_0\mathsf{M}_0)$. Define

$$f_{0,t}=H_{0,t}\phi_0, \quad \phi=\phi_0\circ\pi, \quad \text{and} \quad f_t=H_t\phi.$$

It suffices to show that

$$f_t=f_{0,t}\circ\pi.$$

Since $\phi_0\in\mathcal{C}_c(\Gamma_0\mathsf{M}_0)$ and ϕ is a bounded, uniformly continuous function, it is clear that

$$\lim_{t\to 0} f_t(\xi) = \lim_{t\to 0} f_{0,t}\circ\pi(\xi) = \phi(\xi) \quad \text{for all } \xi\in\Gamma\mathsf{M}.$$

We claim that both $u(t,\xi)=f_t(\xi)$ and $\tilde{u}(t,\xi)=f_{0,t}\circ\pi(\xi)$ are weak solutions of the heat equation on $(0,\infty)\times\Gamma\mathsf{M}$. If we can prove this claim, the desired conclusion will follow from Theorem 4.3, that is, from the uniqueness property for the bounded Cauchy problem, because $\underline{H}_t$ and $H_{0,t}$ are determined on $\mathcal{L}^\infty(\Gamma_0\mathsf{M}_0)$ by their action on $\mathcal{C}_c(\Gamma_0\mathsf{M}_0)$. Note that Theorem 4.3 requires completeness of $\Gamma\mathsf{M}$ and the volume growth condition that we are assuming here.

By hypothesis, $(t,\xi)\mapsto\tilde{u}(t,\xi)=f_{0,t}\circ\pi(\xi)$ is a weak solution on $\Gamma\mathsf{M}$. This yields one half of the claim. To prove the other half, we use Theorem 5.23 to see that the bounded function f_t is a weak solution of the heat equation on $\Gamma\mathsf{M}$. Note that this indeed requires some smoothness estimates on the heat kernel on $\Gamma\mathsf{M}$ since f is not in $\mathcal{L}^2(\Gamma\mathsf{M})$. Theorem 5.23 is more than sufficient for this purpose. This yields the claim and completes the proof. □

6.2. Remarks. (A) Given that $\Gamma_0\mathsf{M}_0$ is the quotient of $\Gamma\mathsf{M}$ by a proper continuous group action, Theorem 6.1 is based on three main hypotheses.

- Hypothesis (1) concerns $\Gamma\mathsf{M}$ and its meaning is quite clear: it implies uniqueness for the bounded Cauchy problem for weak solution of the heat equation.
- Hypothesis (2) is also clear. It is satisfied whenever the action of G on $\Gamma\mathsf{M}$ is by measure-adapted isometries.
- Hypothesis (3) is crucial and concerns the relation between the heat equation on $\Gamma\mathsf{M}$ and a certain semigroup on $\Gamma_0\mathsf{M}_0$. This hypothesis captures a huge amount of information, and it is *a priori* not entirely clear whether it is a reasonable hypothesis, or when it can actually be verified. We thus need study it in more detail.

(B) It can occur that a group acts properly and continuously on a strip complex $\Gamma\mathsf{M}$ equipped with data ϕ, ψ in an isometric, measure adapted way, but that the quotient $\Gamma_0\mathsf{M}_0$ cannot be equipped with corresponding data ϕ_0, ψ_0 such that the quotient semigroup equals the semigroup on $(\Gamma_0\mathsf{M}_0, \phi_0, \psi_0)$. The problem comes from the function ψ_0 that defines the underlying measure. Here is an example.

Let $M = \{0\}$ be trivial. Let Γ be $\mathbb{Z}$ with edge lengths 1, so that $\Gamma^1 = \mathbb{R}$, equipped with $\phi \equiv 1$. Fix $\mathsf{q} > 1$ and let ψ be defined by

$$\psi(s) = \mathsf{q}^{k-1}, \quad \text{if } s \in (2k, 2k+2),$$

so that ψ is constant along pairs of edges sharing an odd integer endpoint. Consider the obvious isometric group action by translation by an even integer. This is measure adapted (translation by $2k$ changes the measure by a constant factor of q^k). The quotient of Γ^1 by this group action is the finite metric graph Γ_0^1 with two vertices a, b and two length 1 edges e, f joining a to b. The vertices a and b correspond to even and odd integers, respectively. The problem comes from the following fact.

Assume that there is a function ψ_0 on Γ_0^1 so that the projected semigroup coincides with the semigroup on (Γ_0, ψ_0). On one hand, inspection shows that ψ_0 must be continuous when passing through a and it must have a jump of size q when going through b. On the other hand, ψ_0 must be constant over edges. These two conditions are, of course, incompatible.

To prepare for the next proposition we make the following observations. Let $\Gamma\mathsf{M} = \Gamma^1 \times M$ and $\Gamma_0\mathsf{M}_0 = \Gamma_0^1 \times M_0$ be two strip complexes and G be a locally compact group that acts continuously and properly by isometries on $\Gamma\mathsf{M}$ with quotient $\Gamma_0\mathsf{M}_0$ (as a topological space). Let π be the quotient map. According to our definition (Definition 3.20), isometries must send bifurcation manifolds to bifurcation manifolds and thus send $\Gamma\mathsf{M}^o$ to $\Gamma\mathsf{M}^o$. Hence the action of G on $\Gamma\mathsf{M}$ induces an action of G on the vertex set V of Γ.

Observe further that for any $s \in \Gamma^1$ and $g \in G$, we must have $g(\{s\} \times M) = \{s'\} \times M$ for some $s' \in \Gamma^1$ because for any $\tau, \tau' \in \Gamma^1$ and $x, y \in M$, $\rho((\tau, x), (\tau', x)) = \rho((\tau, y), (\tau', y))$. Indeed, this distance is equal to the minimum of the integral of $\sqrt{\phi}$ along any path in Γ^1 from τ to τ'. Hence, the action of G on $\Gamma\mathsf{M}$ induces an action of G on Γ^1. Moreover, topologically, the quotient of Γ^1 by this action is Γ_0^1. However, in general, it is not true that the quotient of V by the action of G is V_0 because it might be the case that additional vertices and bifurcation manifolds are needed to turn $\Gamma\mathsf{M}/G$ into the strip complex $\Gamma_0\mathsf{M}_0$. This is best explained by two examples:

(1) Take Γ^1 be the natural graph of $\mathbb{Z}$ ($\equiv \mathbb{R}$ with the integers marked as vertices), $M = \{0\}$, and $G = \mathbb{Z}$ acting by translation. Then the quotient is the circle with one marked point. This is not a strip complex (as a strip complex is required to have no loop) and we need to choose a second marked point to turn it into a strip complex.

(2) Take Γ^1 as in (1) and $G = \{e, \sigma\}$ where e is the identity and σ is the reflexion with respect to $-1/2$. The quotient is a half line with marking at $1/2$ and at the positive integers. To turn this into a strip complex, we need to add a vertex at the origin of the half line.

Fortunately, this difficulty (in the two examples above and in the general case) is solved by adding "dummy" middle vertices and corresponding bifurcation manifolds in every strip $S_e^o \in \Gamma\mathsf{M}$, $e \in E$. This yields a new strip complex $\Gamma\mathsf{M}'$ (isometric with $\Gamma\mathsf{M}$ as metric spaces, and equivalent with $\Gamma\mathsf{M}$ for all analytic purposes) with the same manifold M but the new graph Γ' obtained by subdividing each edge of Γ into two new edges with a new vertex in the middle.

Furthermore, the action of G on $\Gamma\mathsf{M}'$ (resp. $(\Gamma')^1$) is such that if M_v and M_w are two bifurcation manifolds (resp. v, w are two vertices) in the same orbit under the action of G then the pair $\{v, w\}$ cannot be the pair of extremities of an edge e in E'. It follows that $(\Gamma')^1/G$ is naturally a metric graph with vertex set $V_0' = V/G$ and with no loops. Therefore, there is no loss of generality in assuming that $\Gamma_0^1 = (\Gamma')^1/G$.

Consequently, without loss of generality, we can assume that π induces a natural graph homomorphism of Γ onto Γ_0. The latter will also be denoted by π, so that we can speak about the vertices and edges $\pi(v)$ and $\pi(e)$ of Γ_0, where $v \in V$ and $e \in E$, respectively.

Consider a pair of open strips $S^o \subset \Gamma\mathsf{M}$, $S_0^o \subset \Gamma_0\mathsf{M}_0$ with $\pi(S^o) = S_0^o$. Let

$$G_{S^o} = \{g \in G\colon\ g(S^o) = S^o\}/\{g \in G\colon\ g|_{S^o} = id\}$$

be the effective quotient for the action of G on S^o. Since any $g \in G$ such that $g\xi \in S^o$ for some $\xi \in S^o$ must send S^o to S^o, it follows that $\pi(S^o) = S_0^o$ is also the (topological) quotient of S^o by the action of G_{S^o} (see, e.g., Bourbaki [10, I.23]), and for any function u_0 on $\Gamma_0\mathsf{M}_0$, we have

$$u_0 \circ \pi|_{S^o} = u_0|_{S_0^o} \circ \pi^{S^o} \tag{6.3}$$

where π^{S^o} is the projection map from S^o to S_0^o.

Note that G_{S^o} acts by isometries on the manifold S^o. In what follows we will assume that G_{S^o} is a Lie subgroup of the group of isometries of S^o and that

$$\pi^{S^o} : \big(S^o = I \times M,\ \phi\big((ds)^2 + g(\cdot,\cdot)\big)\big) \to \big(S_0^o = I_0 \times M_0, \phi_0\big((d\tau)^2 + g_0(\cdot,\cdot)\big)\big)$$

is a Riemannian submersion. This implies that the action of G_{S^o} on S^0 is free. Moreover, π^{S^o} sends any set of the form $\{s\} \times M$ to some set of the form $\{\tau\} \times M_0$ and, for any $f_0 \in \mathcal{C}^\infty(S_0^o)$ and any $(s, x) \in S^o$ with $\pi^{S^o}(s, x) = (\tau, x_0)$, we have

$$\frac{1}{\phi(s)}\big|\partial_s f_0 \circ \pi^{S^o}(s, x)\big|^2 = \frac{1}{\phi_0(\tau)}\big|\partial_\tau f_0(\tau, x_0)\big|^2 \tag{6.4}$$

and

$$\begin{aligned}&\frac{1}{\phi(s)}\big[\partial_s^2 + \Delta_{Ms} + \big[\partial_s \log \phi(s)^{(n-1)/2}\big]\partial_s\big] f_0 \circ \pi^{S^o}(s, x)\\ &\quad = \frac{1}{\phi_0(\tau)}\big[\partial_\tau^2 + \Delta_{M_0} + \big[\partial_\tau \log \phi_0(\tau)^{(n-1)/2}\big]\partial_\tau\big] f_0(\tau, x_0).\end{aligned} \tag{6.5}$$

This follows from the fundamental property of a Riemannian submersion and the fact that the expressions in (6.5) are the Laplace operators of the relevant Riemannian metrics. Observe that the weight functions ψ and ψ_0 do not appear in this formula.

6.6. Proposition. *Let $\Gamma\mathsf{M}$ and $\Gamma_0\mathsf{M}_0$ be two strip complexes. Assume that there is a locally compact group G that acts continuously and properly on $\Gamma\mathsf{M}$ and such that the quotient of $\Gamma\mathsf{M}$ by G is $\Gamma_0\mathsf{M}_0$. Let π be the quotient map. Assume that $\Gamma\mathsf{M}$ and $\Gamma_0\mathsf{M}_0$ are equipped with the data (ϕ, ψ) and (ϕ_0, ψ_0), respectively, that induce a geometry, measure and a respective Dirichlet form as discussed above. Assume furthermore that the following hypotheses are satisfied.*

(1) *G acts on $\Gamma\mathsf{M}$ by isometries and Γ_0 is the quotient of Γ under the induced action of G.*
(2) *For any edge $e \in E$, the group $G_{S_e^o}$ is a Lie subgroup of the isometry group of S_e^o, the projection map $\pi^{S_e^o}$ is a Riemannian submersion of S_e^o onto $S_0^o = \pi(S_e^o) \subset \Gamma_0\mathsf{M}_0$, and*
(3) *there exists a constant $A(e) \in (0, \infty)$ such that*

$$\psi_e(s) = A(e)\psi_0|_{S_0^o}(\tau)$$

for any s, τ such that $\pi^{S^o}(s, x) = (\tau, x_0)$ for some $x \in M$ and $x_0 \in M_0$.
(4) *For any pair of vertices $v \in V$ and $v_0 \in V_0$ such that $\pi(M_v) = M_{0,v_0}$, there exists a constant $a(v) \in (0, \infty)$ such that*

$$\sum_{e\in E_v:\ \pi(e)=e_0} \psi_e(v) = a(v)\psi_{0,e_0}(v_0) \quad \textit{for all } e_0 \in E_{v_0}.$$

Then, for any $T > 0$ and any function $u_0 \in \mathcal{C}^\infty((0, T) \times \Gamma_0\mathsf{M}_0)$ which is a time regular weak solution of the heat equation on $(0, T) \times \Gamma_0\mathsf{M}_0$, the function $u = u_0 \circ \pi$ is a time regular weak solution of the heat equation on $(0, T) \times \Gamma\mathsf{M}$.

Proof. Because of (6.3) and assumption (2), $u = u_0 \circ \pi$ and its time derivatives $\partial_t^k u$ are in $\mathcal{C}^\infty(\Gamma\mathsf{M})$. For such a function, being a weak solution of the heat equation means:

$$\begin{cases} \partial_t u = \mathfrak{A}u = \dfrac{1}{\phi(s)}\big[\partial_s^2 + \Delta_M + \eta\partial_s\big]u = 0, \quad \text{where } \eta = \partial_s \ln\big(\phi^{(n-1)/2}\psi\big), \\ \displaystyle\sum_{e\in E_v} \psi_e(v)(\mathsf{n}_{v,e}, \nabla u_e) = 0 \quad \text{along } M_v \quad \text{for all } v \in V. \end{cases}$$

That u satisfies the first of those two identities follows by careful inspection using (6.3), assumption (2), (6.5) and assumption (3). The second line identity follows similarly from (6.3), assumption (2), (6.4) and assumption (4). □

6.7. Example. Let $\Gamma\mathsf{M}$ be a strip complex equipped with the data ϕ and ψ. Assume that the isometry group G of (M, g) acts transitively on M. This group also acts on $\Gamma\mathsf{M}$ in an obvious way, and this action is measure adapted (in fact, measure preserving) and isometric.

The quotient of $\Gamma\mathsf{M}$ by this action is the 1-dimensional complex Γ^1. For each open strip S^o, G_{S^o} is isomorphic with G itself, and assumption (2) of Proposition 6.6 is obviously satisfied. Assumptions (3)–(4) of Proposition 6.6 are satisfied if we equip Γ^1 with the data ϕ, $A_0\psi$, where A_0 is any fixed positive constant.

The same applies if G is a subgroup of the isometry group that acts freely and properly on M with quotient M_0. Then there exists a unique Riemannian structure on M_0 that makes the quotient map a Riemannian submersion. The quotient of $\Gamma\mathsf{M}$ under the natural action of G is $\Gamma_0\mathsf{M}_0$ with $\Gamma_0 = \Gamma$. For each open strip S^o, the group G_{S^o} is again isomorphic to G itself, and assumption (2) of Proposition 6.6 is obviously satisfied. Assumptions (3)–(4) of Proposition 6.6 are satisfied if we equip $\Gamma_0\mathsf{M}_0$ with the data ϕ and $A_0\psi$ for any fixed positive constant A_0.

6.8. Example. Let $\Gamma\mathsf{M}$ be a strip complex. Assume that G is a subgroup of the group of automorphisms of the non-oriented version of the graph Γ. By adding dummy vertices in the middle of edges if necessary, we can assume that the quotient $\Gamma_0 = \Gamma/G$ has no loops. For any Riemannian manifold $M = M_0$, the group G has a natural action on $\mathsf{\Gamma M}$ with quotient $\mathsf{\Gamma_0 M_0} = \Gamma_0 \times M_0 = \Gamma_0 \times M$. Let π be the quotient map from Γ^1 to Γ_0^1. In particular, π maps the edge set of Γ onto the edge set of Γ_0. Fix data ϕ_0 and ψ_0 on $\mathsf{\Gamma_0 M_0}$ and equip $\mathsf{\Gamma M}$ with $\phi = \phi_0 \circ \pi$. Then G acts on $\mathsf{\Gamma M}$ by isometries. Next, we consider the conditions (3)–(4) of Proposition 6.6. Condition (3) involves numbers $A(e) > 0$, $e \in E$, such that $\psi_e = A(e)\psi_{0,e_0} \circ \pi|_{S_e^o}$. Given that condition (3) is satisfied, condition (4) requires that

$$\sum_{e \in E_v:\, \pi(e)=e_0} A(e) = a(v) \quad \text{for all } v \in V,\ e_0 \in E_{\pi(v)}.$$

Let us examine some special cases.

(A) First, assume that for any vertex v of Γ, we have $\deg_\Gamma(v) = \deg_{\Gamma_0}(\pi(v))$. Then the restriction of π from E_v to $E_{\pi(v)}$ is bijective, or in other words, π is a graph covering. In this case, the above condition means that $A(e) = A(e')$ if the edges e and e' have a common end vertex. Since our graphs are connected, this actually implies that $A(e) = A$ is a constant, that is, $\psi = A \cdot \psi_0 \circ \pi$.

(B) Second, consider the specific example where $\Gamma = \mathbb{T}_2$ is the regular tree with degree 3, drawn with respect to a reference end ϖ as in Fig. 2. The graph Γ_0 is the two-way-infinite path, which we denote by $\mathbb{Z}$ (which is, more precisely, the vertex set of Γ_0, while the associated 1-complex is $\mathbb{R}$). The group G is the group of all graph automorphisms of the tree that fix every horocycle, and the projection is $\pi = \mathfrak{h}$, the Busemann function with respect to ϖ. Here, the projection is obviously not a graph covering. For simplicity, we assume that all edges have length 1 and that ϕ, $\phi_0 \equiv 1$. Furthermore, we assume that ψ_0 is constant on each edge of $\Gamma_0 = \mathbb{Z}$. Recall that in this specific example, every vertex v has one neighbouring vertex v^- in the "preceding" horocycle and is itself the predecessor of its "forward" neighbours w_1, w_2 that satisfy $w_i^- = v$. (This notation should not be mixed up with the one for the endpoints e^- and e^+ of an edge e.) If $\mathfrak{h}(v) = k$ then $e_v = [v^-, v]$ is the only edge in E_v that projects onto the edge $[k-1, k]$ of $\mathbb{Z}$. Therefore $A(e_v) = a(v)$. On the other hand, both edges e_{w_1} and e_{w_2} project onto the edge $[k, k+1]$ of $\mathbb{Z}$. Therefore the above condition can be rewritten in terms of the positive function $v \mapsto a(v)$. In order to be feasible, it is necessary and sufficient that it satisfies $a(w_1) + a(w_2) = a(v)$ for any vertex v of $\mathbb{T}$, where the w_i are its forward neighbours. Because $\mathbb{T}_2$ is a tree, we can construct infinitely many functions that satisfy this property, and hence there are infinitely many functions ψ, constant on open edges, so that conditions (3) and (4) are satisfied, whenever the function ψ_0 is chosen to have constant value, say b_k, on each open strip $(k-1, k) \times M$ of $\mathsf{\Gamma_0 M_0}$. One solution for ψ is given by

$$\psi|_{S_e^o} \equiv 2^{-k} b_k, \quad \text{when } \pi(e) = [k-1, k].$$

This is the only solution for which the corresponding group action is measure adapted.

(C) Consider the situation described in Theorem 2.23 concerning various projections of $\mathsf{HT(p,q)}$. The hypotheses (1) and (2) of Theorem 6.1 are verified, and hypothesis (3) is also satisfied because of Proposition 6.6. Hence Theorem 2.23 follows from Theorem 6.1 and Proposition 6.6. Note that Proposition 6.6 makes heavy use of the results of Section 5. Further related uniqueness theorems are given in the next two sections.

7. Uniqueness of the heat semigroup

Throughout this section, we use the basic setting of a strip complex with data, distance, measure, Dirichlet form, Laplacian and heat semigroup as already specified at the beginning of Section 4. Our aim is to show that, in some strong sense, there is only one semigroup of operators whose generator coincides with the Laplacian Δ on a certain space of smooth compactly supported functions. This property is important in many applications. We will discuss two different uniqueness results: one concerns uniqueness on $\mathcal{C}_0(\Gamma\mathsf{M})$, whereas the other concerns uniqueness on $\mathcal{L}^2(\Gamma\mathsf{M})$.

A. A candidate for a core of the infinitesimal generator. In this section we introduce a very specific space, $\mathcal{D}_c^\infty$, of compactly supported smooth functions on $\Gamma\mathsf{M}$ that is a good candidate to be a core for the generator of the heat semigroup, either on $\mathcal{C}_0(\Gamma\mathsf{M})$ or on $\mathcal{L}^2(\Gamma\mathsf{M})$. In some cases, we will be able to show that $\mathcal{D}_c^\infty$ is indeed a core. Please note that the spaces $\mathcal{D}^\infty$ and $\mathcal{D}_c^\infty$ introduced below depend on the fixed data (l, ϕ, ψ) on $\Gamma\mathsf{M}$.

7.1. Definition. The space $\mathcal{D}^\infty$ is the space of all functions f in $\mathcal{C}^\infty(\Gamma\mathsf{M})$ such that

(1) For any integer $k = 0, 1, \ldots,$ any $v \in V$ and $e, e' \in E_v$,

$$\mathrm{Tr}_{M_v}^{S_e}\big(\mathfrak{A}^k f\big) = \mathrm{Tr}_{M_v}^{S_{e'}}\big(\mathfrak{A}^k f\big).$$

This means that the functions $\mathfrak{A}^k f$, originally only defined and continuous on $\Gamma\mathsf{M}^o$, are in fact continuous functions on $\Gamma\mathsf{M}$ (after proper extension by continuity) and thus in $\mathcal{C}^\infty(\Gamma\mathsf{M})$.

(2) For any integer $k = 0, 1, \ldots,$ and $v \in V$

$$\sum_{e \in E_v} \psi_e(v)\big(\mathsf{n}_{v,e}, \nabla \mathfrak{A}^k f_e\big) = 0 \quad \text{along } M_v.$$

This means that each function $\mathfrak{A}^k f \in \mathcal{C}^\infty(\Gamma\mathsf{M})$ satisfies the bifurcation condition along any bifurcation manifold M_v, $v \in V$.

The space $\mathcal{D}_c^\infty$ is the subspace of all compactly supported functions in $\mathcal{D}^\infty$.

7.2. Remark. Fix a coordinate chart $(U; x_1, \ldots, x_n)$ in M. Observe that any function f in $\mathcal{C}^\infty(\Gamma\mathsf{M})$ viewed as a function of $(s, x) \in \Gamma^1 \times U$ actually has continuous partial derivatives of all orders $\partial_x^\kappa f(s, x)$ in the x direction, but not in the s direction in general. It follows that the continuity condition on $\mathfrak{A} f$ reduces to the continuity of

$$\partial_s^2 f + \eta(s)\partial_s f$$

across any bifurcation manifold M_v. The bifurcation condition implies that, typically, the function $\partial_s f$ is not continuous across bifurcation manifolds. It follows that, typically, $\partial_s^2 f$ is not continuous and neither are $\partial_s^k f$, $k \geqslant 3$. An important consequence of this is that $\mathcal{D}^\infty$ and $\mathcal{D}_c^\infty$ are not algebras under pointwise multiplication.

7.3. Remark. Note that $\mathcal{D}_c^\infty$ is a subspace of $\mathrm{Dom}(\Delta^k)$ for every $k \geqslant 1$, and

$$\Delta^k = \mathfrak{A}^k \quad \text{on } \mathcal{D}_c^\infty.$$

7.4. Lemma. *The space $\mathcal{D}_c^\infty$ is dense in $\mathcal{C}_0(\Gamma\mathsf{M})$ for the uniform topology.*

Proof. This important result is an immediate corollary of Lemma 3.10 since we have $\mathcal{C}_{c,c}^\infty(\Gamma\mathsf{M}) \subset \mathcal{D}_c^\infty$. Indeed, $\mathcal{C}_{c,c}^\infty(\Gamma\mathsf{M})$ is the subspace of those functions f in $\mathcal{C}_c^\infty(\Gamma\mathsf{M})$ whose partial derivative $\partial_s f$ along Γ^1 vanishes in a neighbourhood of any bifurcation manifold. The desired inclusion thus follows from Remark 7.2 above. □

The following is a simple corollary of Theorem 5.23.

7.5. Theorem. *For every fixed $t > 0$, $\zeta \in \Gamma\mathsf{M}$, $k = 0, 1, \ldots$, every relatively compact coordinate chart $I \times U \ni \zeta$ in $\Gamma\mathsf{M}$ and $\kappa = (\kappa_0, \kappa_1, \ldots, \kappa_n)$, the function*

$$\xi \mapsto \partial_t^k \partial_\zeta^\kappa h(t, \zeta, \xi)$$

belongs to $\mathcal{D}^\infty$.

B. Uniqueness of the heat semigroup on $\mathcal{C}_0(\Gamma\mathsf{M})$. Consider the operator $(\mathfrak{A}, \mathcal{D}_c^\infty)$ as a linear, densely defined operator on $\mathcal{C}_0(\Gamma\mathsf{M})$. Recall that indeed, $\mathcal{C}_c^\infty$ is dense in $\mathcal{C}_0(\Gamma\mathsf{M})$ for the uniform topology, see Lemma 7.4. We claim that $(\mathfrak{A}, \mathcal{D}_c^\infty)$ satisfies the positive maximum principle. That is, if $\xi_0 \in \Gamma\mathsf{M}$ and $f \in \mathcal{D}_c^\infty$ are such that $\max_{\Gamma\mathsf{M}}\{f\} = f(\xi_0) \geqslant 0$, then $\mathfrak{A} f(\xi_0) \leqslant 0$.

Indeed, if ξ_0 is not on a bifurcation manifold, this follows from the usual maximum principle. If $\xi_0 = (v_0, x_0)$ is on a bifurcation manifold, let $(U; x_1, \ldots, x_n)$ be a local coordinate chart in M around x_0. Since $f \in \mathcal{D}_c^\infty$ is maximal at ξ_0, the first order partial derivatives at ξ_0 along M must be 0 and we must have $\partial_{x_i}^2 f(\xi_0) \leqslant 0$, $i = 1, \ldots, n$. It follows that $\Delta_M f(\xi_0) \leqslant 0$.

Moreover, in any strip S_e containing $\xi_0 = (v_0, x_0)$, the outward normal derivatives $(\mathsf{n}_{v,e}, \nabla f_e(\xi_0))$ must be greater or equal to 0. Hence, the bifurcation condition implies that $(\mathsf{n}_{v,e}, \nabla f_e(\xi_0)) = 0$. It follows that in any strip $S_e = I_e \times M$ around ξ_0, we must have $\partial_s^2 f_e(\xi_0) \leqslant 0$. Hence

$$\mathfrak{A} f(\xi_0) = \frac{1}{\phi(\xi_0)} \left[\partial_s^2 f(\xi_0) + \Delta_M f(\xi_0)\right] \leqslant 0.$$

Without further assumption on $\Gamma\mathsf{M}$, we do not know how to show that $(\mathfrak{A}, \mathcal{D}_c^\infty)$ admits an extension that is the infinitesimal generator of a contraction semigroup on $\mathcal{C}_0(\Gamma\mathsf{M})$. The difficulty lies in proving that the range $(\lambda Id - \mathfrak{A})\mathcal{D}_c^\infty$ is dense in $\mathcal{C}_0(\Gamma\mathsf{M})$ for some $\lambda > 0$, that is, that $(\mathfrak{A}, \mathcal{D}_c^\infty)$ is closable in $\mathcal{C}_0(\Gamma\mathsf{M})$. However, by the results of van Casteren and Okitaloshima [36, 26], we have the following [26, Theorem 3.6 and Proposition 3.7]: if $(\mathfrak{A}, \mathcal{D}_c^\infty)$ is closable, then its closure is the only linear extension of $(\mathfrak{A}, \mathcal{D}_c^\infty)$ that is the infinitesimal generator of a Feller semigroup (that is, a strongly continuous semigroup of contractions on $\mathcal{C}_0(\Gamma\mathsf{M})$ preserving positivity). This, together with Theorem 4.4, yields the following result.

7.6. Theorem. *Let $\Gamma\mathsf{M}$ be a strip complex equipped with a geometry and measure as above. Let $h(t, \xi, \zeta)$ be the heat kernel associated with the Dirichlet form $(\mathcal{E}, W_0^1(\Gamma\mathsf{M}))$, where $(t, \xi, \zeta) \in$*

$(0,\infty)\times\Gamma\mathsf{M}\times\Gamma\mathsf{M}$. *Assume that* $(\Gamma\mathsf{M},\rho)$ *is complete and that there are constants* D, P, r_0 *such that* (i) *and* (ii) *hold.*

(i) *For any* $\xi\in\Gamma\mathsf{M}$ *and* $r\in(0,r_0)$, *we have the doubling property* $V(\xi,r)\leqslant DV(\xi,2r)$.
(ii) *For any* $\xi\in\Gamma\mathsf{M}$ *and* $r\in(0,r_0)$, *setting* $B=B(\xi,r)$,

$$\int\limits_B |f-f_B|^2\,d\mu\leqslant Pr^2\int\limits_B|\nabla f|^2\,d\mu\quad\text{for every } f\in\mathcal{W}^1(B),\quad\text{where } f_B=\frac{1}{\mu(B)}\int\limits_B f\,d\mu.$$

Then the densely defined linear operator $(\mathfrak{A},\mathcal{D}_c^\infty)$ *on* $\mathcal{C}_0(\Gamma\mathsf{M})$ *is closable and its closure* $(\overline{\mathfrak{A}},\mathrm{Dom}(\overline{\mathfrak{A}}))$ *is the infinitesimal generator of the Feller semigroup defined by*

$$\mathcal{C}_0(\Gamma\mathsf{M})\ni f\mapsto e^{t\overline{\mathfrak{A}}}f,\quad t>0,\quad\text{where } e^{t\overline{\mathfrak{A}}}f(\xi)=\int\limits_{\Gamma\mathsf{M}}h(t,\xi,\zeta)f(\zeta)\,d\mu(\zeta).$$

Moreover, if $(\widetilde{\mathfrak{A}},\mathrm{Dom}(\widetilde{\mathfrak{A}}))$ *is an extension of* $(\mathfrak{A},\mathcal{D}_c^\infty)$ *and is the infinitesimal generator of a Feller semigroup then* $(\widetilde{\mathfrak{A}},\mathrm{Dom}(\widetilde{\mathfrak{A}}))=(\overline{\mathfrak{A}},\mathrm{Dom}(\overline{\mathfrak{A}}))$.

7.7. Remark. It follows from the results in [36] and [26] that, under the hypotheses of Theorem 7.6, the martingale problem for the operator $(\mathfrak{A},\mathcal{D}_c^\infty)$ is uniquely solvable (for any starting point $\xi\in\Gamma\mathsf{M}$). See [26, Theorem 3.6].

C. Uniqueness of the heat semigroup on $\mathcal{L}^2(\Gamma\mathsf{M})$. Let us observe that, because of the possibility to impose various boundary conditions, uniqueness on $\mathcal{L}^2(\Gamma\mathsf{M})$ cannot hold unless we make the assumption that $(\Gamma\mathsf{M},\rho)$ is complete.

7.8. Definition. We say that a continuous function $\rho_0:\Gamma\mathsf{M}\to(0,\infty)$ is a strip-adapted exhaustion function if it has the following properties.

- The function ρ_0 belongs to $\mathcal{C}^\infty(\Gamma\mathsf{M})$.
- For any edge $e\in E$ and any $x\in M$ the function $s\mapsto\partial_s\rho_e(s,x)$ has compact support in (e^-,e^+).
- The function ρ_0 tends to infinity at infinity.
- The functions $|\nabla\rho_0|$ and $|\mathfrak{A}\rho_0|$ are bounded on $\Gamma\mathsf{M}$.

Note that a strip-adapted exhaustion function is a continuous smooth function on $\Gamma\mathsf{M}$ which is locally constant in the direction of Γ^1 near each bifurcation manifold. The existence of such exhaustion functions is a non-trivial matter that will be discussed in Section 8.

7.9. Definition. A sequence of continuous compactly supported functions ϱ_n is called a strip-adapted approximation of $\mathbf{1}$ if the following holds.

- Each ϱ_n belongs to $\mathcal{C}^\infty_{c,c}(\Gamma\mathsf{M})$.
- Each ϱ_n takes values in $[0,1]$, and $\lim_{n\to\infty}\varrho_n(\xi)=1$ for all $\xi\in\Gamma\mathsf{M}$.

- There is a constant C such that $|\nabla \varrho_n| \leqslant C$, $|\Delta \varrho_n| \leqslant C$, and for all $\xi \in \Gamma\mathsf{M}$,

$$\lim_{n\to\infty} \left|\nabla \varrho_n(\xi)\right| = \lim_{n\to\infty} \left|\Delta \varrho_n(\xi)\right| = 0.$$

7.10. Remarks. (a) If a strip-adapted exhaustion function ρ_0 exists then a strip-adapted approximation of $\mathbf{1}$ is easily obtained by setting $\varrho_n(\xi) = \theta(\rho_0(\xi)/n)$, where θ is a smooth, compactly supported function of *one* variable taking value in $[0, 1]$ and such that $\theta \equiv 1$ in a neighbourhood of 0.

(b) Let ϱ_n be a strip-adapted approximation of $\mathbf{1}$. Then $\varrho_n f \in \mathcal{D}_c^\infty$ for any $f \in \mathcal{D}^\infty$. Compare this with the fact that, in general, $\varrho \in \mathcal{D}_c^\infty$ and $f \in \mathcal{D}^\infty$ does not imply $\varrho f \in \mathcal{D}_c^\infty$.

7.11. Theorem. *The operator $(\mathfrak{A}, \mathcal{D}_c^\infty)$ is symmetric on $\mathcal{L}^2(\Gamma\mathsf{M})$. If $(\Gamma\mathsf{M}, \rho)$ is complete and there exists a strip-adapted approximation of $\mathbf{1}$ then the symmetric operator $(\mathfrak{A}, \mathcal{D}_c^\infty)$ is essentially self-adjoint on $\mathcal{L}^2(\Gamma\mathsf{M})$, and its unique self-adjoint extension is $(\Delta, \mathrm{Dom}(\Delta))$.*

7.12. Remark. When considering Theorem 7.11, the reader should recall that the relevant underlying data include the graph $\Gamma = (V, E)$, the Riemannian manifold (M, g), the function $\phi \in \mathcal{C}^\infty(\Gamma^1)$ which is part of the definition of the geometry on $\Gamma\mathsf{M}$ and plays a crucial role on whether $(\Gamma\mathsf{M}, \rho)$ is complete or not, as well as the function $\psi \in \mathcal{S}^\infty(\Gamma^o)$ which appears in the Definition 3.22 of the underlying measure μ. Indeed, $\mathcal{L}^2(\Gamma\mathsf{M})$ is the $\mathcal{L}^2$-space relative to that specific measure μ. It is interesting to observe how these different parameters enter the definition of Δ and that of $\mathfrak{A}$. Concerning $\mathfrak{A}$, the functions ϕ and ψ appear in the formula defining $\mathfrak{A}$ on each open strip. However, the possible jump discontinuities of ϕ and/or ψ only appear in the definition of $\mathcal{D}_c^\infty$ via the bifurcation condition. This clearly shows that one cannot replace $\mathcal{D}_c^\infty$ by $\mathcal{C}_{c,c}^\infty(\Gamma\mathsf{M})$ in Theorem 7.11 because then the role of the possible jumps of the functions ϕ and ψ is lost.

The proof of Theorem 7.11 requires a number of lemmas. The symmetry of $(\mathfrak{A}, \mathcal{D}_c^\infty)$ on $\mathcal{L}^2(\Gamma\mathsf{M})$ follows from the various definitions by inspection. Let $(\mathfrak{A}^*, \mathrm{Dom}(\mathfrak{A}^*))$ be the adjoint of $(\mathfrak{A}, \mathcal{D}_c^\infty)$.

7.13. Lemma. *For any function $f \in \mathcal{D}^\infty \cap \mathrm{Dom}(\mathfrak{A}^*)$, one has $\mathfrak{A}^* f = \mathfrak{A} f$ in $\mathcal{L}^2(\Gamma\mathsf{M})$.*

Proof. By definition, for any $f \in \mathrm{Dom}(\mathfrak{A}^*)$ and $h \in \mathcal{D}_c^\infty$, we have

$$\langle \mathfrak{A}^* f, h \rangle = \langle f, \mathfrak{A}h \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{L}^2(\Gamma\mathsf{M})$. But for $f \in \mathcal{D}^\infty$ and $h \in \mathcal{D}_c^\infty$, Green's formula in each strip and the bifurcation conditions imposed on f and h yield that

$$\langle f, \mathfrak{A}h \rangle = \langle \mathfrak{A} f, h \rangle.$$

This proves the desired result. $\square$

7.14. Lemma. *Let $f \in \mathrm{Dom}(\mathfrak{A}^*)$, $h \in \mathcal{D}^\infty$, and suppose that $h, \mathfrak{A}h \in \mathcal{L}^2(\Gamma\mathsf{M})$. Then*

$$\langle \mathfrak{A}^* f, h \rangle = \langle f, \mathfrak{A}h \rangle.$$

Proof. Consider the sequence $h_n = \varrho_n h$, where ϱ_n is a strip-adapted approximation of $\mathbf{1}$. Then $h_n \in \mathcal{D}_c^\infty$ and $h_n \to h$ as well as $\mathfrak{A}h_n \to \mathfrak{A}h$ in $\mathcal{L}^2(\Gamma\mathsf{M})$. Hence the desired equality follows from the fact that $\langle \mathfrak{A}^* f, h_n\rangle = \langle f, \mathfrak{A}h_n\rangle$. □

7.15. Lemma. *For any function $f \in \mathrm{Dom}(\mathfrak{A}^*)$ and $t > 0$, the function $f_t = e^{t\Delta} f$ is in $\mathrm{Dom}(\mathfrak{A}^*)$ and*

$$\mathfrak{A}^* f_t = e^{t\Delta}\mathfrak{A}^* f.$$

Proof. For any h in $\mathcal{L}^2(\Gamma\mathsf{M})$, the function $h_t = e^{t\Delta}h$ is a global weak solution of the heat equation that is time regular to infinite order. By Theorem 5.19 this implies that $h_t \in \mathcal{D}^\infty$. Obviously, h_t and $\mathfrak{A}h_t = \Delta h_t$ are also in $\mathcal{L}^2(\Gamma\mathsf{M})$. Now, for $f \in \mathrm{Dom}(\mathfrak{A}^*)$ and $h \in \mathcal{D}_c^\infty$, we have

$$\left\langle e^{t\Delta}\mathfrak{A}^* f, h\right\rangle = \left\langle \mathfrak{A}^* f, e^{t\Delta}h\right\rangle.$$

Since $h_t = e^{t\Delta}h$ is in $\mathcal{D}^\infty$ and both h_t and $\mathfrak{A}h_t$ are in $\mathcal{L}^2(\Gamma\mathsf{M})$, Lemma 7.14 gives

$$\begin{aligned}\left\langle e^{t\Delta}\mathfrak{A}^* f, h\right\rangle &= \left\langle \mathfrak{A}^* f, h_t\right\rangle = \langle f, \mathfrak{A}h_t\rangle = \left\langle f, \Delta e^{t\Delta}h\right\rangle \\ &= \left\langle f, e^{t\Delta}\Delta h\right\rangle = \left\langle e^{t\Delta} f, \mathfrak{A}h\right\rangle = \langle f_t, \mathfrak{A}h\rangle.\end{aligned}$$

This proves that $\mathfrak{A}^* f_t = e^{t\Delta}\mathfrak{A}^* f$ as desired. □

The next lemma will complete the proof of Theorem 7.11.

7.16. Lemma. *$\mathcal{D}_c^\infty$ is dense in $\mathrm{Dom}(\mathfrak{A}^*)$ in the graph norm.*

Proof. Approximate $f \in \mathrm{Dom}(\mathfrak{A}^*)$ by $f_t = e^{t\Delta} f$, where $t \to 0$. Then f_t converges to f in $\mathcal{L}^2(\Gamma\mathsf{M})$ and, by Lemma 7.15, $\mathfrak{A}^* f_t$ also converges to $\mathfrak{A}^* f$ in $\mathcal{L}^2(\Gamma\mathsf{M})$. This shows that $\mathcal{D}^\infty \cap \mathrm{Dom}(\mathfrak{A}^*)$ is dense in $\mathrm{Dom}(\mathfrak{A}^*)$ in the graph norm. Now, we use multiplication by the strip-adapted sequence ϱ_n that approximates $\mathbf{1}$ and set $h_n = f_{1/n}\varrho_n$ to obtain the desired conclusion. □

7.17. Remark. Assume that $M = \{0\}$ is a singleton, so that $(\Gamma\mathsf{M}, \phi, \psi)$ reduces to the metric graph Γ^1 equipped with the data ϕ, ψ. Assume that (Γ^1, ρ) is complete. In this case, the symmetric operator $(\mathfrak{A}, \mathcal{D}_c^\infty)$ is always essentially self-adjoint on $\mathcal{L}^2(\Gamma^1, \mu)$. This is proved in [5] following the argument used for complete Riemannian manifolds by Strichartz [32]. It is not clear that this argument can be adapted to the case when $M \neq \{0\}$. The difficulty lies in showing that any solution $f \in \mathrm{Dom}(\mathfrak{A}^*)$ of the equation $\mathfrak{A}^* f = \lambda f$ is in fact in $\mathcal{W}^1_{\mathrm{loc}}(\Gamma\mathsf{M})$. On a manifold, this follows from local ellipticity. On a graph, it can be checked by an adhoc argument using very much the 1-dimensional nature of the underlying space. See [5].

8. Strip-adapted approximations of 1

Unfortunately, the existence of a strip-adapted approximation of $\mathbf{1}$ is a difficult question in full generality. Even in the case of complete Riemannian manifolds, an adapted approximation of $\mathbf{1}$ is not known to exist in general. The proof of the essential self-adjointness of the Laplacian (see, e.g., [32]) on a complete Riemannian manifold has to avoid the use of an adapted approximation of **1** and, instead, makes use of the fact that the adjoint is an elliptic operator in the sense of distributions. See Remark 7.17 regarding the graph case. Whether or not that approach can be made to work in the present setting is not clear, the main question being whether or not one can prove that

$$\mathrm{Dom}(\mathfrak{A}^*) \subset \mathcal{W}^1_{\mathrm{loc}}(\Gamma\mathsf{M}).$$

This appears to be a rather subtle question although one would conjecture that the answer is "yes".

In this section we construct strip-adapted exhaustion functions (or strip-adapted approximations of **1**) in a number of different special cases. We start with some simple-minded constructions.

8.1. Proposition. *Assume that (M, g) is a complete Riemannian manifold which admits an adapted approximation $(\varrho_{M,n})$ of* **1**. *Assume that the underlying metric graph Γ satisfies*

$$l_* = \inf_E \{l_e\} > 0,$$

that is, edge lengths are bounded below. Assume that $\Gamma\mathsf{M}$ is equipped with its bare strip complex structure, that is, $\phi \equiv 1$ and $\psi \equiv 1$. Then $\Gamma\mathsf{M}$ admits a strip-adapted approximation of 1.

Proof. Let us first construct an edge-adapted exhaustion $s \mapsto \rho_1(s)$ on the one-dimensional complex Γ^1. (Here, the strips are the edges, so that we use "edge-adapted" instead of "strip-adapted".) Fix $\epsilon \in (0, l_*/8)$. On Γ^1, consider a function $\alpha \in \mathcal{C}^\infty(\Gamma^1)$ with the property that for each edge e, the restriction α_e of α to (e^-, e^+) has compact support in $(e^- + \epsilon, e^+ - \epsilon)$, is equal to 1 in $(e^- + 2\epsilon, e^+ - 2\epsilon)$, and satisfies $\sup_{\Gamma^1} |\partial_s \alpha| \leqslant C$. Such a function obviously exists because of the hypothesis $l_* > 0$. Fix an origin vertex v_0 and, minimizing over all paths of the form $\gamma : [0, a] \to \Gamma^1$ from v_0 to $s \in \Gamma^1$, parametrized by arclength, set

$$\rho_*(s) = \min_\gamma \lambda(\gamma), \quad \text{where } \lambda(\gamma) = \int_0^a \alpha\big(\gamma(\tau)\big)\, d\tau.$$

Observe that the function ρ_* tends to infinity at infinity and that it is constant in a neighbourhood of any vertex v. If we had $\rho_* \in \mathcal{C}^\infty(\Gamma^1)$, it would thus be a good candidate for an edge-adapted exhaustion function.

However, this function is not smooth at points s in the interior of an edge (e^-, e^+) with the property that there are two minimizing paths γ_1 and γ_2, one passing through e^-, the other through e^+ and such that ρ_* is not constant in a neighbourhood of s. Observe that in this case, s is a point of local maximum for ρ_*, and $\rho_*(s) \geqslant \max\{\rho_*(e^-), \rho_*(e^+)\}$. It follows that such an edge is never used by minimizing paths except those ending within the edge itself. Thus, changing α along such an edge has no effect on the values of ρ_* elsewhere. Assume without loss of generality that $\rho_*(e^-) \leqslant \rho_*(e^+)$ and replace α_e by a smaller smooth function $\tilde{\alpha}_e$ satisfying $|\partial_s \tilde{\alpha}_e| \leqslant C$ and such that $\int_{e^-}^{e^+} \tilde{\alpha}_e(s)\, ds = \rho_*(e^+) - \rho_*(e^-)$. We can do this along any of those "bad" edges. Globally, this defines a new function $\tilde{\alpha}$, and using the same notation as above, we set

$$\rho_1(s) = \min_{\gamma} \tilde{\lambda}(\gamma), \quad \text{where } \tilde{\lambda}(\gamma) = \int_0^a \tilde{\alpha}\big(\gamma(\tau)\big)\, d\tau.$$

By construction, we have $\rho_1 = \rho_*$ except on edges where $\alpha_e \neq \tilde{\alpha}_e$. In particular, $\rho_1 = \rho_*$ on vertices. Moreover, $\partial_s \rho_1$ has compact support within every open edge. Clearly, ρ_1 tends to infinity at infinity (along with ρ_*) and satisfies $|\partial_s \rho_1| \leqslant 1$ and $|\partial_s^2 \rho_1| \leqslant C$. That is, ρ_1 is an edge-adapted exhaustion function. As explained in Remark 7.10(a), this yields an edge-adapted approximation of $\mathbf{1}$, say $\varrho_{1,n}$, on Γ^1. A strip-adapted approximation of $\mathbf{1}$ on $\Gamma\mathsf{M}$ is obtained by setting

$$\varrho_n(\xi) = \varrho_{1,n}(s)\varrho_{M,n}(x), \quad \xi = (s, x) \in \Gamma\mathsf{M}. \qquad \square$$

8.2. Remark. The conditions $\phi \equiv 1$, $\psi \equiv 1$ can be relaxed to

$$\inf \phi > 0 \quad \text{and} \quad \sup\big|\partial_s \ln\big(\phi^{(n-1)/2}\psi\big)\big| < \infty.$$

Our next result deals with the treebolic spaces $\mathsf{HT}(\mathsf{p}, \mathsf{q})$.

8.3. Proposition. *The treebolic space* $\mathsf{HT}(\mathsf{p}, \mathsf{q})$ *equipped with* ϕ, ψ *as in Example* 3.26 *admits a strip-adapted exhaustion.*

Proof. We will use freely the notation introduced in Section 2. First we construct a smooth function $\eta : (0, \infty) \to (0, \infty)$ such that $\eta \equiv 1$ on $(1 - 1/(8\mathsf{q}), 1 + \mathsf{q}/8)$ and $\eta(\mathsf{q}^k y) = \mathsf{q}^k \eta(y)$ for all $k \in \mathbb{Z}$. Obviously there is a $C > 0$ such that this function satisfies

$$C^{-1} \leqslant y\eta(y) \leqslant C, \quad \sup_{y>0}\big|\eta'(y)\big| \leqslant C, \qquad \sup_{y>0}\big\{y\big|\eta''(y)\big|\big\} \leqslant C.$$

As a first step, consider the case $\mathsf{p} = 1$, $\mathsf{q} > 1$ where $\mathsf{HT}(1, \mathsf{q})$ is the upper half-space with the horizontal lines $\{z = x + \mathfrak{i}y\colon\ y = \mathsf{q}^k\}$ marked as bifurcation lines. Consider the function

$$\delta(z) = \log\left(1 + \frac{1 + x^2 + y^2}{y}\right).$$

Away from the point $\mathfrak{i}$, this is comparable with the hyperbolic distance between z and the point $\mathfrak{i}$. Computing partial derivatives, one easily checks that $y^2(|\partial_x \delta(z)|^2 + |\partial_y \delta(z)|^2) \leqslant C_1$ and $y^2(|\partial_x^2 \delta(z)| + |\partial_y^2 \delta(z)|) \leqslant C_1$ for some $C_1 > 0$. In particular, δ has bounded hyperbolic gradient and bounded hyperbolic Laplacian. Set

$$\rho(z) = \delta\big(x + \mathfrak{i}\eta(y)\big).$$

Then it is not hard to check that ρ is a strip-adapted exhaustion function on $\mathsf{HT}(1, \mathsf{q})$. The role of η is to make ρ constant in y along the lines $\{y = \mathsf{q}^k\}$.

Let us now consider the general case $\mathsf{HT}(\mathsf{p}, \mathsf{q})$, $\mathsf{p} \geqslant 1$, $\mathsf{q} > 1$. Recall that $\mathsf{HT}(\mathsf{p}, \mathsf{q}) = \{(z, w) \in \mathbb{H} \times \mathbb{T}_{\mathsf{p}}\colon\ \mathfrak{h}(w) = \log_{\mathsf{q}} y\}$. Hence, we can consider ρ as a function on $\mathsf{HT}(\mathsf{p}, \mathsf{q})$ by setting $\rho(z, w) =$

$\rho(z)$. This function satisfies all requirements for a strip-adapted exhaustion function, except that it does not tend to ∞ along any fixed horocyclic level $\mathfrak{h}(w) = \mathsf{q}^k$.

To treat this difficulty, fix an end $\mathfrak{u}_0 \in \partial^* \mathbb{T}$. Let $V(\mathfrak{u}_0)$ be the set of all vertices $v \in \mathbb{T}_\mathsf{p}^0$ such that $v \in \overline{\mathfrak{u}_0\, \varpi}$. For any $v \in V(\mathfrak{u}_0)$, let $\mathbf{T}(v)$ be the set of those elements $w \in \mathbb{T}_\mathsf{p}$ such that $w \curlywedge \mathfrak{u}_0 = v$. This set $\mathbf{T}(v)$ is the maximal subtree of $\mathbb{T}_\mathsf{p}$ containing v and intersecting $\overline{\mathfrak{u}_0\, \varpi}$ only at v. The tree $\mathbb{T}_\mathsf{p}$ is the disjoint union

$$\mathbb{T}_\mathsf{p} = \overline{\mathfrak{u}_0 \varpi} \cup \Bigg(\bigcup_{v \in \mathbb{T}_\mathsf{p}^0 \cap \overline{\mathfrak{u}_0\, \varpi}} \mathbf{T}(v) \setminus \{v\} \Bigg),$$

where (recall) $\overline{\mathfrak{u}_0 \varpi}$ is the geodesic between $\mathfrak{u}_0$ and ϖ. By construction, we have $\mathfrak{h}(w) \geqslant \mathfrak{h}(v)$ if $w \in \mathbf{T}(v)$. Thus, for $(z, w) \in \mathsf{HT}(\mathsf{p}, \mathsf{q})$ with $z = x + \mathrm{i}y$ and $w \in \mathbf{T}(v)$, we have $y \geqslant \mathsf{q}^{\mathfrak{h}(v)}$.

We define a function κ on $\mathsf{HT}(\mathsf{p}, \mathsf{q})$ by setting

$$\kappa(z, w) = \begin{cases} 0, & \text{if } w \in \overline{\mathfrak{u}_0 \varpi}, \\ \log(\eta(\mathsf{q}^{-\mathfrak{h}(v)} y)), & \text{if } w \in \mathbf{T}(v). \end{cases}$$

This function κ has the property that it tends to infinity on $\mathsf{HT}(\mathsf{p}, \mathsf{q})$ when its argument (z, w) escapes to infinity along a fixed horocycle $\{(z, w) \in \mathsf{HT}(\mathsf{p}, \mathsf{q})\colon \log_\mathsf{q}(y) = \mathfrak{h}(w) = t\}$, $t \in \mathbb{R}$. This is because, as (z, w) escapes to infinity with $\log_\mathsf{q}(y) = \mathfrak{h}(w) = t$, the vertex $v = v(w) \in \overline{\mathfrak{u}_0 \omega}$ such that $w \in \mathbf{T}(v)$ must tend to ϖ and thus $\mathfrak{h}(v)$ tends to $-\infty$.

Now, we set

$$\rho_1 : \mathsf{HT}(\mathsf{p}, \mathsf{q}) \to (0, \infty), \qquad (z, w) \mapsto \rho_1(z, w) = \rho(z) + \kappa(z, w).$$

From the construction, it is clear that ρ_1 is a strip-adapted exhaustion function. $\square$

Appendix A. Some results concerning $\sqrt{-\Delta_M}$

Let (M, g) be a Riemannian manifold (equipped with its Riemannian measure dx) and let Δ_M be its Laplacian defined on $\mathcal{C}_c^\infty(M)$. Abusing notation, we let Δ_M denote also its Friedrichs extension. Let $h_M(t, x, y)$ be the heat kernel (the smooth positive integral kernel of $e^{t\Delta_M}$) and let $\sqrt{-\Delta_M}$ be defined by spectral theory, that is, $\sqrt{-\Delta_M} = \int_0^\infty \sqrt{\lambda}\, dE_\lambda$, where E_λ is a spectral resolution of $-\Delta_M$. The domain of $\sqrt{-\Delta_M}$ is the Sobolev space $\mathcal{W}_0^1(M) = \mathcal{W}_0^1$.

Let $\mathcal{W}_0^\alpha$ be the dual of $\mathcal{W}_0^{-\alpha}$ (under the identification of $\mathcal{L}^2(M)$ with its own dual). Hence, for $\alpha > \beta > 0$, we have

$$\mathcal{W}_0^\alpha \subset \mathcal{W}_0^\beta \subset \mathcal{L}^2(M) \subset \mathcal{W}_0^{-\beta} \subset \mathcal{W}_0^{-\alpha}.$$

The intersection $\mathcal{W}_0^\infty = \bigcap_\alpha \mathcal{W}_0^\alpha$ is dense in any $\mathcal{W}_0^\alpha$, and the operator $(Id + \sqrt{-\Delta_M}\,)^\gamma$, initially defined on $\mathcal{W}_0^\infty$, extends as a unitary operator from $\mathcal{W}_0^\alpha$ to $\mathcal{W}_0^{\alpha-\gamma}$. Moreover,

$$(Id + \sqrt{-\Delta_M}\,)^\alpha (Id + \sqrt{-\Delta_M}\,)^\beta = (Id + \sqrt{-\Delta_M}\,)^{\alpha+\beta}, \quad \alpha, \beta \in \mathbb{R},$$

and $(Id + \sqrt{-\Delta_M}\,)^0 = Id$. Because $\mathcal{C}_c^\infty(M) \subset \mathcal{W}_0^\infty$ (with continuous embedding when equipped with their natural families of seminorms), it is clear that any $\mathcal{W}^\alpha$ can be understood as a space of distributions.

On $\mathcal{L}^2(M)$, the operator $(Id + \sqrt{-\Delta_M})^{-1}$ has an integral kernel given by

$$G(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} h_M\big(t^2/4u, x, y\big)\, du\, dt. \tag{A.1}$$

It obviously satisfies

$$\int_M G(x,y)\, dy = \int_M G(x,y)\, dx \leqslant 1.$$

It follows that, for any $f \in \mathcal{C}_c^\infty(M)$, we have

$$\begin{aligned}
\sqrt{-\Delta_M} f &= f + (-Id + \sqrt{-\Delta_M})f \\
&= f + (Id + \sqrt{-\Delta_M})^{-1}(Id + \sqrt{-\Delta_M})(-Id + \sqrt{-\Delta_M})f \\
&= f + (Id + \sqrt{-\Delta_M})^{-1}[-f - \Delta_M f] \\
&\in \mathcal{L}^1(M) \cap \mathcal{L}^\infty(M).
\end{aligned}$$

Let now $f \in \mathcal{L}^1(M) + \mathcal{L}^\infty(M)$. The previous observation implies that we can make sense of $\sqrt{-\Delta_M} f$ explicitly as a distribution on M by setting,

$$[\sqrt{-\Delta_M} f](h) = \int_M f[\sqrt{-\Delta_M}\, h]\, dx \quad \text{for } h \in \mathcal{C}_c^\infty(M).$$

By (A.1) and the local regularity of the heat kernel, for any fixed precompact compact coordinate chart $(U; x_1, \ldots, x_n)$ in M and any open set $\Omega \supset \overline{U}$, we have

$$\forall y \in M \setminus \Omega, \quad \sup_{x \in U} \big|\partial_x^m G(x,y)\big| \leqslant C_{U,\Omega,m} \inf_{x \in U} G(x,y) \quad \text{for all } y \in M \setminus \Omega, \tag{A.2}$$

where $m = (m_1, \ldots, m_2)$ and $\partial_x^m f = \partial_{x_1}^{m_1} \ldots \partial_{x_n}^{m_n} f$. Furthermore, if $(U'; y_1, \ldots, y_n)$ is a relatively compact coordinate chart with $U' \subset M \setminus \overline{\Omega}$ then

$$\sup_{x \in U} \sup_{y \in U'} \big|\partial_x^m \partial_y^k G(x,y)\big| \leqslant C_{U,U',m,k}. \tag{A.3}$$

We need the following simple hypoellipticity type result. It is certainly well known but it does not seem very easy to find a precise reference. (See e.g. Bogdan and Byczkowski [9], where (M,g) is Euclidean space.) In particular, note that some care is needed because $\sqrt{-\Delta_M}$ is not a local operator.

A.4. Theorem. *Let $f \in \mathcal{L}^2(M)$ and let F be the distribution $F = (Id + \sqrt{-\Delta_M})f$. Fix two open relatively compact sets $\Omega \subset \Omega' \subset M$ with $\overline{\Omega} \subset \Omega'$. Assume that*

- *$F = 0$ in Ω, that is, $F(u) = 0$ for all $u \in \mathcal{C}_c^\infty(\Omega)$, and*

- *$F|_{X\setminus\Omega'} \in \mathcal{L}^2(M)$, that is, there exists $h \in \mathcal{L}^2(M)$ such that*

$$F(u) = \int_M hu\,dx \quad \text{for all } u \in \mathcal{C}_c^\infty\big(M \setminus \overline{\Omega'}\big).$$

Then $f \in \mathcal{C}_{\mathrm{loc}}^\infty(\Omega)$.

Proof. Without loss of generality, we can assume that $h = 0$ in a neighbourhood of $\overline{\Omega}$. It then follows easily from (A.2) that

$$(I + \sqrt{-\Delta_M})^{-1}h = Gh \in \mathcal{C}_{\mathrm{loc}}^\infty(\Omega).$$

Next, for any two open sets Ω_0, Ω_1 with $\overline{\Omega_0} \subset \Omega_1$ and $\overline{\Omega_1} \subset \Omega$, and any relatively compact neighbourhood Ω_2 of $\overline{\Omega'}$, the distribution $F - h$ is supported in $\Omega_2 \setminus \overline{\Omega_1}$. We can approximate this distribution by functions in $F_j \in \mathcal{C}_c^\infty(M)$ supported in $\Omega_2 \setminus \overline{\Omega_1}$ and such that there exist a constant C, an integer l, and a finite covering of $K = \overline{\Omega_2} \setminus \Omega_1$ by relatively compact charts $(U^i, x_1^i, \ldots, x_n^i)$, $i \in I$, such that for all j

$$\int_M F_j u\,dm \leqslant C \sup\Big\{\big|\partial_{x^i}^k u(x)\big|: x \in U^i,\ i \in I,\ k = (k_1, \ldots, k_n) \text{ with } \sum k_i \leqslant l\Big\}.$$

It then follows from (A.3) that, given any local chart $(U; x_1, \ldots, x_n)$ contained in Ω_0 and any integer m, the functions $(Id + \sqrt{-\Delta_M})^{-1}F_j = GF_j$ satisfy

$$\sup_j \sup\Big\{\big|\partial_x^m GF_j(x)\big|: x \in U,\ m = (m_1, \ldots, m_n),\ \sum m_i \leqslant m\Big\} \leqslant C.$$

This implies that the limit distribution $(Id + \sqrt{-\Delta_M})^{-1}(F - h) = \lim_j GF_j$ can be represented by a smooth function in Ω_0. Hence,

$$f = (Id + \sqrt{-\Delta_M})^{-1}F = (Id + \sqrt{-\Delta_M})^{-1}h + (Id + \sqrt{-\Delta_M})^{-1}(F - h)$$

satisfies

$$f|_\Omega = \big[(Id + \sqrt{-\Delta_M})^{-1}F\big]\big|_\Omega \in \mathcal{C}_{\mathrm{loc}}^\infty(\Omega).$$

This concludes the proof. □

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