A NOTE ON POINCARÉ, SOBOLEV, AND HARNACK INEQUALITIES

L. SALOFF-COSTE

1. Introduction. Let M be a $\mathscr{C}^\infty$-connected manifold. Let L be a second-order differential operator with real $\mathscr{C}^\infty$ coefficients on M and such that $L1 = 0$ (i.e., L has no zero-order term). Assume that there exists a positive $\mathscr{C}^\infty$ measure μ on M such that

$$\langle L\varphi, \psi \rangle = \langle \varphi, L\psi \rangle, \qquad \langle L\psi, \psi \rangle \geqslant 0$$

for all $\varphi, \psi \in \mathscr{C}_0^\infty(M)$, where $\langle\ ,\ \rangle$ is the scalar product on $L^2(M, d\mu)$. We make the technical hypothesis that L is locally subelliptic. Denote also by L the Friedrichs extension of L in $L^2(M, d\mu)$ and consider the symmetric submarkovian semigroup $H_t = e^{-tL}$ acting on the spaces $L^2(M, d\mu)$. The $\mathscr{C}^\infty$ kernel $h_t(x, y)$ of H_t is defined by

$$H_t f(x) = \int_M h_t(x, y) f(y)\, d\mu(y).$$

Since we assume that L is locally subelliptic, there exists a genuine distance function ρ canonically associated with L; see [6, 9]. This distance is continuous and defines the topology of M. We assume that (M, ρ) is a complete metric space. Set $B(x, r) = \{y \in M, \rho(x, y) < r\}$ and $V(x, r) = \mu(B(x, r))$. There is also a notion of gradient associated with L. At any rate, we can set

$$\Gamma(\varphi, \psi) = \tfrac{1}{2}(-L(\varphi\psi) + \varphi L\psi + \psi L\varphi)$$

and define the "length of the gradient" to be $|\nabla f| = \Gamma(f, f)^{1/2}$. ($\Gamma(f, f)$ is the "carré du champ" of Bakry-Emery [1].) See also [26] for an equivalent definition of $|\nabla f|$. It can be shown (see [3, 9] for instance) that, under our hypotheses,

$$\rho(x, y) = \sup\{|f(x) - f(y)|, f \in \mathscr{C}^\infty(M), |\nabla f| \leqslant 1\}.$$

What is really important for us is that, although ρ is not smooth, we can formally apply the inequality $|\nabla \rho(x_0, x)| \leqslant 1$.

Given $0 < r_0 \leqslant +\infty$, consider the two properties

$$V(x, 2r) \leqslant C_1 V(x, r), \qquad 0 < r < r_0, \qquad x \in M \tag{1}$$

Received 18 November 1991.
Communicated by Michael Christ.

and

$$\int_{B(x,r)} |f - f_{x,r}|^2 \, d\mu \leqslant C_2 r^2 \int_{B(x,2r)} |\nabla f|^2 \, d\mu, \quad 0 < r < r_0, \quad x \in M, \quad f \in \mathscr{C}^\infty(M) \tag{2}$$

where $f_{x,r} = \int_{B(x,r)} f \, d\mu$. Property (1) is the usual doubling property. The inequality appearing in property (2) is a (weak) form of Poincaré inequality. It follows from the work of D. Jerison [8] that (1) and (2) imply the (strong) Poincaré inequality where the integral on the right-hand side of (2) is taken over the ball $B(x, r)$ instead of $B(x, 2r)$.

In this paper we show that a parabolic Harnack inequality is equivalent to the above two properties (see Section 3). In [15, 16], J. Moser proved a Harnack inequality for parabolic equations associated with second-order uniformly elliptic divergence form operators in Euclidean space. His approach has been used in many other situations because it rests only on two functional inequalities usually referred to as Sobolev and Poincaré inequalities. Here, we show that the doubling property (1) and the family of Poincaré inequalities (2) imply a family of Sobolev inequalities which is good enough to run Moser's iteration. It is well known that Harnack inequality is a powerful tool. Selected applications are presented which illustrate this fact.

One aspect of this work is that it unifies important results which were obtained in different settings by different means. For instance, consider the question whether or not harmonic positive functions are constant. S.-T. Yau proved that the answer is yes on manifolds with nonnegative Ricci curvature (here, L is the Laplace operator); see [27]. Y. Guivarc'h in [7], T. Lyons and D. Sullivan in [13] also gave a positive answer for manifold which are normal covering of a compact manifold with nilpotent deck transformation group. In [24], N. Varopoulos obtained a similar result in the setting of Lie groups having polynomial volume growth. As explained in the last section, all these results can be seen as corollaries of Theorem 4.3 below.

2. Sobolev inequality. In this section we show that (1) and (2) imply a family of Sobolev inequalities on balls.

THEOREM 2.1. *Assume that M, L, are as above and that (1), (2), hold for some fixed $r_0 > 0$. Then there exist $\nu > 2$ and $C_3 > 0$ depending only on C_1, C_2 such that*

$$\left(\int |f|^{2\nu/(\nu-2)} \, d\mu\right)^{(\nu-2)/\nu} \leqslant C_3 V(x, r)^{-2/\nu} r^2 \left(\int (|\nabla f|^2 + r^{-2}|f|^2) \, d\mu\right),$$

$$f \in \mathscr{C}_0^\infty(B(x, r))$$

for all $x \in M$ and all $0 < r < r_0$.

Note that for $0 < s \leqslant r$ we have

$$V(x, r) \leqslant 2V(x, s)(r/s)^{\nu_0} \tag{3}$$

for some $\nu_0 > 0$ depending only on C_1, C_2. Indeed, consider the integer n such that $2^{n-1} < r/s \leqslant 2^n$. From the doubling property it follows that

$$V(x, r) \leqslant V(x, 2^n s) \leqslant C_1^n V(x, s) \leqslant 2V(x, s)(r/s)^{\nu_0}$$

where $\nu_0 = \log(C_1)/\log(2)$. The real ν appearing in Theorem 2.1 can be taken to be any number greater or equal to ν_0 and strictly greater than 2. The first ingredient of the proof of Theorem 2.1 is an abstract result.

THEOREM 2.2. *Let e^{-tA} be a symmetric submarkovian semigroup acting on the spaces $L^p(M, d\mu)$. Given $\nu > 2$, the three following properties are equivalent.*

1. $\|e^{-tA}f\|_\infty \leqslant C_4 t^{-\nu/2}\|f\|_1$ *for* $0 < t < t_0$.
2. $\|f\|^2_{2\nu/(\nu-2)} \leqslant C_5(\|A^{1/2}f\|_2^2 + t_0^{-1}\|f\|_2^2)$.
3. $\|f\|_2^{2+4/\nu} \leqslant C_6(\|A^{1/2}f\|_2^2 + t_0^{-1}\|f\|_2^2)\|f\|_1^{4/\nu}$.

Moreover, 3. *implies* 1. *with* $C_4 = (\nu C C_6)^{\nu/2}$ *and* 1. *implies* 2. *with* $C_5 = C C_4^{2/\nu}$, *where* C *is some numerical constant.*

The proof of 1. implies 2. follows easily from [22], Theorem 1. The equivalence with 3. follows from [3]. The other ingredients in the proof of Theorem 2.1 are the two following lemmas. Denote by $f_s(x)$ the mean of f over the ball $B(x, s)$. Set $\chi_s(x, z) = V(x, s)^{-1}1_{B(x,s)}(z)$ so that

$$f_s(x) = \int \chi_s(x, z) f(z)\, d\mu(z).$$

LEMMA 2.3. *There exists a constant C_7 depending only of C_1 such that for all $y \in M$ and all $0 < s \leqslant r < r_0$ we have*

$$\|f_s\|_2 \leqslant C_7 V^{-1/2}(r/s)^{\nu_0/2}\|f\|_1, \qquad \text{for all } f \in \mathscr{C}_0^\infty(B)$$

where $B = B(y, r)$, $V = V(y, r)$.

Proof. Note that $\chi_s(x, z) \leqslant C_1\chi_s(z, x)$. This shows that $\|f_s\|_1 \leqslant C_1\|f\|_1$. Moreover, if $B \cap B(x, s) \neq \varnothing$ with $0 < s \leqslant r$, (3) yields

$$V(x, s)^{-1} \leqslant 2V(x, 2r + s)^{-1}(2r/s + 1)^{\nu_0} \leqslant V^{-1}(4r/s)^{\nu_0}.$$

Hence, $\|f\|_\infty \leqslant V^{-1}(4r/s)^{\nu_0}\|f\|_1$ for all $f \in \mathscr{C}_0^\infty(B)$. The lemma follows by interpolation.

LEMMA 2.4. *There exists C_8 depending only on C_1, C_2, such that*

$$\|f - f_s\|_2 \leqslant C_8 s \|\nabla f\|_2, \qquad f \in \mathscr{C}_0^\infty(M)$$

for all $0 < s < r_0/4$.

Proof. Fix $0 < s < r_0/4$. Let $\{B_j, j \in J\}$ be a collection of balls of radius $s/2$ such that $B_i \cap B_j = \varnothing$ if $i \neq j$ and $M = \bigcup_{i \in J} 2B_i$, where $tB = B(x, tr)$ if $B = B(x, r)$. Such a collection always exists. Moreover, the doubling property implies that the overlapping number $N(z) = \#\{i \in J, z \in 8B_i\}$ is bounded by a number N_0 depending only on C_1. Now, write

$$\|f - f_s\|_2^2 \leqslant \sum_{i \in J} \int_{2B_i} |f(x) - f_s(x)|^2 \leqslant \sum_{i \in J} \left(\int_{2B_i} |f(x) - f_{4B_i}|^2 + |f_{4B_i} - f_s(x)|^2 \right)$$

where all the integrations are taken with respect to μ and where f_B is the mean of f over the ball B. Poincaré inequality (2) implies

$$\int_{2B_i} |f(x) - f_{4B_i}|^2 \leqslant \int_{4B_i} |f(x) - f_{4B_i}|^2 \leqslant C_2 s^2 \int_{8B_i} |\nabla f|^2.$$

Using (1) and (2), we also have

$$\begin{aligned}\int_{2B_i} |f_{4B_i} - f_s(x)|^2 &\leqslant \int_{2B_i} \int \chi_s(x, z) |f_{4B_i} - f(z)|^2 \, d\mu(z) \, d\mu(x) \\ &\leqslant C_9 V_i^{-1} \int_{2B_i} \int_{4B_i} |f_{4B_i} - f(z)|^2 \, d\mu(z) \, d\mu(x) \\ &\leqslant C_{10} s^2 \int_{8B_i} |\nabla f|^2.\end{aligned}$$

Hence, we obtain

$$\|f - f_s\|_2^2 \leqslant C_{11} s^2 \sum_{i \in J} \int_{8B_i} |\nabla f|^2 \leqslant C_{11} N_0 s \|\nabla f\|_2^2.$$

This ends the proof of Lemma 2.4.

Proof of Theorem 2.1. Fix $x \in M$, $0 < r < r_0$, and set $v = \max\{3, v_0\}$. Assume that $0 < s \leqslant r/4$ and $f \in \mathscr{C}_0^\infty(B(x, r))$. Following an idea of Robinson [17], write

$$\|f\|_2 \leqslant \|f - f_s\|_2 + \|f_s\|_2.$$

Using the above two lemmas, we obtain

$$\|f\|_2 \leqslant C_8 s \|\nabla f\|_2 + C_7 V^{-1/2}(r/s)^{\nu/2} \|f\|_1$$

where $V = V(x, r)$. Hence, for all $s > 0$ and $f \in \mathscr{C}_0^\infty(B(x, r))$, we have

$$\|f\|_2 \leqslant 4C_8 s(\|\nabla f\|_2 + r^{-1}\|f\|_2) + C_7 V^{-1/2}(r/s)^{\nu/2}\|f\|_1 .$$

Optimizing over $s > 0$ yields

$$\|f\|_2^{2+4/\nu} \leqslant C_{12} V^{-2/\nu} r^2 (\|\nabla f\|_2^2 + r^{-2}\|f\|_2^2)\|f\|_1^{4/\nu}.$$

Theorem 2.1 follows from the above and Theorem 2.2. In [5], Th. Coulhon and the author use variations of the above arguments to study isoperimetric questions on Riemannian manifolds. In the present setting, the method of [5] shows that (1) and the L^1 version of (2), namely

$$\int_{B(x,r)} |f - f_r(x)| \leqslant C_2' r \int_{B(x,2r)} |\nabla f|, \qquad 0 < r < r_0, \qquad x \in M, \qquad f \in \mathscr{C}^\infty(M)$$

imply the L^1 version of Theorem 2.1 which reads

$$\left(\int |f|^{\nu/(\nu-1)}\, d\mu\right)^{(\nu-1)/\nu} \leqslant C_3' V(x, r)^{-1} r \left(\int (|\nabla f| + r^{-1}|f|)\, d\mu\right), \qquad f \in \mathscr{C}_0^\infty(B(x, r))$$

for all $x \in M$ and all $0 < r < r_0$.

3. Harnack inequality. The power of properties (1) and (2) is better understood through the result presented below. Indeed, we show in this section that the conjunction of (1) and (2) is equivalent to a parabolic Harnack inequality.

THEOREM 3.1. *Let M and L be as in Section* 1. *The following two properties are equivalent.*

1. *The properties* (1) *and* (2) *hold for M, L, and some $r_0 > 0$.*
2. *There exists $r_1 > 0$, and there exists a constant C depending only on the parameters $0 < \varepsilon < \eta < \delta < 1$, such that, for any $x \in M$, any real s, and any $0 < r < r_1$, any nonnegative solution u of $(\partial_t + L)u = 0$ in $Q =]s - r^2, s[\times B(x, r)$ satisfies*

$$\sup_{Q_-} \{u\} \leqslant C \inf_{Q_+} \{u\}$$

where $Q_- = [s - \delta r^2, s - \eta r^2] \times B(x, \delta r)$ and $Q_+ = [s - \varepsilon r^2, s[\times B(x, \delta r)$.

Proof of 1. *implies* 2. This part of the theorem follows from Moser's iteration: assuming that (1) and (2) hold, Theorem 2.1 yields the family of Sobolev inequalities

$$\|f\|_{2\nu/(\nu-2)}^2 \leqslant CV^{-2/\nu}r^2(\|\nabla f\|_2^2 + r^{-2}\|f\|_2^2),$$

$$f \in \mathscr{C}_0^\infty(B(x, r)), \qquad y \in M, \qquad 0 < r < r_0.$$

As explained in [20] in a Riemannian setting, such a family of Sobolev inequalities is enough to run the first part of Moser's iteration. Hence, we have (see [15]) the following theorem.

THEOREM 3.2. *Assume that* (1), (2), *hold for some* $r_0 > 0$. *Given* $0 < \delta < 1$, *there exists a constant C depending on* C_1, C_2, *and* δ, *such that, for any* $x \in M$, *any real s, and any* $0 < r < r_0$, *any nonnegative solution of* $(\partial_t + L)u \leqslant 0$ *in* $Q =]s - r^2, s[\times B(x, r)$ *satisfies*

$$\sup_{Q_\delta} \{u^2\} \leqslant C(r^2V)^{-1} \int_Q u^2$$

where $Q_\delta =]s - \delta r^2, s[\times B(x, \delta r)$.

In order to obtain the full Harnack inequality stated in Theorem 3.1, we first note that the technique presented in [21] applies here and allows us to deduce from (1) and (2) the following weighted form of Poincaré inequality. Set $\Phi_{x,r}(z) = (1 - \rho(x, z)/r)^2$ for $z \in B(x, r)$ and $\Phi_{x,r}(z) = 0$ otherwise. Also, set $\tilde{f}_r(x) = \int f\Phi_{x,r}$. We have

$$\int |f - \tilde{f}_r(x)|^2 \Phi_{x,r} \leqslant Cr^2 \int |\nabla f|^2 \Phi_{x,r}$$

for all $x \in M$, $0 < r < r_0$ and $f \in \mathscr{C}^\infty(M)$. Once we have such a weighted Poincaré inequality, we can prove statement 2. of Theorem 3.1 by using Moser's technique; see [15, 16, 20].

Proof of 2. *implies* 1. First, we show that 2. implies the doubling property of the volume. Recall that h_t is the kernel of $H_t = e^{-tL}$. Applying 2. to h_t, we obtain

$$V(x, r)h_{r^2}(x, x) \leqslant C \int_{B(x,r)} h_{2r^2}(x, y)\, d\mu(y) \leqslant C.$$

Consider now the function defined by $u(s, z) = H_s 1_{B(x,r)}(z)$ when $s > 0$, and $u(s, z) = 1$ when $s \leqslant 0$. This function is a nonnegative solution of $(\partial_t + L)u = 0$ in $]-\infty, +\infty[\times B(x, r)$. Hence, we have

$$1 = u(-r^2/4, x) \leqslant Cu(r^2/2, x) = \int_{B(x,r)} h_{r^2/2}(x, y)\, d\mu(y) \leqslant C^2 V(x, r)h_{r^2}(x, x).$$

The above yields

$$(C'V(x, r))^{-1} \leqslant h_{r^2}(x, x) \leqslant C'V(x, r)^{-1}.$$

Hence, 2. implies that $V(x, 2r) \leqslant C''V(x, r)$ for all $x \in M$ and all $0 < r < r_1/2$.

The fact that 2. implies Poincaré inequality on balls follows from a remark of Kusuoka-Stroock [11] which we now explain. Denote by $H_{B,t}$ the semigroup associated with the operator L and Neumann boundary condition on the ball $B = B(x, r)$, where $x \in M$ and $0 < r < r_1$. Let $h_{B,t}$ be the kernel of this semigroup. Applying Harnack inequality to $h_{B,t}$ as above, we find that

$$h_{B,r^2}(z, y) \geqslant (CV)^{-1} \qquad \text{for all } y, z \in B(x, r/2)$$

where $V = V(x, r)$. Hence, for $y \in B(x, r/2)$ we have

$$\begin{aligned} H_{B,r^2}(f - H_{B,r^2}f(y))^2(y) &\geqslant (CV)^{-1} \int_{B(x,r/2)} |f(z) - H_{B,r^2}f(y)|^2 \, d\mu(z) \\ &\geqslant (CV)^{-1} \int_{B(x,r/2)} |f - f_{r/2}(x)|^2 \, d\mu. \end{aligned}$$

Integrating over $B(x, r/2)$, we obtain

$$\int_B H_{B,r^2}(f - H_{B,r^2}f(y))^2(y) \, d\mu(y) \geqslant C'^{-1} \int_{B(x,r/2)} |f - f_{r/2}(x)|^2 \, d\mu.$$

But, we also have

$$\begin{aligned} \int_B H_{B,r^2}(f - H_{B,r^2}f(y))^2(y) \, d\mu(y) &= \|f\|_{2,B}^2 - \|H_{B,r^2}f\|_{2,B}^2 \\ &= -\int_0^{r^2} \partial_s \|H_{B,s}f\|_{2,B}^2 \, ds \\ &\leqslant 2r^2 \int_B |\nabla f|^2 \, d\mu. \end{aligned}$$

This proves (2) with $r_0 = r_1/2$ and also ends the proof of Theorem 3.1.

Remark. One can wonder whether the parabolic Harnack inequality 2. could be replaced by an elliptic Harnack inequality for L-harmonic functions on balls. I do not know the answer to this question.

4. Applications. The preceding section made it clear that (1) and (2) are enough to obtain powerful results concerning the operator L. In this section we present some further consequences of the hypothesis that L satisfies (1) and (2). Since these results are obtained by arguments which have been explained elsewhere, I will be sketchy. A classical corollary of Harnack inequality is the Hölder continuity of the solutions of the given equation. Namely, we have the following theorem.

THEOREM 4.1. *Assume that* (1), (2), *hold for some* $r_0 > 0$. *Fix* $0 < \delta < 1$. *There exist* $0 < \alpha < 1$ *and* C *depending only on* C_1, C_2, δ, *and such that, for any* $x \in M$, $s \in]-\infty, +\infty[$, *and any* $0 < r < r_0$, *any solution* u *of* $(\partial_t + L)u = 0$ *in* $Q =]s - r^2, s[\times B(x, r)$ *satisfies*

$$|u(t', y') - u(t, y)| \leqslant C(\bar{\rho}/r)^{\alpha} \|u\|_{\infty, Q}$$

where $\bar{\rho} = \max\{|t - t'|^{1/2}, \rho(y, y')\}$ *and* $(t, y), (t', y') \in Q_\delta$.

See Moser's article [15] for a proof. Another important corollary of Theorems 3.1 and 3.2 is as follows.

THEOREM 4.2. *Assume that* (1), (2), *hold for some* $r_0 > 0$. *Then there exist constants* C_k, $k = 0, 1, 2, \ldots$, *such that*

$$|\partial_t^k h_t(x, x')| \leqslant C_k V(x, t^{1/2} \wedge r_0)^{-1} t^{-k} (1 + \rho^2/t)^{\nu_0/2 + k} \exp(-\rho^2/4t)$$

for all $x, x' \in M$, *all* $t > 0$, *and* $\rho = \rho(x, x')$. *Also, there exist* C, C', *such that*

$$h_t(x, x') \geqslant (CV(x, t^{1/2}))^{-1} \exp(-C'\rho^2/t)$$

for all $x, x' \in M$, $t > 0$, *such that* $\rho \leqslant r_0$ *and* $t \leqslant r_0^2$.

The proof can be adapted from the arguments in [20]. Note that, when $r_0 = +\infty$, we obtain a global two-sided Gaussian estimate for h_t. This implies that, under the hypothesis that (1), (2), hold with $r_0 = +\infty$, the Green function $G(x, y)$ of L exists if and only if $\int_1^{+\infty} V(x, t^{1/2})^{-1}\, dt < +\infty$. Moreover, G satisfies

$$C^{-1} \int_{\rho^2}^{+\infty} V(x, t^{1/2})^{-1}\, dt \leqslant G(x, y) \leqslant C \int_{\rho^2}^{+\infty} V(x, t^{1/2})^{-1}\, dt$$

where $\rho = \rho(x, y)$; see [12].

Consider the bottom of the spectrum of L defined by $\lambda_0 = \inf\{\langle Lf, f\rangle / \|f\|_2^2, f \in \mathscr{C}_0^\infty(M)\}$. In the case when $r_0 < +\infty$, it is possible that $\lambda_0 > 0$. See [20] for a Gaussian upper bound on h_t which can be adapted to the present setting and takes λ_0 into account.

In [10], Koranyi and Taylor give elegant arguments which show that the uniqueness property for the positive Cauchy problem associated with $\partial_t + L$ follows from a local uniform Harnack inequality. Hence, their results apply to operators L which satisfy (1), (2) for some $r_0 > 0$. In the process they show that any minimal solution $u \geqslant 0$ of $(\partial_t + L)u = 0$ on $]-\infty, s[\, \times M$ is of the form $u(t, x) = e^{\gamma t}v(x)$, where v is a minimal solution of $Lv = \gamma v$ on M. (Recall that a solution $u \geqslant 0$ is minimal if any solution v such that $0 \leqslant v \leqslant u$ is a constant multiple of u.)

Concerning L-harmonic functions, Theorems 3.1 and 4.1 yield the following theorem.

THEOREM 4.3. *Assume that* (1), (2), *hold with* $r_0 = +\infty$. *Then any solution of* $Lv = 0$ *on* M *which is bounded below is constant. Moreover, there exists* $0 < \alpha \leqslant 1$ *depending only on* C_1, C_2, *such that any solution* v *of* $Lv = 0$ *which satisfies*

$$\lim_{r \to +\infty} \left(r^{-\alpha} \sup_{B(x_0, r)} \{|v|\} \right) = 0$$

for some fixed $x_0 \in M$ *is constant.*

Finally, there is a further idea which, together with Theorem 3.2, yields interesting results. Namely, consider the wave equation $(\partial_t^2 + L)u = 0$. When $L = \Delta$ is the Laplace operator of a complete Riemannian manifold, it is well known that, if $u(t, .)$ is supported in $B(x_0, r)$ and $s > t$, then $u(s, .)$ is supported in $B(x_0, r + s - t)$. In other words, waves have finite propagation speed. We claim that this is still true for the operator L considered in this paper. Indeed, this can be seen by replacing L by $L + \varepsilon\Delta$, where Δ is the Laplace operator for some fixed Riemannian structure on M, and letting ε tend to zero; see [14]. The main point in this argument is to show that the distance associated with $L + \varepsilon\Delta$ tends to the distance associated with L when ε tends to zero; this follows from the qualitative hypothesis that L is locally subelliptic. Once the above finite propagation speed property has been proved for L, we can follow Section 2 of [4] and obtain estimates on the kernels of operators $f(L^{1/2})$, where f is a (nice) even function; see also [20], Section 8. Instead of writing a general theorem, we note the following application of this technique. (See [20] for more details in a Riemannian setting.)

THEOREM 4.4. *Assume that* (1), (2), *hold with* $r_0 = +\infty$. *Fix a positive integer* σ. *Then the kernel* $h_{\sigma,t}$ *of the semigroup* e^{-tL^σ} *satisfies*

$$|\partial_t^k h_{\sigma,t}(x, x')| \leqslant C_k' V(x, t^{1/2\sigma})^{-1} t^{-k} \exp(-(\rho/C_k t^{1/2\sigma})^{2\sigma/(2\sigma-1)})$$

for all $t > 0$, $x, x' \in M$, $\rho = \rho(x, x')$, *and any fixed integer* k. *In particular, we have* $\|\partial_t^k h_{\sigma,t}(x, .)\|_1 \leqslant C_k'' t^{-k}$ *for all* $t > 0$ *and all* $x \in M$, *which shows that the semigroup* e^{-tL^σ} *is bounded analytic on* L^p *for all* $p \in [1, +\infty[$.

Remarks. It is worth emphasizing the fact that the above results are very stable. For instance, if we assume that (1) and (2) hold for an operator L, then all that has been said about L is also valid for any operator L' symmetric with respect to a measure μ' and such that $C^{-1}\Gamma(f, f) \leqslant \Gamma'(f, f) \leqslant C\Gamma(f, f)$ for all $f \in \mathscr{C}^\infty(M)$ and $C^{-1}V(x, r) \leqslant V'(x, r) \leqslant CV(x, r)$. Here, L' does not even need to have smooth coefficients. For instance, Theorem 2.1 shows that Harnack inequality is stable under quasi-isometric changes of a metric on a Riemannian manifold.

Another instance of the stability of the above is as follows. Assume that M, L, μ and M', L', μ', are as in Section 1. Assume that L satisfies (1), (2), for some $r_0 > 0$. Assume also that $\pi: M \to M'$ is a surjection such that $L(u \circ \pi) = L'u \circ \pi$ for any smooth function u on M'. Then (1), (2), also hold for L' and some $r_0' > 0$. This is because Harnack inequality projects easily from M, L to M', L'. Note that it does not seem easy to see more directly that the doubling property holds for M', L', μ'.

In the same spirit as the above remarks, note that operators of the form $L +$ *lower-order terms* can also be studied using Moser's iteration; see [20] and the references given there. In fact, most of the results described in [20] could be adapted to the present setting.

5. Examples. In this section we describe different settings where the above results apply.

Example 1. Let M, g, be a complete Riemannian manifold of dimension n and $L = \Delta$ be the corresponding Laplace operator. Assume that there exists $K \geqslant 0$ such that the Ricci curvature satisfies $Ric \geqslant -Kg$ on M. Classical comparison theorems imply that $V(x, r)/V(x, s) \leqslant (r/s)^n e^{r\sqrt{(n-1)K}}$ for $r \geqslant s$; see [4], for instance. Moreover, P. Buser proved in [2] that

$$\int_{B(x,r)} |f - f_r(x)|^2 \, dv \leqslant r^2 C^{rK^{1/2}} \int_{B(x,r)} |\nabla f|^2 \, dv$$

for $f \in \mathscr{C}^\infty(M)$, $x \in M$, and $r > 0$. Hence, we can apply the above in this setting. Note that, when $K = 0$, (1) and (2) hold with $r_0 = +\infty$. This gives an alternative approach to most of the results of Li-Yau [12]. (The above method cannot yield gradient estimates but only Hölder continuity estimates.) At the same time, we also recover the results of [19] concerning manifolds which are quasi-isometric to a manifold with nonnegative Ricci curvature.

Example 2. Let G be a Lie group having polynomial volume growth; see [23, 18], for instance. Let $L = -\sum_1^k X_i^2$, where $\{X_1, \ldots, X_k\}$ is a family of left invariant vector fields having the Hörmander property (see [23]). Then (1) and (2) hold with $r_0 = \infty$; see [23, 24]. Hence, we recover the results of [23, 24, 18, 21]. The conclusion of Theorem 4.4 is new in this setting. Note that, in this context, Poincaré inequality is very easy to obtain; see [24].

Example 3. Let N be a compact Riemannian manifold and let M be a normal covering of N. Assume that the deck transformation group G of this covering has polynomial volume growth. It follows from the arguments in [25] that (1), (2), hold with $r_0 = \infty$ for the Laplace operator on M. Moreover, thanks to the second remark at the end of last section, if $H \subset G$ is closed subgroup of G (not necessarily normal), (1), (2), also holds on $M' = M/H$ with $r_0 = +\infty$. Many of the results obtained above, including the uniform Harnack inequality which follows from Theorem 3.1, are new in this setting.

Example 4. Consider again a normal covering M of a compact Riemannian manifold N with the deck transformation group G having polynomial volume growth. Let L_0 be the Laplace operator on M, μ_0 be the Riemannian volume, and ∇_0 be the Riemannian gradient. Now let L, μ be as in Section 1. Assume that L is uniformly subelliptic with respect to the Laplace operator L_0 and that $d\mu = m d\mu_0$ with $C^{-1} \leqslant m \leqslant C$. Also, assume that $|\nabla f| \leqslant C|\nabla_0 f|$. (In the case when M is just the euclidean space, this last hypothesis means that L has bounded coefficients.) It follows from the local results concerning subelliptic operators (see [9] for details and references) and the arguments in [25] that (1), (2), holds for L with $r_0 = \infty$. Note that our hypotheses are satisfied whenever L is the pullback of a subelliptic operator on the compact manifold N.

References

[1] D. Bakry and M. Emery, "Diffusions hypercontractives" in *Séminaire de Probabilités XIX, 1983/84*, Lecture Notes in Math. **1123**, Springer, Berlin, 1985, 177–207.

[2] P. Buser, *A note on the isoperimetric constant*, Ann. Sci. École Norm. Sup. **15** (1982), 213–230.

[3] E. Carlen, S. Kusuoka, and D. Stroock, *Upper bounds for symmetric Markov transition functions*, Ann. Inst. H. Poincaré Non Linéaire **23** (1987), 245–287.

[4] J. Cheeger, M. Gromov, and M. Taylor, *Finite propagation speed, kernel estimates for functions of the Laplace operator, and geometry of complete Riemannian manifolds*, J. Differential Geom. **17** (1982), 15–53.

[5] Th. Coulhon and L. Saloff-Coste, *Isopérimetrie pour les groupes et les variétés*, preprint, 1991.

[6] C. Fefferman and D. H. Phong, "Subelliptic eigenvalue problems" in *Proceedings of the Conference in Harmonic Analysis in Honor of Antoni Zygmund*, Wadsworth Math. Ser., Wadsworth, Belmont, California, 1981, 590–606.

[7] Y. Guivarc'h, *Mouvement brownien sur les revêtements d'une variété compacte*, C.R. Acad. Sci. Paris Sér. I. Math. **292** (1981), 851–853.

[8] D. Jerison, *The Poincaré inequality for vector fields satisfying Hörmander condition*, Duke Math. J. **53** (1986), 503–523.

[9] D. Jerison and A. Sanchez-Calle, "Subelliptic second order differential operators" in *Complex Analysis III, Proceedings, Univ. of Md.*, Lecture Notes in Math. **1277**, Springer, Berlin, 1986, 47–77.

[10] A. Koranyi and J. C. Taylor, *Minimal solutions of the heat equation and uniqueness of the positive Cauchy problem on homogeneous spaces*, Proc. Amer. Math. Soc. **94** (1985), 273–279.

[11] S. Kusuoka and D. Stroock, *Applications of Malliavin calculus, part 3*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), 391–442.

[12] P. Li and S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986), 153–201.

[13] T. Lyons and D. Sullivan, *Function theory, random paths and covering spaces*, J. Differential Geom. **19** (1984), 299–323.

[14] R. Melrose, "Propagation for the wave group of a positive subelliptic second-order differential operator" in *Hyperbolic Equations and Related Topics, Katata/Kyoto, 1984*, Academic, Boston, 1986, 181–192.

[15] J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **17** (1964), 101–134.

[16] ———, *On pointwise estimate for parabolic differential equations*, Comm Pure Appl. Math. **24** (1971), 727–740.

[17] D. Robinson, *Elliptic Operators and Lie Groups*, Oxford Univ. Press, Oxford, 1991.
[18] L. Saloff-Coste, *Analyse sur les groupes de Lie à croissance polynômiale*, Ark. Mat. **28** (1990), 315–331.
[19] ———, *Opérateurs uniformément elliptiques sur les variétés riemanniennes*, C.R. Acad. Sci. Paris Sér. I. Math. **312** (1991), 25–30.
[20] ———, *Uniformly elliptic operators on Riemannian manifolds*, to appear in J. Differential Geom.
[21] L. Saloff-Coste and D. Stroock, *Opérateurs uniformément sous-elliptiques sur des groupes de Lie*, J. Funct. Anal. **98** (1991), 97–121.
[22] N. Varopoulos, *Hardy-Littlewood theory for semigroups*, J. Funct. Anal. **63** (1985), 240–260.
[23] ———, *Analysis on Lie groups*, J. Funct. Anal. **76** (1988), 346–410.
[24] ———, *Fonctions harmoniques sur les groupes de Lie*, C.R. Acad. Sci. Paris Sér. I Math. **304** (1987), 519–521.
[25] ———, "Random walks and Brownian motion on manifolds" in *Analisi Armonica, Spazi Simmetrici e Teoria della Probabilità, Cortona, 1984*, Sympos. Math. **29**, Academic, London, 1987, 97–109.
[26] ———, *Distances associées aux opérateurs sous-elliptiques du second ordre*, C.R. Acad. Sci. Paris Sér. I Math. **309** (1989), 663–667.
[27] S.-T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228.

Analyse Complexe et Géométrie, Université Paris VI, 4, Place Jussieu, 75252 Paris Cedex 05, France