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Random walks on free solvable groups

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Abstract For any finitely generated group G, let $n \mapsto \Phi_G(n)$ be the function that describes the rough asymptotic behavior of the probability of return to the identity element at time $2n$ of a symmetric simple random walk on G (this is an invariant of quasi-isometry). We determine this function when G is the free solvable group $\mathbf{S}_{d,r}$ of derived length d on r generators and some related groups.

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1 Introduction

1.1 The random walk group invariant Φ_G

Let G be a finitely generated group. Given a probability measure μ on G, the random walk driven by μ (started at the identity element e of G) is the G-valued random process $X_n = \xi_1 \dots \xi_n$ where $(\xi_i)_1^\infty$ is a sequence of independent identically distributed G-valued random variables with law μ. If $u * v(g) = \sum_h u(h)v(h^{-1}g)$ denotes the convolution of two functions u and v on G then the probability that $X_n = g$ is given by $\mathbf{P}_e^\mu(X_n = g) = \mu^{(n)}(g)$ where $\mu^{(n)}$ is the n-fold convolution of μ.

Given a symmetric set of generators S, the word-length $|g|$ of $g \in G$ is the minimal length of a word representing g in the elements of S. The associated volume growth function, $r \mapsto V_{G,S}(r)$, counts the number of elements of G with $|g| \le r$. The word-length induces a left invariant metric on G which is also the graph metric on the Cayley graph (G, S). A

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quasi-isometry between two Cayley graphs (G_i, S_i), $i = 1, 2$, say, from G_1 to G_2, is a map $q : G_1 \to G_2$ such that

$$C^{-1} d_2(q(x), q(y)) \le d_1(x, y) \le C(1 + d_2(q(x), q(y)))$$

and $\sup_{g,\in G_2}\{d_2(g, q(G_1)) \le C$ for some finite positive constant C. This induces an equivalence relation on Cayley graphs. In particular, (G, S_1), (G, S_2) are quasi-isometric for any choice of generating sets S_1, S_2. See, e.g., [5] for more details.

Given two monotone functions ϕ, ψ, write $\phi \simeq \psi$ if there are constants $c_i \in (0, \infty)$, $1 \le i \le 4$, such that $c_1\psi(c_2 t) \le \phi(t) \le c_3\psi(c_4 t)$ (using integer values if ϕ, ψ are defined on $\mathbb{N}$). We denote by $\lesssim$ and $\gtrsim$ the associated inequalities. If S_1, S_2 are two symmetric generating sets for G, then $V_{G,S_1} \simeq V_{G,S_2}$. We use the notation V_G to denote either the $\simeq$-equivalence class of $V_{G,S}$ or any one of its representatives. The volume growth function V_G is one of the simplest quasi-isometry invariant of a group G.

By [16, Theorem 1.4], if μ_i, $i = 1, 2$, are symmetric (i.e., $\mu_i(g) = \mu_i(g^{-1})$ for all $g \in G$) finitely supported probability measures with generating support, then the functions $n \mapsto \phi_i(n) = \mu_i^{(2n)}(e)$ satisfy $\phi_1 \simeq \phi_2$. By definition, we denote by Φ_G any function that belongs to the $\simeq$-equivalence class of $\phi_1 \simeq \phi_2$. In fact, Φ_G is an invariant of quasi-isometry. Further, if μ is a symmetric probability measure with generating support and finite second moment $\sum_G |g|^2\mu(g) < \infty$ then $\mu^{(2n)}(e) \simeq \Phi_G(n)$. See [16].

1.2 Free solvable groups

This work is concerned with finitely generated solvable groups. Recall that $G^{(i)}$, the derived series of G, is defined inductively by $G^{(0)} = G$, $G^{(i)} = [G^{(i-1)}, G^{(i-1)}]$. A group is solvable if $G^{(i)} = \{e\}$ for some i and the smallest such i is the derived length of G. A group G is polycyclic if it admits a normal descending series $G = N_0 \supset N_1 \supset \cdots \supset N_k = \{e\}$ such that each of the quotient N_i/N_{i+1} is cyclic. The lower central series $\gamma_j(G)$, $j \ge 1$, of a group G is obtained by setting $\gamma_1(G) = G$ and $\gamma_{j+1} = [G, \gamma_j(G)]$. A group G is nilpotent of nilpotent class c if $\gamma_c(G) \ne \{e\}$ and $\gamma_{c+1}(G) = \{e\}$. Finitely generated nilpotent groups are polycyclic and polycyclic groups are solvable.

Recall the following well-known facts. If G is a finitely generated solvable group then either G has polynomial volume growth $V_G(n) \simeq n^D$ for some $D = 0, 1, 2, \dots,$ or G has exponential volume growth $V_G(n) \simeq \exp(n)$. See, e.g., [5] and the references therein. If $V_G(n) \simeq n^D$ then G is virtually nilpotent and $\Phi_G(n) \simeq n^{-D/2}$. If G is polycyclic with exponential volume growth then $\Phi_G(n) \simeq \exp(-n^{1/3})$. See [1,12,22–24] and the references given there. However, among solvable groups of exponential volume growth, many other behaviors than those described above are known to occur. See, e.g., [8,15,20]. Our main result is the following theorem. Set

$$\log_{[1]} n = \log(1 + n) \text{ and } \log_{[i]}(n) = \log(1 + \log_{[i-1]} n).$$

Theorem 1.1 *Let $\mathbf{S}_{d,r}$ be the free solvable group of derived length d on r generators, that is, $\mathbf{S}_{d,r} = \mathbf{F}_r/\mathbf{F}_r^{(d)}$ where $\mathbf{F}_r$ is the free group on r generators, $r \ge 2$.*

- *If $d = 2$ (the free metabelian case) then*

$$\Phi_{\mathbf{S}_{2,r}}(n) \simeq \exp\left(-n^{r/(r+2)}(\log n)^{2/(r+2)}\right).$$

- *If $d > 2$ then*

$$\Phi_{\mathbf{S}_{d,r}}(n) \simeq \exp\left(-n\left(\frac{\log_{[d-1]} n}{\log_{[d-2]} n}\right)^{2/r}\right).$$

In the case $d = 2$, this result is known and due to Anna Erschler who computed the Følner function of $\mathbf{S}_{2,r}$ in an unpublished work based on the ideas developed in [8]. We give a different proof.

Note that if G is r-generated and solvable of length at most d then there exists $c, k \in (0, \infty)$ such that $\Phi_G(n) \geq c\Phi_{\mathbf{S}_{d,r}}(kn)$.

1.3 On the groups of the form $\mathbf{F}_r/[N, N]$

The first statement in Theorem 1.1 can be generalized as follows. Let N be a normal subgroup of $\mathbf{F}_r$ and consider the tower of r generated groups $\Gamma_d(N)$ defined by $\Gamma_d(N) = \mathbf{F}_r/N^{(d-1)}$. Given information about $\Gamma_1(N) = \mathbf{F}_r/N$, more precisely, about the pair $(\mathbf{F}_r, N)$, one may hope to determine $\Phi_{\Gamma_d(N)}$ (in Theorem 1.1, $N = [\mathbf{F}_r, \mathbf{F}_r]$ and $\Gamma_1(N) = \mathbb{Z}^r$). Here, it is important to note that the groups $\Gamma_d(N)$, $d \geq 2$, depend not only of the group $\Gamma_1(N)$ but also of the particular presentation $\mathbf{F}_r/N = \Gamma_1(N)$ of that group that is chosen. See Example 2.3. The following theorem captures some of the results we obtain in this direction when $d = 2$. Further examples are given in Sect. 5.3.

Theorem 1.2 *Let $N \trianglelefteq \mathbf{F}_r$, $\Gamma_1(N) = \mathbf{F}_r/N$ and $\Gamma_2(N) = \mathbf{F}_r/[N, N]$ as above.*

- *Assume that $r \geq 2$ and that $\Gamma_1(N)$ be infinite nilpotent of volume growth of degree D. Then we have*

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n^{D/(D+2)}(\log n)^{2/(D+2)}\right).$$

- *Assume that*
  - *either $\Gamma_1(N) = \mathbb{Z}_q \wr \mathbb{Z}$ with presentation $\langle a, t | a^q, [a, t^{-n}at^n], n \in \mathbb{Z}\rangle$,*
  - *or $\Gamma_1(N) = BS(1, q)$ with presentation $\langle a, b | a^{-1}ba = b^q\rangle$.*

  Then we have

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-\frac{n}{(\log n)^2}\right).$$

In Sect. 5, Theorem 5.4, we treat polycyclic groups of exponential volume growth equipped with their standard polycyclic presentation.

Obtaining results for $d \geq 3$ is not easy. We treat a few examples beyond the case $N = [\mathbf{F}_r, \mathbf{F}_r]$ of Theorem 1.1. These examples include the case when $N = \gamma_c(\mathbf{F}_r)$. See Theorem 6.13.

Remark 1.3 Fix the presentation $\mathbf{F}_r/N = \Gamma_1(N)$. Let $\boldsymbol{\mu}$ be the probability measure driving the lazy simple random walk $(\xi_n)_0^\infty$ on $\mathbf{F}_r$ so that

$$\mathbf{P}^{\boldsymbol{\mu}}_{\mathbf{e}}(\xi_n = \mathbf{g}) = \boldsymbol{\mu}^{(n)}(\mathbf{g}).$$

Let $X = (X_n)_0^\infty$ and $Y = (Y_n)_0^\infty$ be the projections on $\Gamma_2(N)$ and $\Gamma_1(N)$, respectively so that

$$\Phi_{\Gamma_2(N)}(n) \simeq \mathbf{P}^{\boldsymbol{\mu}}_{\mathbf{e}}(X_n = e) \text{ and } \Phi_{\Gamma_1(N)}(n) \simeq \mathbf{P}^{\boldsymbol{\mu}}_{\mathbf{e}}(Y_n = \bar{e})$$

where e (resp. $\bar{e}$) is the identity element in $\Gamma_2(N)$ (resp. $\Gamma_1(N)$.) By the flow interpretation of the group $\Gamma_2(N)$ developed in [6,14] and reviewed in Sect. 2.2 below,

$$\mathbf{P}_{\mathbf{e}}^{\boldsymbol{\mu}}(X_n = e) = \mathbf{P}_{\mathbf{e}}^{\boldsymbol{\mu}}(Y \in \mathfrak{B}_n)$$

where $\mathfrak{B}_n$ is the event that, at time n, every oriented edge of the marked Cayley graph $\Gamma_1(N)$ has been traversed an equal number of times in both directions. For instance, if $\Gamma_1(N) = \mathbb{Z}^r$, the estimate $\Phi_{\Gamma_2(N)}(n) \simeq \exp(-n^{r/(2+r)}(\log n)^{2/(2+r)})$ also gives the order of magnitude of the probability that a simple random walk on $\mathbb{Z}^r$ returns to its starting point at time n having crossed each edge an equal number of time in both direction.

1.4 Other random walk invariants

Let $|g|$ be the word-length of G with respect to some fixed finite symmetric generating set and $\rho_\alpha(g) = (1 + |g|)^\alpha$. In [3], for any finitely generated group G and real $\alpha \in (0, 2)$, the non-increasing function

$$\widetilde{\Phi}_{G,\rho_\alpha} : \mathbb{N} \ni n \to \widetilde{\Phi}_{G,\rho_\alpha}(n) \in (0, \infty)$$

is defined in such a way that it provides the best possible lower bound

$$\exists\, c, k \in (0, \infty),\ \forall n,\quad \mu^{(2n)}(e) \geq c\widetilde{\Phi}_{G,\rho_\alpha}(kn),$$

valid for every symmetric probability measure μ on G satisfying the weak-ρ_α-moment condition

$$W(\rho_\alpha, \mu) = \sup_{s>0}\{s\mu(\{g : \rho_\alpha(g) > s\})\} < \infty.$$

It is well known and easy to see (using Fourier transform techniques) that

$$\widetilde{\Phi}_{\mathbb{Z}^r,\rho_\alpha}(n) \simeq n^{-r/\alpha}.$$

It is proved in [3] that $\widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq n^{-D/\alpha}$ if G has polynomial volume growth of degree D and that $\widetilde{\Phi}_{G,\rho_\alpha}(n) \simeq \exp\left(-n^{-1/(1+\alpha)}\right)$ if G is polycyclic of exponential volume growth. We prove the following result.

Theorem 1.4 *For any $\alpha \in (0, 2)$,*

$$\widetilde{\Phi}_{\mathbf{S}_{2,r},\rho_\alpha}(n) \simeq \exp\left(-n^{r/(r+\alpha)}(\log n)^{\alpha/(r+\alpha)}\right).$$

The lower bound in this theorem follows from Theorem 1.1 and [3]. Indeed, for $d > 2$, Theorem 1.1 and [3, Theorem 3.3] also give

$$\widetilde{\Phi}_{\mathbf{S}_{d,r},\rho_\alpha}(n) \geq c \exp\left(-Cn\left(\frac{\log_{[d-1]} n}{\log_{[d-2]} n}\right)^{\alpha/r}\right).$$

The upper bound in Theorem 1.4 is obtained by studying random walks driven by measures that are not finitely supported. The fact that the techniques we develop below can be applied successfully in certain cases of this type is worth noting. Proving an upper bound matching the lower bound given above for $\widetilde{\Phi}_{\mathbf{S}_{d,r},\rho_\alpha}$ with $d > 2$ is an open problem.

1.5 Wreath products and Magnus embedding

Let H, K be countable groups. Recall that the wreath product $K \wr H$ (with base H) is the semidirect product of the algebraic direct sum $K_H = \sum_{h\in H} K_h$ of H-indexed copies of K by H where H acts on K_H by translation of the indices. More precisely, elements of $K \wr H$ are pair $(f, h) \in K_H \times H$ and

$$(f, h)(f', h') = (f\tau_h f', hh')$$

where $\tau_h f_x = f_{h^{-1}x}$ if $f = (f_x)_{x\in H} \in K_H$ (recall that, by definition, only finitely many f_x are not the identity element e_K in K). In the context of random walk theory, the group H is called the base-group and the group K the lamp-group of $K \wr H$ (an element $(f, h) \in K \wr H$ can understood as a finite lamp configuration f over H together with the position h of the "lamplighter" on the base H). Given probability measures η on K and μ on H, the switch-walk-switch random walk on $K \wr H$ is driven by the measure $\eta * \mu * \eta$ and has the following interpretation. At each step, the lamplighter switches the lamp at its current position using an η-move in K, then the lamplighter makes a μ-move in H according to μ and, finally, the lamplighter switches the lamp at its final position using an η-move in K. Each of these steps are performed independently of each others. See, e.g., [15,19] for more details. When we write $\eta * \mu * \eta$ in $K \wr H$, we identify η with the probability measure on $K \wr H$ with is equal to η on the copy of K above the identity of H and vanishes everywhere else, and we identify μ with the a probability measure on $K \wr H$ supported on the obvious copy of H in $K \wr H$.

Thanks to [4,8,15,19], quite a lot is known about the random walk invariant $\Phi_{K\wr H}$. Further, the results stated in Theorems 1.1–1.2 can in fact be rephrased as stating that

$$\Phi_{\Gamma_2(N)} \simeq \Phi_{\mathbb{Z}^a \wr \Gamma_1(N)}$$

for some/any integer $a \geq 1$. It is relevant to note here that for Γ of polynomial volume growth of degree $D > 0$ or Γ infinite polycyclic (and in many other cases as well), we have $\Phi_{\mathbb{Z}^a\wr\Gamma} \simeq \Phi_{\mathbb{Z}^b\wr\Gamma}$ for any integers $a, b \geq 1$. Indeed, the proofs of Theorems 1.1–1.2–1.4 make use of the Magnus embedding which provides us with an injective homomorphism $\bar{\psi} : \Gamma_2(N) \hookrightarrow \mathbb{Z}^r \wr \Gamma_1(N)$. This embedding is use to prove a lower bound of the type

$$\Phi_{\Gamma_2(N)}(n) \geq c\Phi_{\mathbb{Z}^r\wr\Gamma_1(N)}(kn)$$

and an upper bound that can be stated as

$$\Phi_{\Gamma_2(N)}(Cn) \leq C\Phi_{\mathbb{Z}\wr\overline{\Gamma}}(n)$$

where $\overline{\Gamma} < \Gamma_1(N)$ is a subgroup which has a similar structure as $\Gamma_1(N)$. For instance, in the easiest cases including when $\Gamma_1(N)$ is nilpotent, $\overline{\Gamma}$ is a finite index subgroup of $\Gamma_1(N)$. The fact that the wreath product is taken with $\mathbb{Z}^r$ in the lower bound and with $\mathbb{Z}$ in the upper bound is not a typo. It reflects the nature of the arguments used for the proof. Hence, the fact that the lower and upper bounds that are produced by our arguments match up depends on the property that, under proper hypotheses on $\overline{\Gamma} < \Gamma_1(N)$ and $\Gamma_1(N)$,

$$\Phi_{\mathbb{Z}^a\wr\Gamma_1(N)} \simeq \Phi_{\mathbb{Z}^b\wr\overline{\Gamma}}$$

for any pair of positive integers a, b.

1.6 A short guide

Section 2 of the paper is devoted to the algebraic structure of the group $\Gamma_2(N) = \mathbf{F}_r/[N, N]$. It describes the Magnus embedding as well as the interpretation of $\Gamma_2(N)$ in terms of flows on $\Gamma_1(N)$. See [6,14,25]. The Magnus embedding and the flow representation play key parts in the proofs of our main results.

Section 3 describes two methods to obtain lower bounds on the probability of return of certain random walks on $\Gamma_2(N)$. The first method is based on a simple comparison argument and the notion of Følner couples introduced in [4] and already used in [8]. This method works for symmetric random walks driven by a finitely supported measure. The second method allows us to treat some measures that are not finitely supported, something that is of interest in the spirit of Theorem 1.4.

Section 4 focuses on upper bounds for the probability of return. This section also makes use of the Magnus embedding, but in a somewhat more subtle way. We introduce the notion of exclusive pair. These pairs are made of a subgroup Γ of $\Gamma_2(N)$ and an element $\boldsymbol{\rho}$ in the free group $\mathbf{F}_r$ that projects to a cycle on $\Gamma_1(N)$ with the property that the traces of Γ and $\boldsymbol{\rho}$ on $\Gamma_1(N)$ have, in a sense, minimal interaction. See Definition 4.3. Every upper bound we obtain is proved using this notion.

Section 5 presents a variety of applications of the results obtained in Sects. 3 and 4. In particular, the statement regarding $\Phi_{\mathbf{S}_{2,r}}$ as well as Theorems 1.2–1.4 and assorted results are proved in Sect. 5.

Section 6 is devoted to the result concerning $\mathbf{S}_{d,r}$, $d \geq 3$. Both the lower bound and the upper bound methods are re-examined to allow iteration of the procedure.

Section 7 presents assorted results regarding the L^2-isoperimetric profile (or Faber-Krahn function) and the isoperimetric profile (equivalently, Følner function). In particular, the isoperimetric profile of the free solvable group $\mathbf{S}_{d,r}$ is computed (up to $\simeq$-equivalence).

Throughout this work, we will have to distinguish between convolutions in different groups. We will use $*$ to denote either convolution on a generic group G (when no confusion can possibly arise) or, more specifically, convolution on $\Gamma_2(N)$. When $*$ is used to denote convolution on $\Gamma_2(N)$, we use e_* to denote the identity element in $\Gamma_2(N)$. We will use $\star$ to denote convolution on various wreath products such as $\mathbb{Z}^r \wr \Gamma_1(N)$. When this notation is used, $e_\star$ will denote the identity element in the corresponding group. When necessary, we will decorate $\star$ with a subscript to distinguish between different wreath products. So, if μ is a probability measure on $\Gamma_2(N)$ and ϕ a probability measure on $\mathbb{Z}^r \wr \Gamma_1(N)$, we will write $\mu^{*n}(e_*) = \phi^{\star n}(e_\star)$ to indicate that the n-fold convolution of μ on $\Gamma_2(N)$ evaluated at the identity element of $\Gamma_2(N)$ is equal to the n-fold convolution of ϕ on $\mathbb{Z} \wr \Gamma_1(N)$ evaluated at the identity element of $\mathbb{Z} \wr \Gamma_1(N)$.

2 $\Gamma_2(N)$ and the Magnus embedding

This work is concerned with random walks on the groups $\Gamma_\ell(N) = \mathbf{F}_r/N^{(\ell-1)}$ where $\mathbf{F}_r$ is the free group on r generators and N is a normal subgroup of $\mathbf{F}_r$. In fact, it is best to think of $\Gamma_\ell(N)$ as a marked group, that is, a group equipped with a generating tuple. In the case of $\Gamma_\ell(N)$, the generating r-tuple is always provided by the images of the free generators of $\mathbf{F}_r$. Ideally, one would like to obtain results based on hypotheses on the nature of $\Gamma_1(N)$ viewed as an unmarked group. However, as pointed out in Remark 2.8 below, the unmarked group $\Gamma_1(N)$ is not enough to determine either $\Gamma_2(N)$ or the random walk invariant $\Phi_{\Gamma_2(N)}$. That is, in general, one needs information about the pair $(\mathbf{F}_r, N)$ itself to obtain precise information about $\Phi_{\Gamma_2(N)}$. Note however that when $\Gamma_1(N)$ is nilpotent with volume growth of degree at least 2, Theorem 1.2 provides a result that does not require further information on N.

2.1 The Magnus embedding

In 1939, Magnus [13] introduced an embedding of $\Gamma_2(N) = \mathbf{F}_r/[N, N]$ into a matrix group with coefficients in a module over $\mathbb{Z}(\Gamma_1(N)) = \mathbb{Z}(\mathbf{F}_r/N)$. In particular, the Magnus embedding is used to embed free solvable groups into certain wreath products.

Let $\mathbf{F}_r$ be the free group on the generators $\mathbf{s}_i$, $1 \le i \le r$. Let N be a normal subgroup of $\mathbf{F}_r$ and let $\pi = \pi_N$ and $\pi_2 = \pi_{2,N}$ be the canonical projections

$$\pi : \mathbf{F}_r \to \mathbf{F}_r/N = \Gamma_1(N), \quad \pi_2 : \mathbf{F}_r \to \mathbf{F}_r/[N, N] = \Gamma_2(N).$$

We also let

$$\bar{\pi} : \Gamma_2(N) \to \Gamma_1(N)$$

the projection from $\Gamma_2(N)$ onto $\Gamma_1(N)$, whose kernel can be identified with $N/[N, N]$, has the property that $\pi = \bar{\pi} \circ \pi_2$. Set

$$s_i = \pi_2(\mathbf{s}_i), \quad \bar{s}_i = \pi(\mathbf{s}_i) = \bar{\pi}(s_i).$$

When it is necessary to distinguish between the identity element in $e \in \Gamma_2(N)$ and the identity element in $\Gamma_1(N)$, we write $\bar{e}$ for the latter.

Let $\mathbb{Z}(\mathbf{F}_r)$ be the integral group ring of the free group $\mathbf{F}_r$. By extension and with some abuse of notation, let π denote also the ring homomorphism

$$\pi : \mathbb{Z}(\mathbf{F_r}) \to \mathbb{Z}(\mathbf{F}_r/N)$$

determined by $\pi(\mathbf{s}_i) = \bar{s}_i$, $1 \le i \le r$.

Let Ω be the free left $\mathbb{Z}(\mathbf{F}_r/N)$-module of rank r with basis $(\lambda_{\mathbf{s}_i})_1^r$ and set

$$M = \begin{bmatrix} \mathbf{F}_r/N & \Omega \\ 0 & 1 \end{bmatrix}$$

which is a subgroup of the group of the 2×2 upper-triangular matrices over Ω. The map

$$\psi(\mathbf{s}_i) = \begin{bmatrix} \pi(\mathbf{s}_i) & \lambda_{\mathbf{s}_i} \\ 0 & 1 \end{bmatrix} \tag{2.1}$$

extends to a homomorphism ψ of $\mathbf{F}_r$ into M. We denote by $a(\mathbf{u})$, $\mathbf{u} \in \mathbf{F}_r$, the (1, 2)-entry of the matrix $\psi(\mathbf{u})$, that is

$$\psi(\mathbf{u}) = \begin{bmatrix} \pi(\mathbf{u}) & a(\mathbf{u}) \\ 0 & 1 \end{bmatrix}. \tag{2.2}$$

Theorem 2.1 (Magnus [13]) The kernel of the homomorphism $\psi : \mathbf{F}_r \to M$ defined as above is

$$\ker(\psi) = [N, N].$$

Therefore ψ induces a monomorphism

$$\bar{\psi} : \mathbf{F}_r/[N, N] \hookrightarrow M.$$

It follows that $\mathbf{F}_r/[N, N]$ is isomorphic to the subgroup of M generated by

$$\begin{bmatrix} \pi(\mathbf{s}_i) & \lambda_{\mathbf{s}_i} \\ 0 & 1 \end{bmatrix}, \quad i = 1, \dots, r.$$

Remark 2.2 For $g \in \mathbf{F}_r/[N,N]$, we write

$$\bar{\psi}(g) = \begin{bmatrix} \bar{\pi}(g) & \bar{a}(g) \\ 0 & 1 \end{bmatrix} \tag{2.3}$$

where $\bar{a}(\pi_2(\mathbf{u})) = a(\mathbf{u})$, $\mathbf{u} \in \mathbf{F}_r$.

Remark 2.3 The free left $\mathbb{Z}(\mathbf{F}_r/N)$-module Ω with basis $\{\lambda_{\mathbf{s}_i}\}_{1\le i\le d}$ is isomorphic to the direct sum $\sum_{x\in\mathbf{F}_r/N}(\mathbb{Z}^r)_x$. More precisely, if we regard the elements in $\sum_{x\in\mathbf{F}_r/N}(\mathbb{Z}^r)_x$ as functions $f=(f_1,\ldots,f_r) : \mathbf{F}_r/N \to \mathbb{Z}^r$ with finite support, the map

$$\sum_{x\in\mathbf{F}_r/N}(\mathbb{Z}^r)_x \to \Omega :$$

$$f \mapsto \left(\sum_{x\in\mathbf{F}_r/N} f_1(x)x\right)\lambda_{\mathbf{s}_1} + \cdots + \left(\sum_{x\in\mathbf{F}/N} f_r(x)x\right)\lambda_{\mathbf{s}_r}$$

is a left $\mathbb{Z}(\mathbf{F}_r/N)$-module isomorphism. We will identity Ω with $\sum_{x\in\mathbf{F}_r/N}(\mathbb{Z}^r)_x$. Using the above interpretation, one can restate the Magnus embedding theorem as an injection from $\mathbf{F}_r/[N,N]$ into the wreath product $\mathbb{Z}^r \wr (\mathbf{F}_r/N)$.

The entry $a(g) \in \Omega$ under the Magnus embedding is given by Fox derivatives which we briefly review. Let G be a group and $\mathbb{Z}(G)$ be its integral group ring. Let M be a left $\mathbb{Z}(G)$-module. An additive map $d : \mathbb{Z}(G) \to M$ is called a *left derivation* if for all $x, y \in G$,

$$d(xy) = xd(y) + d(x).$$

As a consequence of the definition, we have $d(e) = 0$ and $d(g^{-1}) = -g^{-1}d(g)$.

For the following two theorems of Fox, we refer the reader to the discussion in [14, Sect. 2.3] and the references given there.

Theorem 2.4 (Fox) *Let $\mathbf{F}_r$ be the free group on r generators $\mathbf{s}_i$, $1 \le i \le r$. For each i, there is a unique left derivation*

$$\partial_{\mathbf{s}_i} : \mathbb{Z}(\mathbf{F}_r) \to \mathbb{Z}(\mathbf{F}_r)$$

satisfying

$$\partial_{\mathbf{s}_i}(\mathbf{s}_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \ne j. \end{cases}$$

Further, if N is a normal subgroup of $\mathbf{F}_r$, then $\pi(\partial_{\mathbf{s}_i}\mathbf{u}) = 0$ in $\mathbb{Z}(\mathbf{F}_r/N)$ for all $1 \le i \le r$ if and only if $\mathbf{u} \in [N,N]$.

Example 2.1 For $\mathbf{g} = \mathbf{s}_{i_1}^{\varepsilon_1}\ldots\mathbf{s}_{i_n}^{\varepsilon_n}$, $\varepsilon_j \in \{\pm 1\}$,

$$\begin{aligned}\partial_{\mathbf{s}_i}(\mathbf{g}) &= \sum_{j=1}^{n} \mathbf{s}_{i_1}^{\varepsilon_1}\ldots\mathbf{s}_{i_{j-1}}^{\varepsilon_{j-1}}\partial_{\mathbf{s}_i}(\mathbf{s}_{i_j}^{\varepsilon_j}) \\ &= \sum_{j:i_j=i,\varepsilon_j=1} \mathbf{s}_{i_1}^{\varepsilon_1}\ldots\mathbf{s}_{i_{j-1}}^{\varepsilon_{j-1}} - \sum_{j:i_j=i,\varepsilon_j=-1} \mathbf{s}_{i_1}^{\varepsilon_1}\ldots\mathbf{s}_{i_{j-1}}^{\varepsilon_{j-1}}\mathbf{s}_{i_j}^{\varepsilon_j}.\end{aligned}$$

Theorem 2.5 (Fox) The Magnus embedding

$$\bar{\psi} : \mathbf{F}_r/[N, N] \hookrightarrow M$$

is given by

$$\bar{\psi}(g) = \begin{bmatrix} \bar{\pi}(g) & \sum_{i=1}^{r} \pi(\partial_{\mathbf{s}_i} \mathbf{g})\lambda_{\mathbf{s}_i} \\ 0 & 1 \end{bmatrix} \tag{2.4}$$

where $\mathbf{g} \in \mathbf{F}_r$ is any element such that $\pi_2(\mathbf{g}) = g$.

Example 2.2 In the special case that $N = [\mathbf{F}_r, \mathbf{F}_r]$, we have $\mathbf{F}_r/N \simeq \mathbb{Z}^r$ and $\mathbb{Z}(\mathbf{F}_r/N)$ is the integral group ring over the free abelian group $\mathbb{Z}^r$. The integral group ring $\mathbb{Z}(\mathbb{Z}^r)$ is quite similar to the multivariate polynomial ring with integer coefficients, except that we allow negative powers like $Z_1^{-3} Z_2 \dots Z_r^{-5}$. The monomials $\{Z_1^{x_1} Z_2^{x_2} \dots Z_r^{x_r} : x \in \mathbb{Z}^r\}$ are $\mathbb{Z}$-linear independent in $\mathbb{Z}(\mathbb{Z}^r)$.

2.2 Interpretation in terms of flows

Following [6,14,25], one can also think of elements of $\Gamma_2(N) = \mathbf{F}_r/[N, N]$ in terms of flows on the (labeled) Cayley graph of $\Gamma_1(N) = \mathbf{F}_r/N$. To be precise, Let $\mathbf{s}_1, \dots, \mathbf{s}_k$ be the generators of $\mathbf{F}_r$ and $\bar{s}_1, \dots, \bar{s}_k$ their images in $\Gamma_1(N)$. The Cayley graph of $\Gamma_1(N)$ is the marked graph with vertex set $V = \Gamma_1(N)$ and marked edge set $\mathfrak{E} \subset V \times V \times \{\mathbf{s}_1, \dots, \mathbf{s}_k\}$ where $(x, y, \mathbf{s}_i) \in \mathfrak{E}$ if and only if $y = x\bar{s}_i$ in $\Gamma_1(N)$. Note that each edge $\mathfrak{e} = (x, y, \mathbf{s}_i)$ as an origin $o(\mathfrak{e}) = x$, an end (or terminus) $t(\mathfrak{e}) = y$ and a label or mark $\mathbf{s}_i$.

Given a function $\mathfrak{f}$ on the edge set $\mathfrak{E}$ and a vertex $v \in V$, define the net flow $\mathfrak{f}^*(v)$ of $\mathfrak{f}$ at v by

$$\mathfrak{f}^*(v) = \sum_{o(\mathfrak{e})=v} f(\mathfrak{e}) - \sum_{t(\mathfrak{e})=v} f(\mathfrak{e}).$$

A *flow* (or $\mathbb{Z}$-flow) with source s and sink t is a function $\mathfrak{f} : \mathfrak{E} \to \mathbb{Z}$ such that

$$\forall\, v \in V \backslash \{s, t\}, \quad \mathfrak{f}^*(v) = 0,$$
$$\mathfrak{f}^*(s) = 1, \;\; \mathfrak{f}^*(t) = -1.$$

If $\mathfrak{f}^*(v) = 0$ holds for all $v \in V$, we say that $\mathfrak{f}$ is a *circulation*.

For each edge $\mathfrak{e} = (x, y, \mathbf{s}_i)$, introduce its formal inverse $(y, x, \mathbf{s}_i^{-1})$ and let $\mathfrak{E}^*$ be the set of all edges and their formal inverses. A finite path on the Cayley graph of $\Gamma_1(N)$ is a finite sequence $p = (\mathfrak{e}_1, \dots, \mathfrak{e}_\ell)$ of edges in $\mathfrak{E}^*$ so that the origin of $\mathfrak{e}_{i+1}$ is the terminus of $\mathfrak{e}_i$. We call $o(\mathfrak{e}_1)$ (resp. $t(\mathfrak{e}_\ell)$) the origin (resp. terminus) of the path p and denote it by $o(p)$ (resp. $t(p)$). Note that reading the labels along the edge of a path determines a word in the generators of $\mathbf{F}_r$ and that, conversely, any finite word ω in the generators of $\mathbf{F}_r$ determines a path p_ω starting at the identity element in $\Gamma_1(N)$.

A (finite) path p determines a flow $\mathfrak{f}_p$ with source $o(p)$ and sink $t(p)$ by setting $\mathfrak{f}_p(e)$ to be the algebraic number of time the edge $\mathfrak{e} \in \mathfrak{E}$ is crossed positively or negatively along p. Here, the edge $\mathfrak{e} = (x, y, \mathbf{s}_\alpha) \in \mathfrak{E}$ is crossed positively at the i-step along p if $\mathfrak{e}_i = (x, y, \mathbf{s}_\alpha)$. It is crossed negatively if $\mathfrak{e}_i = (y, x, \mathbf{s}_\alpha^{-1})$. We note that $\mathfrak{f}_p$ has finite support and that either $o(p) = t(p)$ and $\mathfrak{f}_p$ is a circulation or $o(p) \neq t(p)$ and $\mathfrak{f}_p^*(o(p)) = 1$, $\mathfrak{f}_p^*(t(p)) = -1$.

Given a word $\omega = \mathbf{s}_{i_1}^{\varepsilon_1} \dots \mathbf{s}_{i_n}^{\varepsilon_n}$ in the generators of $\mathbf{F}_r$, let $\mathfrak{f}_\omega$ denote the flow function on the Cayley graph of $\Gamma_1(N)$ defined by the corresponding path starting at the identity element in $\Gamma_1(N)$. We note that it is obvious from the definition that $\mathfrak{f}_\omega = \mathfrak{f}_{\omega'}$ if ω' is the reduced word in $\mathbf{F}_r$ associated with ω.

Theorem 2.6 ([14, Theorem 2.7]) Two elements $\mathbf{u}, \mathbf{v} \in \mathbf{F}_r$ project to the same element in $\Gamma_2(N) = \mathbf{F}_r/[N, N]$ if and only if they induce the same flow on $\Gamma_1(N) = \mathbf{F}_r/N$. In other words,

$$\mathbf{u} \equiv \mathbf{v} \mod [N, N] \iff \mathfrak{f}_{\mathbf{u}} = \mathfrak{f}_{\mathbf{v}}.$$

This theorem shows that an element $g \in \Gamma_2(N)$ corresponds to a unique flow $\mathfrak{f}_\omega$ on $\mathbf{F}_r/N$, defined by the path p_ω associated with any word $\omega \in \mathbf{F}_r$ such that ω projects to g in $\Gamma_2(N)$. For $g \in \Gamma_2(N)$, $\mathfrak{f}_g := \mathfrak{f}_\omega$ is well defined (i.e., is independent of the word ω projecting to g, and we call $\mathfrak{f}_g$ the flow of g. Hence, in a certain sense, we can regard elements of $\Gamma_2(N)$ as flows on $\Gamma_1(N)$. In fact, the flow $\mathfrak{f}_\omega$ is directly related to the description of the image of the element $g = \omega \bmod [N, N]$ under the Magnus embedding through the following geometric interpretation of Fox derivatives.

Lemma 2.7 ([14, Lemma 2.6]) Let $\omega \in \mathbf{F}_r$, then for any $g \in \mathbf{F}_r/N$ and $\mathbf{s}_i$, the value of $\mathfrak{f}_\omega$ on the edge $(g, gs_i, \mathbf{s}_i)$, is equal to coefficient in front of g in the Fox derivative $\pi(\partial_{\mathbf{s}_i}\omega) \in \mathbb{Z}(\mathbf{F}/N)$, i.e.

$$\pi(\partial_{\mathbf{s}_i}\omega) = \sum_{g \in \mathbf{F}/N} \mathfrak{f}_\omega((g, gs_i, \mathbf{s}_i))g. \tag{2.5}$$

There is also a characterization of geodesics on $\Gamma_2(N)$ in terms of flows (see [14, Theorem 2.11]) which is closely related to the description of geodesics on wreath products. See [18, Theorem 2.6] where it is proved that the Magnus embedding is bi-Lipschitz with small explicit universal distortion.

Remark 2.8 In [7], it is asserted that the group $\Gamma_2(N)$ depends only of $\Gamma_1(N)$ (in [7], $\Gamma_1(N)$ is denoted by A and $\Gamma_2(N)$ by C_A). This assertion is correct only if one interprets $\Gamma_1(N)$ as a marked group, i.e., if information about $\pi : \mathbf{F}_r \to \Gamma_1(N)$ is retained. Indeed, $\Gamma_2(N)$ depends in some essential ways of the choice of the presentation $\Gamma_1(N) = \mathbf{F}_r/N$. We illustrate this fact by two examples that are very good to keep in mind.

Example 2.3 Consider two presentations of $\mathbb{Z}$, namely, $\mathbb{Z} = \mathbf{F}_1$ and $\mathbb{Z} = \langle a, b | b \rangle$. In the first presentation, the kernel N_1 is trivial, therefore $\mathbf{F}_1/[N_1, N_1] \simeq \mathbb{Z}$. In the second presentation, the kernel N_2 is the normal closure of $\langle b \rangle$ in the free group $\mathbf{F}_2$ on generators a, b. Hence, N_2 is generated by $\{a^i b a^{-i}, i \in \mathbb{Z}\}$. We can then write down a presentation of $\mathbf{F}_2/[N_2, N_2]$ in the form

$$\mathbf{F}_2/[N_2, N_2] = \left\langle a, b | [a^i b a^{-i}, a^j b a^{-j}], \quad i, j \in \mathbb{Z} \right\rangle.$$

This is, actually, a presentation of the wreath product $\mathbb{Z} \wr \mathbb{Z}$. Therefore $\mathbf{F}_2/N_2' \simeq \mathbb{Z} \wr \mathbb{Z}$. We encourage the reader to recognize the structure of both $\mathbf{F}_1/[N_1, N_1] \simeq \mathbb{Z}$ and $\mathbf{F}_2/[N_2, N_2] \simeq \mathbb{Z} \wr \mathbb{Z}$ using flows on the labeled Cayley graphs associated with $\mathbf{F}_1/N_1$ and $\mathbf{F}_2/N_2$. The Cayley graph of $\mathbf{F}_2/N_2$ is the usual line graph of $\mathbb{Z}$ decorated with an oriented loop at each vertex. In the flow representation of an element of $\mathbf{F}_2/[N_2, N_2]$, the algebraic number of times the flow goes around each of these loops is recorded thereby creating the wreath product structure of $\mathbb{Z} \wr \mathbb{Z}$.

Example 2.4 Consider the following two presentations of $\mathbb{Z}^2$,

$$\mathbb{Z}^2 = \langle a, b | [a, b] \rangle$$
$$\mathbb{Z}^2 = \langle a, b, c | [a, b], c = ab \rangle.$$

Call $N_1 \subset \mathbf{F}_2$ and $N_2 \subset \mathbf{F}_3$ be the associated normal subgroups. We claim that $\mathbf{F}_2/[N_1, N_1]$ is a proper quotient of $\mathbf{F}_3/[N_2, N_2]$. Let $\theta : \mathbf{F}_3 \to \mathbf{F}_2$ be the homomorphism determined by $\theta(a) = a$, $\theta(b) = b$, $\theta(c) = ab$. Obviously, $N_2 = \theta^{-1}(N_1)$, $[N_2, N_2] \subset \theta^{-1}([N_1, N_1])$, and θ induces a surjective homomorphism $\theta' : \mathbf{F}_3/[N_2, N_2] \to \mathbf{F}_2/[N_1, N_1]$. The element abc^{-1} is nontrivial in $\mathbf{F}_3/[N_2, N_2]$, but $\theta'(abc^{-1}) = e$. A Hopfian group is a group that cannot be isomorphic to a proper quotient of itself. Finitely generated metabelian groups are Hopfian. Hence $\mathbf{F}_2/[N_1, N_1]$ is not isomorphic to $\mathbf{F}_3/[N_2, N_2]$.

3 Return probability lower bounds

3.1 Measures supported by the powers of the generators

The group $\Gamma_2(N) = \mathbf{F}_r/[N, N]$ comes equipped with the generators $(s_i)_1^r$ which are the images of the generators $(\mathbf{s}_i)_1^r$ of $\mathbf{F}_r$. Accordingly, we consider a special class of symmetric random walks defined as follows. Given probability measures p_i, $1 \le i \le r$ on $\mathbb{Z}$, we define a probability measure $\boldsymbol{\mu}$ on $\mathbf{F}_r$ by

$$\forall\, \mathbf{g} \in \mathbf{F}_r, \quad \boldsymbol{\mu}(\mathbf{g}) = \sum_{i=1}^{r} \frac{1}{r} \sum_{m \in \mathbb{Z}} p_i(m) \mathbf{1}_{\{\mathbf{s}_i^m\}}(\mathbf{g}). \tag{3.1}$$

This probability measure induces push-forward measures $\bar{\mu}$ and μ on $\Gamma_1(N) = \mathbf{F}_r/N$ and $\Gamma_2(N) = \mathbf{F}_r/[N, N]$, namely,

$$\begin{cases} \forall\, \bar{g} \in \Gamma_1(N), & \bar{\mu}(\bar{g}) = \boldsymbol{\mu}(\pi^{-1}(\bar{g})) \\ \forall\, g \in \Gamma_2(N), & \mu(g) = \boldsymbol{\mu}(\pi_2^{-1}(g)). \end{cases} \tag{3.2}$$

In fact, we will mainly consider two cases. In the first case, each p_i is the measure of the lazy random walk on $\mathbb{Z}$, that is $p_i(0) = 1/2$, $p_i(\pm 1) = 1/4$. In this case, $\boldsymbol{\mu}$ is the measure of the lazy simple random walk on $\mathbf{F}_r$, that is,

$$\boldsymbol{\mu}(e) = 1/2, \quad \boldsymbol{\mu}(\mathbf{s}_i^{\pm 1}) = 1/4r. \tag{3.3}$$

The second case can be viewed as a generalization of the first. Let $a = (\alpha_1)_1^r \in (0, \infty]^r$ be a r-tuple of extended positive reals. For each i, consider the symmetric probability measure p_{α_i} on $\mathbb{Z}$ with $p_{\alpha_i}(m) = c_i(1 + |m|)^{-1-\alpha_i}$ (if $\alpha_i = \infty$, set $p_\infty(0) = 1/2$, $p_\infty(\pm 1) = 1/4$). Let $\boldsymbol{\mu}_a$ be the measure on $\mathbf{F}_r$ obtained by setting $p_i = p_{\alpha_i}$ in (3.1). When a is such that $\alpha_i = \infty$ for all i we recover (3.3). In particular, starting with (3.3), μ is given by

$$\forall\, g \in \Gamma_2(N), \quad \mu(g) = \frac{1}{2} \mathbf{1}_g(e) + \frac{1}{4r} \sum_{1}^{r} \mathbf{1}_{s_i}(g).$$

The formula for $\bar{\mu}$ is exactly similar. For any fixed $a \in (0, \infty]^r$, we let μ_a and $\bar{\mu}_a$ be the push-forward of $\boldsymbol{\mu}_a$ on $\Gamma_2(N)$ and $\Gamma_1(N)$, respectively.

3.2 Lower bound for simple random walk

In this section, we explain how, in the case of the lazy simple random walk measure μ on $\Gamma_2(N)$ associated with $\boldsymbol{\mu}$ at (3.3), one can obtained lower bounds for the probability of return $\mu^{(2n)}(e)$ by using well-known arguments and the notion of Følner couples introduced in [4].

Definition 3.1 (See [4, Definition 4.7] and [8, Proposition 2]) Let G be a finitely generated group equipped with a finite symmetric generating set T and the associated word distance d. Let $\mathcal{V}$ be a positive increasing function on $[1,\infty)$ whose inverse is defined on $[\mathcal{V}(1),\infty)$. We say that a sequence of pairs of nonempty sets $((\Omega_k,\Omega'_k))_1^\infty$ is a a sequence of Følner couples adapted to $\mathcal{V}$ if

1. $\Omega'_k\subset\Omega_k$, $\#\Omega'_k\ge c_0\#\Omega_k$, $d(\Omega'_k,\Omega_k^c)\ge c_0k$.
2. $v_k=\#\Omega_k\nearrow\infty$ and $v_k\le\mathcal{V}(k)$.

Let ν be a symmetric finitely supported measure on G and $\lambda_\nu(\Omega)$ be the lowest Dirichlet eigenvalue in Ω for the convolution by $\delta_e-\nu$, namely,

$$\lambda_\nu(\Omega)=\inf\left\{\frac{1}{2}\sum_{x,y}|f(xy)-f(x)|^2\nu(y):\mathrm{supp}(f)\in\Omega,\sum|f|^2=1\right\}.$$

If (Ω_k,Ω'_k) is a pair satisfying the first condition in Definition 3.1 then plugging $f=d(\cdot,\Omega_k^c)$ in the definition of of $\lambda_\nu(\Omega_k)$ immediately gives $\lambda_\nu(\Omega_k)\le\frac{C}{k^2}$.

Given a function $\mathcal{V}$ as in Definition 3.1, let γ be defined implicitly by

$$\int_{\mathcal{V}(1)}^{\gamma(t)}([\mathcal{V}^{-1}(s)]^2\frac{ds}{s}=t. \tag{3.4}$$

This is the same as stating that γ is a solution of the differential equation

$$\frac{\gamma'}{\gamma}=\frac{1}{[\mathcal{V}^{-1}\circ\gamma]^2},\quad \gamma(0)=\mathcal{V}(1). \tag{3.5}$$

Following [8], we say that γ is δ-regular if $\gamma'(s)/\gamma(s)\ge\delta\gamma'(t)/\gamma(t)$ for all s,t with $0<t<s<2t$.

With this notation, Erschler [8, Proposition 2] gives a modified version of [4, Theorem 4.7] which contains the following statement.

Proposition 3.2 *If the group G admits a sequence of Følner couples adapted to the function $\mathcal{V}$ as in Definition 3.1 and the function γ associated to $\mathcal{V}$ by (3.4) is δ-regular for some $\delta>0$ then there exist $c,C\in(0,\infty)$ such that*

$$\Phi_G(n)\ge\frac{c}{\gamma(Cn)}.$$

A key aspect of this statement is that it allows for very fast growing $\mathcal{V}$ as long as one can check that γ is δ-regular. Erschler [8] gives a variety of examples showing how this works in practice but it seems worth explaining why the δ-regularity of γ is a relatively mild assumption. Suppose first that $\mathcal{V}$ is regularly varying of positive finite index. Then the same is true for $\mathcal{V}^{-1}$ and $\int_{\mathcal{V}(1)}^T\mathcal{V}^{-1}(s)^2\frac{ds}{s}\sim c\mathcal{V}^{-1}(T)^2$. In this case, it follows from (3.5) that $\gamma'(s)/\gamma(s)\simeq 1/s$. If instead we assume that $\log\mathcal{V}$ is of regular variation of positive index (resp. rapid variation) then $\mathcal{V}^{-1}\circ\exp$ is of regular variation of positive index (resp. slow variation) and we can show that

$$\int_{\mathcal{V}(1)}^T\mathcal{V}^{-1}(s)^2\frac{ds}{s}\simeq\mathcal{V}^{-1}(T)^2\log T.$$

In this case, it follows again that γ is δ-regular. All the examples treated in [8] and in the present paper fall in these categories.

The following proposition regarding wreath products is key.

Proposition 3.3 (Proof of [8, Theorem 2]) Assume that the group G is infinite, finitely generated, and admits a sequence of Følner couples adapted to the function $\mathcal{V}$ as in Definition 3.1. Set

$$\begin{aligned}\Theta_k &= \{(f,x)\in\mathbb{Z}^r\wr G : x\in\Omega_k,\ \ \operatorname{supp}(f)\subset\Omega_k, |f|_\infty\le k\#\Omega_k\},\\ \Theta'_k &= \{(f,x)\in\mathbb{Z}^r\wr G : x\in\Omega'_k,\ \ \operatorname{supp}(f)\subset\Omega_k, |f|_\infty\le k\#\Omega_k-k\}.\end{aligned}$$

Set

$$\mathcal{W}(v):=\exp\left(C\mathcal{V}(v)\log\mathcal{V}(v)\right).$$

Then (Θ_k,Θ'_k) is a sequence of Følner couples on $\mathbb{Z}^r\wr G$ adapted to $\mathcal{W}$ (for an appropriate choice of the constant C).

Proof By construction (and with an obvious choice of generators in $\mathbb{Z}^r\wr G$ based on a given set of generators for G), the distance between Θ'_k and Θ^c_k in $\mathbb{Z}^r\wr G$ is greater or equal to the minimum of k and the distance between Ω'_k and Ω^c_k in G. Also, we have

$$\#\Theta_k=\#\Omega_k(k\#\Omega_k)^{r\#\Omega_k},\ \ \#\Theta'_k=\#\Omega'_k(k\#\Omega_k-k)^{r\#\Omega_k}$$

so that

$$\frac{\#\Theta'_k}{\#\Theta_k}\ge\left(1-(\#\Omega_k)^{-1}\right)^{r\#\Omega_k}\frac{\#\Omega'_k}{\#\Omega_k}\ge\frac{1}{e^r}\frac{\#\Omega'_k}{\#\Omega_k}$$

and

$$\#\Theta_k=\exp\left(\log\#\Omega_k+r\#\Omega_k(\log\#\Omega_k+\log k)\right)\le\exp\left(C\mathcal{V}(k)\log\mathcal{V}(k)\right).$$

□

Proposition 3.4 (Computations) *Let $\mathcal{V}$ be given. Define $\mathcal{W}$ and $\gamma=\gamma_{\mathcal{W}}$ by*

$$\mathcal{W}=\exp(C\mathcal{V}\log\mathcal{V})\ \textit{and}\ \gamma^{-1}(t)=\int_{\mathcal{W}(1)}^t[\mathcal{W}^{-1}(s)]^2\frac{ds}{s}.$$

1. *Assume that $\mathcal{V}(t)\simeq t^D$. Then we have*

$$\gamma(t)\simeq\exp\left(t^{D/(2+D)}[\log t]^{2/(2+D)}\right).$$

2. *Assume that $\mathcal{V}(t)\simeq\exp(t^\alpha\ell(t))$, $\alpha>0$, where $\ell(t)$ is slowly varying with $\ell(t^a)\simeq\ell(t)$ for any fixed $a>0$. Then γ satisfies*

$$\gamma(t)\simeq\left(t\left(\frac{\ell(\log t)}{\log t}\right)^{2/\alpha}\right).$$

3. *Assume that $\mathcal{V}(t)\simeq\exp(\ell^{-1}(t))$ where $\ell(t)$ is slowly varying with $\ell(t^a)\simeq\ell(t)$ for any fixed $a>0$. Then γ satisfies*

$$\gamma(t)\simeq\left(t/[\ell(\log t)]^2\right).$$

Note that if $\ell^{-1}(t)=\exp\circ\cdots\circ\exp(t\log t)$ with m exponentials then

$$\ell(t)\simeq\frac{\log_m t}{\log_{m+1}(t)}.$$

Theorem 3.5 *Let N be a normal subgroup of $\mathbf{F}_r$. Assume that the group $\Gamma_1(N) = \mathbf{F}_r/N$ admits a sequence of Følner couples adapted to the function $\mathcal{V}$ as in Definition 3.1. Let $\mathcal{W}$ and $\gamma = \gamma_{\mathcal{W}}$ be related to $\mathcal{V}$ as in Proposition 3.4. Then we have*

$$\Phi_{\Gamma_2(N)}(n) \geq \frac{c}{\gamma(Cn)}.$$

Proof By the Magnus embedding, $\Gamma_2(N)$ is a subgroup of $\mathbb{Z}^r \wr \Gamma_1(N)$. By [16, Theorem 1.3], it follows that $\Phi_{\Gamma_2(N)} \geq \Phi_{\mathbb{Z}^r \wr \Gamma_1(N)}$. The conclusion then follows from Propositions 3.2–3.3. □

Example 3.1 Assume $\Gamma_1(N)$ has polynomial volume growth of degree D. Then $\Phi_{\Gamma_2(N)}(n) \geq \exp\left(-cn^{D/(2+D)}[\log n]^{2/(2+D)}\right)$.

Example 3.2 Assume $\Gamma_1(N)$ is either polycyclic or equal to the Baumslag–Solitar group $\mathrm{BS}(1,q) = \langle a, b | a^{-1}ba = b^q\rangle$, or equal to the lamplighter group $F \wr \mathbb{Z}$ with F finite. Then $\Phi_{\Gamma_2(N)}(n) \geq \exp\left(-cn/[\log n]^2\right)$.

Example 3.3 Assume $\Gamma_1(N) = F \wr \mathbb{Z}^D$ with F finite. Then

$$\Phi_{\Gamma_2(N)}(n) \geq \exp\left(-cn/[\log n]^{2/D}\right).$$

If instead $\Gamma_1(N) = \mathbb{Z}^b \wr \mathbb{Z}^D$ for some integer $b \geq 1$ then

$$\Phi_{\Gamma_2(N)}(n) \geq \exp\left(-cn\left(\frac{\log\log n}{\log n}\right)^{2/D}\right).$$

3.3 Another lower bound

The aim of this subsection is to provide lower bounds for the probability of return $\mu^{*n}(e_*)$ on $\Gamma_2(N)$ when μ at (3.2) is the push-forward of a measure $\boldsymbol{\mu}$ on $\mathbf{F}_r$ of the form (3.1), that is, supported on the powers of the generators $\mathbf{s}_i$, $1 \leq i \leq r$, possibly with unbounded support. Our approach is to construct symmetric probability measure ϕ on $\mathbb{Z}^r \wr \Gamma_1(N)$ such that the return probability $\phi^{\star n}(e_\star)$ of the random walk driven by ϕ coincides with $\mu^{*n}(e_*)$. Please note that we will use the notation $\star$ for convolution on the wreath product $\mathbb{Z}^r \wr \Gamma_1(N)$ and $*$ for convolution on $\Gamma_2(N)$. We also decorate the identity element e_* of $\Gamma_2(N)$ with a $*$ to distinguish it from the identity element $e_\star$ of $\mathbb{Z}^r \wr \Gamma_1(N)$. Recall that the identity element of $\Gamma_1(N)$ is denoted by $\bar{e}$. We will use $(\epsilon_i)_1^r$ for the canonical basis of $\mathbb{Z}^r$.

Fix r symmetric probability measures p_i, $1 \leq i \leq r$ on $\mathbb{Z}$. Recall that, by definition, μ is the push-forward of $\boldsymbol{\mu}$, the probability measure on $\mathbf{F}_r$ which gives probability $r^{-1}p_i(n)$ to $\mathbf{s}_i^n$, $1 \leq i \leq r$, $n \in \mathbb{Z}$. See (3.1)–(3.2).

On $\mathbb{Z}^r \wr \Gamma_1(N)$, consider the measures ϕ_i supported on elements of the form

$$g = (\delta^i, 0)(0, \overline{s}_i^m)(-\delta^i, 0),$$

where $\delta^i : \mathbf{F}_r/N = \Gamma_1(N) \to \mathbb{Z}^r$ is the function that's identically zero except that at identity e of $\Gamma_1(N)$, $\delta^i(e) = \epsilon_i \in \mathbb{Z}^r$. For such g, set (compare to (3.1))

$$\phi_i(g) = p_i(m).$$

Note that

$$g^{-1} = (\delta^i, 0)(0, \overline{s}_i^{-m})(-\delta^i, 0)$$

is an element of the same form, and $\phi_i(g^{-1}) = \phi_i(g) = p_i(m)$. Set

$$\phi = \frac{1}{r}\sum_{i=1}^{r} \phi_i.$$

More formally, ϕ can be written as

$$\forall\, g \in \mathbb{Z}^r \wr \Gamma_1(N), \quad \phi(g) = \sum_{1\le i\le r} \frac{1}{r} \sum_{m\in\mathbb{Z}} p_i(m) \mathbf{1}_{\{(\delta^i,0)(0,\bar{s}_i^m)(-\delta^i,0)\}}(g). \tag{3.6}$$

Let $(U_n)_1^\infty$ be a sequence of $\mathbf{F}_r$-valued i.i.d. random variables with distribution $\boldsymbol{\mu}$ and $Z_n = U_1 \cdots U_n$. Note that the projection of U_n to $\mathbf{F}_r/[N, N] = \Gamma_2(N)$ (resp. $\mathbf{F}_r/N = \Gamma_1(N)$) is an i.i.d. sequence of $\Gamma_2(N)$-valued (resp. $\Gamma_1(N)$-valued) random variables with distribution μ (resp. $\overline{\mu}$). Let X_i denote the projection of U_i on $\Gamma_1(N)$and $T_j = X_1 \cdots X_j$. Consider the $\mathbb{Z}^r \wr \Gamma_1(N)$-valued random variable defined by

$$V_n = (\delta^i, 0)(0, \bar{s}_i^m)(-\delta^i, 0) \text{ if } U_n = \mathbf{s}_i^m.$$

Then $(V_n)_1^\infty$ is a sequence of i.i.d. random variables on $\mathbb{Z}^r \wr \Gamma_1(N)$ with distribution ϕ. Write

$$W_n = V_1 \ldots V_n.$$

Then W_n is the random walk on $\mathbb{Z}^r \wr \Gamma_1(N)$ driven by ϕ.

The following proposition is based on Theorem 2.6, that is, [14, Theorem 2.7], which states that two words in $\mathbf{F}_r$ projects to the same element in $\Gamma_2(N)$ if and only if they induce the same flow on $\Gamma_1(N)$. In particular, the random walk on $\Gamma_2(N)$ returns to identity if and only if the path on $\Gamma_1(N)$ induces the zero flow function.

Proposition 3.6 *Fix a measure $\boldsymbol{\mu}$ on $\mathbf{F}_r$ of the form (3.1). Suppose none of the $\bar{s}_i$ are torsion elements in $\Gamma_1(N) = \mathbf{F}_r/N$. Let μ be the probability measure on $\Gamma_2(N)$ defined at (3.2). Let ϕ be the probability measure on $\mathbb{Z}^r \wr \Gamma_1(N)$ defined at (3.6). It holds that*

$$\mu^{*n}(e_*) = \phi^{\star n}(e_\star).$$

Remark 3.7 It's important here that the probability measure μ is supported on powers of generators, so that each step is taken along one dimensional subgraphs $g\,\langle \bar{s}_i\rangle$. The statement is not true for arbitrary measure on $\mathbf{F}/N'$.

Proof The random walk W_n on $\mathbb{Z}^r \wr (\mathbf{F}/N)$ driven by ϕ can be written as

$$W_n = (f_n, T_n) = ((f_n^1, \ldots, f_n^r), T_n).$$

By definition of W_n, for any $x \in \Gamma_1(N)$, $f_n^i(x)$ counts the algebraic sums of the i-arrivals and i-departures of the random walk T_n at x where by i-arrival (resp. i-departure) at x, we mean a time ℓ at which $T_\ell = x$ and $U_\ell \in \langle \mathbf{s}_i\rangle$ (resp. $U_{\ell+1} \in \langle \mathbf{s}_i\rangle$). The condition $T_n = x \neq \bar{e}$ implies that the vector $f_n(x)$ must have at least one non-zero component because the total number of arrivals and departures at x must be odd. Hence, we have

$$\phi^{\star n}(e_\star) = \mathbf{P}((f_n, T_n) = e_\star) = \mathbf{P}(f_n^i(x) = 0,\ 1 \le i \le r,\ x \in \Gamma_1(N)).$$

We also have

$$\mu^{*n}(e_*) = \mathbf{P}(\mathfrak{f}_{Z_n}(x, x\bar{s}_i, \mathbf{s}_i) = 0,\ 1 \le i \le r,\ x \in \Gamma_1(N)).$$

Given a flow $\mathfrak{f}$ on $\Gamma_1(N)$ (i.e., a function the edge set $\mathfrak{E} = \{(x, x\bar{s}_i, \mathbf{s}_i), x \in \Gamma_1(N), 1 \le i \le r\} \subset \Gamma_1(N) \times \Gamma_1(N) \times S$, for each i, $1 \le i \le r$, introduce the i-partial total flow $\partial_i \mathfrak{f}(x)$ at $x \in \Gamma_1(N)$ by setting

$$\partial_i \mathfrak{f}(x) = \mathfrak{f}((x, x\bar{s}_i, \mathbf{s}_i)) - \mathfrak{f}(x\bar{s}_i^{-1}, x, \mathbf{s}_i).$$

It is easy to check (e.g., by induction on n) that

$$\forall\, x \in \Gamma_1(N), \quad f_n^i(x) = \partial_i \mathfrak{f}_{Z_n}(x). \tag{3.7}$$

Obviously, $\mathfrak{f}_{Z_n} \equiv 0$ implies $f_n^i \equiv 0$ for all $1 \le i \le r$ so

$$\phi^{\star n}(e_\star) \ge \mu^{*n}(e_*).$$

But, in fact, under the assumption that none of the $\bar{s}_i$ are torsion elements in $\Gamma_1(N)$, each edge $(x, x\bar{s}_i, \mathbf{s}_i)$ in the Cayley graph of $\Gamma_1(N)$ is contained in the one dimensional infinite linear subgraph $\{x\bar{s}_i^k : k \in \mathbb{Z}\}$ and, since f_n and $\mathfrak{f}_{Z_n}$ are finitely supported, the Eq. (3.7) shows that $f_n^i \equiv 0$ implies that $\mathfrak{f}_{Z_n}(x, x\bar{s}_i, \mathbf{s}_i) = 0$ for all $x \in \Gamma_1(N)$. In particular, if $f_n^i \equiv 0$ for all $1 \le i \le r$ then we must have $\mathfrak{f}_{Z_n} \equiv 0$. Hence, if none of the $\bar{s}_i$ is a torsion element in $\Gamma_1(N)$, we have $f_n^i \equiv 0$, $1 \le i \le r \iff \mathfrak{f}_{Z_n} \equiv 0$ and thus $\mu^{*n}(e_*) = \phi^{\star n}(e_\star)$. □

In general, the probability measure ϕ on $\mathbb{Z}^r \wr \Gamma_1(N)$ does not have generating support because of the very specific and limited nature of the lamp moves and how they correlate to the base moves. To fix this problem, let η_r be the probability measure of the lazy random walk on $\mathbb{Z}^r$ so that $\eta_r(0) = 1/2$ and $\eta_r(\pm\epsilon_i) = 1/(4r)$, $1 \le i \le r$. With this notation, let

$$q = \eta_r \star \bar{\mu} \star \eta_r \tag{3.8}$$

be the probability measure of the switch-walk-switch random walk on the wreath product $\mathbb{Z}^r \wr \Gamma_1(N)$ associated with the walk-measure $\bar{\mu}$ on the base-group $\Gamma_1(N)$ and the switch-measure η_r on the lamp-group $\mathbb{Z}^r$. See [15,19] and Sect. 1.5 for further details.

Proposition 3.8 *Fix a measure* $\boldsymbol{\mu}$ *on* $\mathbf{F}_r$ *of the form (3.1). Suppose that none of the* $\bar{s}_i$ *are torsion elements in* $\Gamma_1(N) = \mathbf{F}_r/N$. *Referring to the notation introduced above, there are* $c, N \in (0, \infty)$ *such that the probability measure* μ *on* $\Gamma_2(N)$ *defined by (3.2) and the measure* q *on* $\mathbb{Z}^r \wr \Gamma_1(N)$ *defined at (3.8) satisfy*

$$\mu^{*2n}(e_*) \ge c q^{\star 2Nn}(e_\star).$$

Proof On a group G, the Dirichlet form associated with a symmetric measure p is defined by

$$\mathcal{E}_p(f, f) = \frac{1}{2} \sum_{x,y \in G} |f(xy) - f(x)|^2 p(y).$$

From the definition, it easily follows that $\mathbb{Z}^r \wr \Gamma_1(N)$, we have the comparison of Dirichlet forms

$$\mathcal{E}_\phi \le (2r)^2 \mathcal{E}_{\eta_r \star \overline{\mu} \star \eta_r} = (2r)^2 \mathcal{E}_q.$$

Therefore, by [16, Theorem 2.3],

$$\phi^{\star 2n}(e_\star) \ge c q^{\star 2Nn}(e_\star).$$

From Proposition 3.6 we conclude that

$$\mu^{*2n}(e_*) = \phi^{\star 2n}(e_\star) \ge c q^{\star 2Nn}(e_\star).$$

□

Corollary 3.9 *Fix $a = (\alpha_1, \dots, \alpha_r) \in (0, 2)^r$ and let $\boldsymbol{\mu}_a$ be defined by (3.1) with $p_i(m) = c_i(1 + |m|)^{-1-\alpha_i}$. Let $N = [\mathbf{F}_r, \mathbf{F}_r]$ so that $\Gamma_1(N) = \mathbb{Z}^r$ and $\Gamma_2(N) = \mathbf{S}_{2,r}$. Let μ_a be the probability measure on $\mathbf{S}_{2,r}$ associated to $\boldsymbol{\mu}_a$ by (3.2). Then we have*

$$\mu_a^{*n}(e_*) \geq \exp\left(-Cn^{r/(r+\alpha)}[\log n]^{\alpha/(r+\alpha)}\right)$$

where

$$\frac{1}{\alpha} = \frac{1}{r}\left(\frac{1}{\alpha_1} + \cdots + \frac{1}{\alpha_r}\right).$$

Remark 3.10 Later we will prove a matching upper bound.

Proof Proposition 3.8 yields

$$\mu_a^{*n}(e_*) \geq c[\eta_r \star \bar{\mu}_a \star \eta_r]^{\star n}(e_\star)$$

where the probability $\bar{\mu}_a$ on $\Gamma_1(N) = \mathbb{Z}^r$ is defined at (3.2) and is given explicitly by

$$\bar{\mu}_a(g) = \frac{1}{r}\sum_1^r p_i(m)\mathbf{1}_{\bar{s}_i{}^n}(g)$$

where $\bar{s}_i$ canonical generators of $\mathbb{Z}^r$ and we have retained the multiplicative notation so that $\bar{s}_1^n = (n, 0, \dots, 0), \dots, \bar{s}_r^n = (0, \dots, 0, n)$.

The behavior of the random walk on the wreath product $\mathbb{Z}^r \wr \mathbb{Z}^r$ associated with the switch-walk-switch measure $q = \eta_r \star \bar{\mu}_a \star \eta_r$ is studied in [19] where it is proved that

$$q^{\star 2n}(e_\star) \simeq \exp\left(-n^{r/(r+\alpha)}[\log n]^{\alpha/(r+\alpha)}\right).$$

Corollary 3.9 follows. □

4 Return probability upper bounds

This section explains how to use the Magnus embedding (defined at (2.1))

$$\bar{\psi} : \mathbf{F}_r/[N, N] = \Gamma_2(N) \hookrightarrow \mathbb{Z}^r \wr \Gamma_1(N), \quad \Gamma_1(N) = \mathbf{F}_r/N,$$

to produce, in certain cases, an upper bound on the probability of return $\mu^{*2n}(e_*)$ on $\Gamma_2(N)$. Recall from (2.3) that the Magnus embedding ψ is described more concretely by

$$\begin{aligned} &\Gamma_2(N) \hookrightarrow \mathbb{Z}^r \wr \Gamma_1(N) \\ &g \mapsto \bar{\psi}(g) = (\bar{a}(g), \overline{g}), \quad \overline{g} = \bar{\pi}(g). \end{aligned}$$

Here $\bar{a}(g)$ is an element of $\sum_{x \in \Gamma_1(N)} \mathbb{Z}^r_x$, equivalently, a $\mathbb{Z}^r$-valued function with finite support defined on $\Gamma_1(N)$, equivalently, an element of the $\mathbb{Z}(\Gamma_1(N))$-module $\mathbb{Z}^r(\Gamma_1(N))$. In any group G, we let $\tau_g x = gx$ be the translation by $g \in G$ on the left as well as its extension to any $\mathbb{Z}(G)$ module. We will need the following lemma.

Lemma 4.1 *For any $g, h \in \Gamma_2(N)$ with $\bar{g} = \bar{\pi}(g) \in \Gamma_1(N)$, we have*

$$\bar{a}(gh) = \bar{a}(g) + \tau_{\bar{g}}\bar{a}(h).$$

In particular, if $g \in \Gamma_2(N)$ and $\boldsymbol{\rho} \in N$ with $\rho = \pi_2(\boldsymbol{\rho}) \in \Gamma_2(N)$, we have

$$\bar{a}(g\rho g^{-1}) = \tau_{\bar{g}}\bar{a}(\rho).$$

Proof The first formula follows from the Magnus embedding by inspection. The second formula is an easy consequence of the first and the fact that $\pi(\boldsymbol{\rho})$ is the identity element in $\Gamma_1(N)$. □

Remark 4.2 The identities stated in Lemma 4.1 can be equivalently written in terms of flows on $\Gamma_1(N)$. Namely, for $\mathbf{u}, \mathbf{v} \in \mathbf{F}_r$, we have

$$\mathfrak{f}_{\mathbf{uv}} = \mathfrak{f}_{\mathbf{u}} + \tau_{\pi(\mathbf{u})}\mathfrak{f}_{\mathbf{v}} \text{ and } \mathfrak{f}_{\mathbf{uvu}^{-1}} = \tau_{\pi(\mathbf{u})}\mathfrak{f}_{\mathbf{v}}.$$

4.1 Exclusive pairs

Definition 4.3 Let Γ be a subgroup of $\Gamma_2(N)$ and $\boldsymbol{\rho}$ be a reduced word in $N\backslash[N, N] \subset \mathbf{F}_r$. Let $\overline{\Gamma} = \bar{\pi}(\Gamma)$. Set $\rho = \pi_2(\boldsymbol{\rho}) \in \Gamma_2(N)$. We say the pair $(\Gamma, \boldsymbol{\rho})$ is *exclusive* if the following two conditions are satisfied:

(i) The collection $\{\tau_{\bar{g}}(\bar{a}(\rho))\}_{\bar{g}\in\overline{\Gamma}}$ is $\mathbb{Z}$-independent in the $\mathbb{Z}$-module $\sum_{\Gamma_1(N)}(\mathbb{Z}^r)_x$.
(ii) In the $\mathbb{Z}$-module $\sum_{\Gamma_1(N)}(\mathbb{Z}^r)_x$, the $\mathbb{Z}$-submodule generated by $\{\tau_{\bar{g}}(\bar{a}(\rho))\}_{\bar{g}\in\overline{\Gamma}}$, call it $A = A(\Gamma, \boldsymbol{\rho})$, has trivial intersection with the subset $B = \{\bar{a}(g) : g \in \Gamma\}$, that is

$$A \cap B = \{\mathbf{0}\}.$$

Remark 4.4 Condition (i) implies that the $\mathbb{Z}$-submodule $A(\Gamma, \boldsymbol{\rho})$ of $\sum_{\Gamma_1(N)}(\mathbb{Z}^r)_x$ is isomorphic to $\sum_{\bar{g}\in\overline{\Gamma}}(\mathbb{Z})_{\bar{g}}$.

Example 4.1 In the free metabelian group $S_{2,r} = \mathbf{F}_r/[N, N]$, $N = [\mathbf{F}_r, \mathbf{F}_r]$, set $\Gamma = \langle s_1^2, \ldots, s_r^2 \rangle$, and $\boldsymbol{\rho} = [\mathbf{s}_1, \mathbf{s}_2]$. Then $(\Gamma, \boldsymbol{\rho})$ is an exclusive pair. The conditions (i)–(ii) are easy to check because the monomials $\{Z_1^{x_1} Z_2^{x_2} \ldots Z_r^{x_r} : x \in \mathbb{Z}^d\}$ are $\mathbb{Z}$-linear independent in $\mathbb{Z}(\mathbb{Z}^r)$. A similar idea was used in the proof of [7, Theorem 3.2].

We now formulate a sufficient condition for a pair $(\Gamma, \boldsymbol{\rho})$ to be exclusive. This sufficient condition is phrased in terms of the representation of the elements of $\Gamma_2(N)$ as flows on $\Gamma_1(N)$. Recall that $\Gamma_1(N)$ come equipped with a marked Cayley graph structure as described in Sect. 2.2.

Lemma 4.5 *Fix* $\Gamma < \Gamma_2(N)$ *and* $\boldsymbol{\rho}$ *as in Definition 4.3. Set*

$$U = \bigcup_{g\in\Gamma} supp(\mathfrak{f}_g),$$

that is the union of the support of the flows on $\Gamma_1(N)$ *induced by elements of* Γ. *Assume that* $\boldsymbol{\rho} = \mathbf{usv}$ *with* $\mathbf{s} \in \{\mathbf{s}_1, \ldots, \mathbf{s}_r\}$ *and that:*

1. $\mathfrak{f}_{\boldsymbol{\rho}}((\overline{u}, \overline{us}, \mathbf{s})) \neq 0$;
2. *For all* $x \in \overline{\Gamma}\backslash\{\bar{e}\}$, $\mathfrak{f}_{\boldsymbol{\rho}}((x\overline{u}, x\overline{us}, \mathbf{s})) = 0$;
3. *The edge* $(\overline{u}, \overline{us}, \mathbf{s})$ *is not in* U.

Then the pair $(\Gamma, \boldsymbol{\rho})$ *is exclusive.*

Remark 4.6 The first assumption insures that the given edge is really active in the loop associated with ρ on the Cayley graph $\Gamma_1(N)$. The proof given below shows that conditions 1-2 above imply condition (i) of Definition 4.3. All three assumptions above are used to obtain condition (ii) of Definition 4.3.

Proof Condition (i). Suppose there is a nontrivial linear relation such that

$$c_1\tau_{\overline{g}_1}(\bar{a}(\rho)) + \cdots + c_n\tau_{\overline{g}_n}(\bar{a}(\rho)) = 0, \quad c_i \in \mathbb{Z},$$

where some c_j, say c_1, is not zero and the element $\overline{g}_j \in \overline{\Gamma}$ are pairwise distinct. Let $\mathbf{g}_j$ be representative of $\bar{g}_j$ in $\mathbf{F}_r$. Let b denote the coefficient of $\sum_{i=1}^n c_i\tau_{\overline{g}_i}(\bar{a}(\rho))$ in front of the term $\overline{g}_1\overline{u}\lambda_{\mathbf{s}}$. By formula (2.5),

$$b = \sum_{i=1}^{n} c_i \mathfrak{f}_{\mathbf{g}_i\boldsymbol{\rho}\mathbf{g}_i^{-1}}((\overline{g}_1\overline{u}, \overline{g}_1\overline{us}, \mathbf{s})).$$

Note that

$$\mathfrak{f}_{\mathbf{g}_i\boldsymbol{\rho}\mathbf{g}_i^{-1}}((\overline{g}_1\overline{u}, \overline{g}_1\overline{us}, \mathbf{s})) = \mathfrak{f}_{\boldsymbol{\rho}}((\overline{g}_i^{-1}\overline{g}_1\overline{u}, \overline{g}_i^{-1}\overline{g}_1\overline{us}, \mathbf{s})).$$

Therefore, since $\overline{g}_i^{-1}\overline{g}_1 \neq \overline{e}$ for all $i \neq 1$, the hypothesis stated in Lemma 4.5(2) gives

$$\forall\, i \neq 1, \quad \mathfrak{f}_{\mathbf{g}_i\boldsymbol{\rho}\mathbf{g}_i^{-1}}((\overline{g}_1\overline{u}, \overline{g}_1\overline{us}, \mathbf{s})) = 0.$$

By hypothesis (1) of Lemma 4.5, this implies

$$b = c_1\mathfrak{f}_{\boldsymbol{\rho}}((\overline{u}, \overline{us}, \mathbf{s})) \neq 0$$

which provides a contradiction.

We now verify that Condition (ii) of Definition 4.3 holds. Fix $x \in A \cap B$ and assume that x is nontrivial. From Condition (i), x can be written uniquely as

$$x = c_1\tau_{\overline{g}_1}\overline{a}(\rho) + \cdots + c_n\tau_{\overline{g}_n}\overline{a}(\rho),$$

where $c_j \in \mathbb{Z}\backslash\{0\}$ and the elements $\overline{g}_j$ are pairwise distinct. On the other hand, since $x \in B$, there exists some $h \in \Gamma$ such that $x = \bar{a}(h)$. By formula (2.5), $\bar{a}(h) = \sum_{i=1}^n c_i\tau_{\overline{g}_i}\overline{a}(\rho)$ is equivalent to

$$\mathfrak{f}_h = \sum_{i=1}^{n} c_i\mathfrak{f}_{g_i\rho g_i^{-1}}.$$

Therefore $\mathfrak{f}_{g_1^{-1}hg_1} = \sum_{i=1}^n c_i\mathfrak{f}_{g_1^{-1}g_i\rho g_i^{-1}g_1}$. By hypothesis (2), it follows that

$$\mathfrak{f}_{g_1^{-1}hg_1}((\overline{u}, \overline{us}, \mathbf{s})) = c_1\mathfrak{f}_{\rho}((\overline{u}, \overline{us}, \mathbf{s})) \neq 0.$$

However, since $g_1^{-1}hg_1 \in \Gamma$, this implies that $(\overline{u}, \overline{us}, \mathbf{s}_i)) \in U$, a conclusion which contradicts assumption (3). Hence $A \cap B = \{\mathbf{0}\}$ as desired. □

4.2 Existence of exclusive pairs

This section discuss algebraic conditions that allow us to produce appropriate exclusive pairs.

Lemma 4.7 *Assume $\Gamma_1(N) = \mathbf{F}_r/N$ is residually finite and $r \geq 2$. Fix an element $\boldsymbol{\rho}$ in $N\backslash[N, N]$. There exists a finite index normal subgroup $K = K_{\boldsymbol{\rho}} \lhd \Gamma_1(N)$ such that, for any edge $(\mathbf{u}, \mathbf{us})$ in $\boldsymbol{\rho} = \mathbf{usv}$ with $\mathbf{s} \in \{\mathbf{s}_1, \ldots, \mathbf{s}_r\}$ and any subgroup $H < \Gamma_2(N)$ with $\overline{\pi}(H) < K$,*

$$\forall\, x \in \bar{\pi}(H)\backslash\{\overline{e}\}, \quad \mathfrak{f}_{\boldsymbol{\rho}}((x\overline{u}, x\overline{us}, \mathbf{s})) = 0.$$

Remark 4.8 Since $\boldsymbol{\rho} \notin [N, N]$, the flow induced by $\boldsymbol{\rho}$ is not identically zero. Therefore, after changing $\boldsymbol{\rho}$ to $\boldsymbol{\rho}^{-1}$ if necessary, there exists a reduced word $\mathbf{u}$ and $i \in \{1, \dots, r\}$ such that $\boldsymbol{\rho} = \mathbf{u}\mathbf{s}_i\mathbf{u}'$ and $\mathfrak{f}_{\boldsymbol{\rho}}((\overline{u}, \overline{us}_i, \mathbf{s}_i)) \neq 0$. Hence Lemma 4.7 provides a way to verify conditions 1 and 2 of Lemma 4.5.

Proof For any element $\boldsymbol{\rho}$ in $N\backslash[N, N]$, view $\boldsymbol{\rho}$ as a reduced word in $\mathbf{F}_r$. Let $B_{\boldsymbol{\rho}}$ be the collection of all proper subword $\mathbf{u}$ of $\boldsymbol{\rho}$ such that $\bar{\pi}(\mathbf{u})$ is not trivial in $\Gamma_1(N)$. Since $\Gamma_1(N)$ is residually finite, there exists a normal subgroup $K \lhd \Gamma_1(N)$ such that $\Gamma_1(N)/K$ is finite and $\bar{\pi}(B_\rho) \cap K = \emptyset$.

Suppose there exists $x \in \bar{\pi}(H)$ such that x is not trivial and $\mathfrak{f}_\rho((x\overline{u}, x\overline{us}, \mathbf{s})) \neq 0$. Therefore, there is a proper subword $\mathbf{v}$ of $\boldsymbol{\rho}$ such that $\boldsymbol{\rho} = \mathbf{vw}$ and $\overline{v} = x\overline{u}$. Since both $\mathbf{u}$ and $\mathbf{v}$ are prefixes of $\boldsymbol{\rho}$ and x is not trivial, $\mathbf{vu}^{-1}$ is the conjugate of a proper subword of $\boldsymbol{\rho}$ with non-trivial image in $\Gamma_1(N)$. By construction this implies that $\bar{\pi}(\mathbf{vu}^{-1}) \notin K$, a contradiction since $\bar{\pi}(\mathbf{vu}^{-1}) = x \in \bar{\pi}(H) < K$. □

Remark 4.9 Subgroups of finitely residually finite groups are residually finite. By a classical result of Hirsch, polycyclic groups are residually finite, [17, 5.4.17]. By a result of P. Hall, a finitely generated group which is an extension of an abelian group by a nilpotent group is residually finite. In particular, all finitely generated metabelian groups are residually finite, [17, 15.4.1]. Gruenberg [10], proves that $A \wr G$ is residually finite if either A is abelian or G is residually finite. The free solvable groups $\mathbf{S}_{d,r}$ are residually finite as subgroups of wreath products. Note that all finitely generated residually finite groups are Hopfian, [17, 6.1.11].

Our next task is to find ways to verify condition 3 of Lemma 4.5. To this end, let A be the abelian group $\Gamma_1(N)/[\Gamma_1(N), \Gamma_1(N)]$. Fix $m = (m_1, \dots, m_r) \in \mathbb{N}^r$ and let A_m be the subgroup of A generated by the images of the elements $\bar{s}_i^{m_i}$, $1 \leq i \leq r$, in A. Let T_m be the finite abelian group $T_m = A/A_m$. Let $\pi_{T_m} : \mathbf{F}_r \to T_m$ be the projection from $\mathbf{F}_r$ onto T_m. Set

$$H_m = \langle s_i^{m_i}, 1 \leq i \leq r \rangle < \Gamma_2(N).$$

Lemma 4.10 *Fix a reduced word* $\boldsymbol{\rho} \in N\backslash[N, N]$. *Assume that* $\boldsymbol{\rho} = \mathbf{usv}$, *where* $\mathbf{s} \in \{\mathbf{s}_1, \dots, \mathbf{s}_r\}$ *and* $\mathfrak{f}_\rho((\overline{u}, \overline{us}, \mathbf{s})) \neq 0$. *Fix* $m \in \mathbb{N}^r$ *and assume that, in the finite abelian group* T_m, $\pi_{T_m}(\mathbf{u}) \notin \langle \pi_{T_m}(\mathbf{s}) \rangle$. *Then the edge* $(\overline{u}, \overline{us}, \mathbf{s}))$ *is not in*

$$U(H_m) = \bigcup_{g \in H_m} \operatorname{supp}(\mathfrak{f}_g).$$

Proof Assume that $(\overline{u}, \overline{us}, \mathbf{s}) \in U(H_m)$. Then there must exist $x \in \pi(H_m)$ and $q \in \mathbb{Z}$ such that $x\bar{s}^q = \overline{u}$. But, projecting on T_m, this contradicts the assumption $\pi_{T_m}(\mathbf{u}) \notin \langle \pi_{T_m}(\mathbf{s}) \rangle$. □

We now put together these two lemmas and state a proposition that will allow us to produce exclusive pairs.

Proposition 4.11 *Fix* $N \lhd \mathbf{F}_r$ *and* $\boldsymbol{\rho} \in N\backslash[N, N]$, *in reduced form. Let* $\mathbf{u}$ *be a prefix of* $\boldsymbol{\rho}$ *such that* $\boldsymbol{\rho} = \mathbf{usv}$, $\mathbf{s} \in \{\mathbf{s}_1, \dots, \mathbf{s}_r\}$ *and* $\mathfrak{f}_{\boldsymbol{\rho}}((\bar{u}, \bar{u}\bar{s}, \mathbf{s})) \neq 0$. *Assume that the group* $\Gamma_1(N)$ *is residually finite and there exists an integer vector* $m = (m_1, \dots, m_r) \in \mathbb{N}^r$ *such that, in the finite group* T_m, $\pi_{T_m}(\mathbf{u}) \notin \langle \pi_{T_m}(\mathbf{s}) \rangle$. *Then there is* $m' = (m'_1, \dots, m'_r)$ *such that the pair* $(H_{m'}, \boldsymbol{\rho})$ *is an exclusive pair in* $\Gamma_2(N)$.

Proof Let $K_{\boldsymbol{\rho}}$ the the finite index normal subgroup of $\Gamma_1(N)$ given by Lemma 4.7. Since $K_{\boldsymbol{\rho}}$ is of finite index in $\Gamma_1(N)$, we can pick m'_i to be a multiple of m_i such that $\bar{s}_i^{m'_i} \in K_{\boldsymbol{\rho}}$.

Observe that the assumption $\pi_{T_m}(\mathbf{u}) \notin \langle \pi_{T_m}(\mathbf{s}) \rangle$ implies the same property with m replaced by m'. Applying Lemmas 4.7, 4.10 and Lemma 4.5 yields that $(H_{m'}, \boldsymbol{\rho})$ is an exclusive pair in $\Gamma_2(N)$. □

We conclude this section with a concrete application of Proposition 4.11.

Proposition 4.12 *Assume that $\Gamma_1(N) = \mathbf{F}_r/N$ is an infinite nilpotent group and $r \geq 2$. Then there exists an exclusive pair (Γ, ρ) in $\Gamma_2(N)$ with Γ finitely generated such that $\pi(\Gamma)$ is a subgroup of finite index in $\Gamma_1(N)$.*

Proof First we construct an exclusive pair using Proposition 4.11. Suppose that $\Gamma_1(N)$ is not virtually $\mathbb{Z}$. Then the torsion-free rank of $\Gamma_1(N)/[\Gamma_1(N), \Gamma_1(N)]$ is at least 2. Choose two generators s_{i_1}, s_{i_2} such that their projections in the abelianization are $\mathbb{Z}$-independent. Choose $\boldsymbol{\rho}$ to be an element of minimal length in $N \cap \langle \mathbf{s}_{i_1}, \mathbf{s}_{i_2} \rangle$. Note that since $\Gamma_1(N)$ is nilpotent, this intersection contains commutators of $\mathbf{s}_{i_1}, \mathbf{s}_{i_2}$ with length greater than the nilpotency class, therefore it is non-empty. Proposition 4.11 applies and yields an integer m such that $(\Gamma = \langle s_1^m, \ldots, s_r^m \rangle, \boldsymbol{\rho})$ is an exclusive pair.

In the special case when $\Gamma_1(N)$ is virtually $\mathbb{Z}$, choose $\boldsymbol{\rho}$ to be an element of minimal length in N, and a generator $\mathbf{s}_{i_1}$ such that $\bar{s}_{i_1}$ is not a torsion element in $\Gamma_1(N)$. Set $\Gamma = \left\langle s_{i_1}^m \right\rangle$ with $m = [\Gamma_1(N) : K_{\boldsymbol{\rho}}]$. Then by Lemmas 4.5, Lemma 4.7 and inspection, $(\Gamma, \boldsymbol{\rho})$ is an exclusive pair.

Next we use induction on nilpotency class c to show that, for any $m \in \mathbb{N}$, $\pi(\Gamma) = \langle \bar{s}_1^m, \ldots, \bar{s}_r^m \rangle$ is a subgroup of finite index in $\Gamma_1(N)$. When $c = 1$, observe that the statement is obviously true for finitely generated abelian groups. Suppose $\Gamma_1(N)$ is of nilpotency class c. Let $H = \gamma_c(\Gamma_1(N))$. Using the induction hypothesis, it suffices to prove that $H \cap \pi(\Gamma)$ is a finite index subgroup of H. Note that H is contained in the center of $\Gamma_1(N)$ and is generated by commutators of length c. Further,

$$[s_{i_c}[\ldots[s_{i_2}, s_{i_1}]]]^{m^c} = [s_{i_c}^m[\ldots[s_{i_2}^m, s_{i_1}^m]]].$$

Therefore $H/H \cap \pi(\Gamma)$ is a finitely generated torsion abelian group, hence finite, as desired. □

4.3 Random walks associated with exclusive pairs

The following result captures the main idea and construction of this section.

Theorem 4.13 *Let μ be a symmetric probability measure on $\Gamma_2(N)$. Let $\Gamma < \Gamma_2(N)$ and $\boldsymbol{\rho}$ be an exclusive pair as in Definition 4.3. Set $\rho = \pi_2(\boldsymbol{\rho}) \in \Gamma_2(N)$. Let ν be the probability measure on $\Gamma_2(N)$ such that*

$$\nu(\rho^{\pm 1}) = 1/2.$$

Let φ be a symmetric probability measure on Γ such that

$$\mathcal{E}_{\nu * \varphi * \nu} \leq C_0 \mathcal{E}_\mu. \tag{4.1}$$

Let $\bar{\varphi}$ be the symmetric probability on $\overline{\Gamma} = \bar{\pi}(\Gamma) < \Gamma_1(N)$ defined by

$$\forall\, \bar{g} \in \Gamma_1(N), \quad \bar{\varphi}(\bar{g}) = \varphi(\bar{\pi}^{-1}(\bar{g})).$$

On the wreath product $\mathbb{Z} \wr \overline{\Gamma}$ (whose group law will be denoted here by $\star$), consider the switch-walk-switch measure $q = \eta \star \bar{\varphi} \star \eta$ with $\eta(\pm 1) = 1/2$ on $\mathbb{Z}$. Then there are constants $C, k \in (0, \infty)$ such that

$$\mu^{*2kn}(e_*) \leq C q^{\star 2n}(e_\star).$$

Proof By [16, Theorem 2.3], the comparison assumption between the Dirichlet forms of μ and $\nu * \varphi * \nu$ implies that there is a constant C and an integer k such that

$$\forall n, \quad \mu^{*kn}(e_*) \leq C[\nu * \varphi * \nu]^{*2n}(e_*).$$

Hence, the desired conclusion easily follows from the next proposition. □

Proposition 4.14 *Let $\Gamma < \Gamma_2(N)$ and $\boldsymbol{\rho}$ be an exclusive pair as in Definition 4.3. Let $\rho = \pi(\boldsymbol{\rho})$ and let ν be the probability measure on $\Gamma_2(N)$ such that $\nu(\rho) = \nu(\rho^{-1}) = \frac{1}{2}$. Let φ be a probability measure supported on Γ. Let $\bar{\varphi}$ be the push-forward of φ on $\bar{\pi}(\Gamma) = \overline{\Gamma} < \Gamma_1(N)$. Let η be the probability measure on $\mathbb{Z}$ such that $\eta(\pm 1) = 1/2$. Let $q = \eta \star \bar{\varphi} \star \eta$ be the switch-walk-switch measure on $\mathbb{Z} \wr \overline{\Gamma}$. Then*

$$(\nu * \varphi * \nu)^{*n}(e_*) \leq (\eta \star \overline{\varphi}' \star \eta)^{\star n}(e_\star) = q^{\star n}(e_\star).$$

To prove this proposition, we will use the following lemma.

Lemma 4.15 *Let φ be a probability measure on $\Gamma_2(N)$. Let ν be the uniform measure on $\{r_0^{\pm 1}\}$ where $r_0 \in \Gamma_2$ and $r_0 \neq r_0^{-1}$. Let $(Y_i)_1^\infty$ and $(\varepsilon_i)_1^\infty$ be i.i.d. sequence with law φ and ν respectively. Let $S_n = Y_1 \cdots Y_n$ and $\overline{S}_n = \bar{\pi}(S_n)$. Then we have*

$$\begin{aligned}&(\nu * \varphi * \nu)^{*n}(e_*)\\ &\quad = \mathbf{P}\left(\overline{S}_n = \bar{e}, \sum_{j=1}^{n}(\varepsilon_{2j-1} + \varepsilon_{2j})\tau_{\overline{S}_j}\bar{a}(r_0) + \bar{a}(S_n) = 0\right).\end{aligned}$$

Proof The product $r_0^{\varepsilon_1} Y_1 r_0^{\varepsilon_2+\varepsilon_3} Y_2 r_0^{\varepsilon_4} \cdots r_0^{\varepsilon_{2n-1}} Y_n r_0^{\varepsilon_{2n}}$ has distribution

$$(\nu * \varphi * \nu)^{*n}.$$

Therefore, we have

$$\begin{aligned}(\nu * \mu * \nu)^{*n}(e_*) &= \mathbf{P}(r_0^{\varepsilon_1} Y_1 r_0^{\varepsilon_2+\varepsilon_3} Y_2 \cdots Y_n r_0^{\varepsilon_n} = e_*)\\ &= \mathbf{P}(Y_1 r_0^{\varepsilon_2+\varepsilon_3} Y_2 \cdots Y_n r_0^{\varepsilon_{2n}+\varepsilon_1} = e_*).\end{aligned}$$

Using the Magnus embedding

$$\psi : \mathbf{F}/[N, N] \hookrightarrow \mathbb{Z}^r \wr \Gamma_1(N)$$

(and re-indexing of the ε_i) this yields

$$(\nu * \mu * \nu)^{*n}(e_*) = \mathbf{P}(\overline{S}_n = \bar{e}, \bar{a}(Y_1 r_0^{\varepsilon_1+\varepsilon_2} Y_2 \cdots Y_n r_0^{\varepsilon_{2n-1}+\varepsilon_{2n}}) = 0).$$

However, we have

$$\begin{aligned}&\bar{a}(Y_1 r_0^{\varepsilon_1+\varepsilon_2} Y_2 \cdots Y_n r_0^{\varepsilon_{2n-1}+\varepsilon_{2n}})\\ &\quad = \bar{a}(S_1 r_0^{\varepsilon_1+\varepsilon_2} S_1^{-1} \cdots S_n r_0^{\varepsilon_{2n-1}+\varepsilon_{2n}} S_n^{-1} S_n)\\ &\quad = \sum_{j=1}^{n}(\varepsilon_{2j-1} + \varepsilon_{2j})\bar{a}(S_j r_0 S_j^{-1}) + \bar{a}(S_n)\\ &\quad = \sum_{j=1}^{n}(\varepsilon_{2j-1} + \varepsilon_{2j})\tau_{\overline{S}_j}\bar{a}(r_0) + \bar{a}(S_n).\end{aligned}$$

The last equality above from Lemma 4.1. □

Proof Proof of Proposition 4.14 By Lemma 4.15,

$$(\nu * \mu * \nu)^{*n}(e_*) = \mathbf{P}\left(\overline{S}_n = \bar{e}, \sum_{j=1}^{n}(\varepsilon_{2j-1}+\varepsilon_{2j})\tau_{\overline{S}_j}\bar{a}(r_0)) + \bar{a}(S_n) = 0\right).$$

Under the assumption that (Γ, ρ) is an exclusive pair, (ii) of Definition 4.3 gives

$$\left\{\sum_{j=1}^{n}(\varepsilon_{2j-1}+\varepsilon_{2j})\tau_{\overline{S}_j}\bar{a}(r_0) + \bar{a}(S_n) = 0\right\} = \left\{\sum_{j=1}^{n}(\varepsilon_{2j-1}+\varepsilon_{2j})\tau_{\overline{S}_j}\bar{a}(r_0) = 0\right\} \cap \{\bar{a}(S_n) = 0\} \tag{4.2}$$

Further, (i) of Definition 4.3 gives

$$\left\{\sum_{j=1}^{n}(\varepsilon_{2j-1}+\varepsilon_{2j})\tau_{\overline{S}_j}\bar{a}(r_0) = 0\right\} = \bigcap_{x\in\overline{\Gamma}}\left\{\sum_{j=1}^{n}(\varepsilon_{2j-1}+\varepsilon_{2j})\mathbf{1}_{\{x\}}(\overline{S}_j) = 0\right\}.$$

Therefore, dropping $\{\bar{a}(S_n) = 0\}$ in (4.2) yields

$$(\nu * \varphi * \nu)^{*n}(e_*) \le \mathbf{P}\left(\overline{S}_n = \bar{e}, \sum_{j=1}^{n}(\varepsilon_{2j-1}+\varepsilon_{2j})\mathbf{1}_{\{x\}}(\overline{S}_j) = 0 \text{ for all } x \in \overline{\Gamma}\right).$$

On the other hand, the return probability of the random walk on

$$\mathbb{Z} \wr \overline{\Gamma} < \mathbb{Z} \wr \Gamma_1(N)$$

driven by $\eta \star \overline{\varphi}' \star \eta$ is exactly

$$(\eta \star \overline{\varphi} \star \eta)^{\star n}(e_\star) = \mathbf{P}\left(\overline{S}_n = \bar{e}, \sum_{j=1}^{n}(\varepsilon_{2j-1}+\varepsilon_{2j})\mathbf{1}_{\{x\}}(\overline{S}_j) = 0 \text{ for all } x \in \overline{\Gamma}\right).$$

□

5 Examples of two sided bounds on $\Phi_{\Gamma_2(N)}$

5.1 The case of nilpotent groups

Our first application of the techniques developed above yields the following Theorem.

Theorem 5.1 *Assume that $\Gamma_1(N) = \mathbf{F}_r/N$ is an infinite nilpotent group and $r \ge 2$. Let D be the degree of polynomial volume growth of $\Gamma_1(N)$. Then*

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n^{D/(2+D)}[\log n]^{2/(2+D)}\right).$$

Proof Example 3.1 provides the desired lower bound. By Proposition 4.12, we have an exclusive pair $(\Gamma, \boldsymbol{\rho})$ in $\Gamma_2(N)$ such that $\overline{\Gamma} = \bar{\pi}(\Gamma)$ is of finite index in $\Gamma_1(N)$. Applying Theorem 4.13 gives

$$\Phi_{\Gamma_2(N)}(kn) \le C\Phi_{\mathbb{Z}\wr\overline{\Gamma}}(n).$$

Since $\overline{\Gamma}$ has finite index in $\Gamma_1(N)$, it has the same volume growth degree D and, by [8, Theorem 2],

$$\Phi_{\mathbb{Z}\wr\overline{\Gamma}}(n) \le \exp\left(-cn^{D/(2+D)}[\log n]^{2/(2+D)}\right).$$

□

5.2 Application to the free metabelian groups

This section is devoted to the free metabelian group $\mathbf{S}_{2,r} = \mathbf{F}/[N, N]$, $N = [\mathbf{F}_r, \mathbf{F}_r]$.

Theorem 5.2 *The free metabelian group* $\mathbf{S}_{2,r}$ *satisfies*

$$\Phi_{\mathbf{S}_{2,r}}(n) \simeq \exp\left(-n^{r/(2+r)}[\log n]^{2/(2+r)}\right) \tag{5.1}$$

and, for any $\alpha \in (0, 2)$*,*

$$\widetilde{\Phi}_{\mathbf{S}_{2,r},\rho_\alpha}(n) \simeq \exp\left(-n^{r/(\alpha+r)}[\log n]^{\alpha/(\alpha+r)}\right). \tag{5.2}$$

Further, for $a = (\alpha_1, \dots, \alpha_r) \in (0, 2)^r$*, let* $\boldsymbol{\mu}_a$ *be defined by (3.1) with* $p_i(m) = c_i(1 + |m|)^{-1-\alpha_i}$*. Let* μ_a *be the probability measure on* $\mathbf{S}_{2,r}$ *associated to* $\boldsymbol{\mu}_a$ *by (3.2). Then we have*

$$\mu_a^{(n)}(e) \simeq \exp\left(-n^{r/(r+\alpha)}[\log n]^{\alpha/(r+\alpha)}\right) \tag{5.3}$$

where

$$\frac{1}{\alpha} = \frac{1}{r}\left(\frac{1}{\alpha_1} + \cdots + \frac{1}{\alpha_r}\right).$$

Proof The lower bound in (5.1) follows from Theorem 3.5 (in particular, Example 3.1). The lower bound in (5.2) then follows from [3, Theorem 3.3]. The lower bound in (5.3) is Corollary 3.9. If we consider the measure μ_a with $a = (\alpha, \alpha, \dots, \alpha)$, $\alpha \in (0, 2)$, it is easy to check that this measure satisfies

$$\sup_{s>0}\left\{s\mu_a(g : (1 + |g|)^\alpha > s)\right\} < \infty,$$

that is, has finite weak ρ_α-moment with $\rho_\alpha(g) = (1 + |g|)^\alpha$. This implies that $\mu_a^{(2n)}(e)$ provides an upper bound for $\widetilde{\Phi}_{\mathbf{S}_{2,r},\rho_\alpha}(n)$. See the definition of $\widetilde{\Phi}_{G,\rho}$ in Sect. 1.4 and [3]. The upper bound in (5.2) is thus a consequence of the upper bound in (5.3).

We are left with proving the upper bounds contained in (5.1)–(5.3). The proofs follow the same line of reasoning and we focus on the upper bound (5.3).

Lemma 5.3 *Set* $\Gamma = \langle s_1^2, \dots, s_r^2 \rangle < \mathbf{S}_{2,r}$ *and* $\boldsymbol{\rho} = [\mathbf{s}_1, \mathbf{s}_2] \in \mathbf{F}_r$*. The pair* $(\Gamma, \boldsymbol{\rho})$ *is an exclusive pair in the sense of Definition 4.3*

Proof This was already observed in Example 4.1. □

In order to apply Proposition 4.14 to the pair $(\Gamma, \boldsymbol{\rho})$, we now define an appropriate measure φ on the subgroup $\Gamma = \langle s_1^2, \dots, s_r^2 \rangle$ of $\mathbf{S}_{2,r} = \mathbf{F}_r/[N,N] = \Gamma_2(N)$, $N = [\mathbf{F}_r, \mathbf{F}_r]$. In this context, $\overline{\Gamma} = (2\mathbb{Z})^r \subset \mathbb{Z}^r = \Gamma_1(N)$. The measure φ is simply given by

$$\varphi(g) = \sum_{i=1}^{r} \frac{1}{r} \sum_{m \in \mathbb{Z}} c_i (1+|m|)^{-1-\alpha_i} \mathbf{1}_{\{s_i^{2m}\}}(g).$$

With this definition, it is clear that, on $\mathbf{S}_{2,r}$, we have the Dirichlet form comparison

$$\mathcal{E}_{\mu_a} \geq c\mathcal{E}_{\nu*\varphi*\nu}.$$

Then by Proposition 4.14,

$$\mu_a^{*n}(e_*) \preceq (\eta \star \overline{\varphi} \star \eta)^{\star n}(e_\star).$$

Here as in the previous section, $*$ denotes convolution in $\Gamma_2(N)$ and $\star$ denotes convolution on $\mathbb{Z} \wr \overline{\Gamma}$ (or $\mathbb{Z} \wr \Gamma_1(N)$). Here, $\overline{\Gamma} = (2\mathbb{Z})^r$ which is a subgroup of (but also isomorphic to) $\Gamma_1(N) = \mathbb{Z}^r$. The switch-walk-switch measure $q = \eta \star \overline{\varphi} \star \eta$ on $\mathbb{Z} \wr (2\mathbb{Z})^r$ has been studied by the authors in [19] where it is proved that

$$q^{\star n}(e_\star) \leq \exp\left(-cn^{\frac{r}{r+\alpha}} (\log n)^{\frac{\alpha}{r+\alpha}}\right).$$

The proof of this result given in [19] is based on an extension of the Donsker-Varadhan Theorem regarding the Laplace transform of the number of visited points. This extension treats random walks on $\mathbb{Z}^r$ driven by measures that are in the domain of normal attraction of an operator stable law. See [19, Theorem 1.3]. □

5.3 Miscellaneous applications

This section describes further applications of the results of Sects. 4.1–4.3. Namely, we consider a number of examples consisting of a group $G = \Gamma_1(N) = \mathbf{F}_r/N$ given by an explicit presentation. We identify an exclusive pair (Γ, ρ) with the property that the subgroup $\overline{\Gamma}$ of $\Gamma_1(N)$ is either isomorphic to $\Gamma_1(N)$ or has a similar structure so that $\Phi_{\mathbb{Z}^d \wr \Gamma_1(N)} \simeq \Phi_{\mathbb{Z} \wr \overline{\Gamma}}$. In each of these examples, the results of Sects. 3.2–3.3 and those of Sect. 4.1–4.3 provide matching lower and upper bounds for $\Phi_{\Gamma_2(N)}$ where $\Gamma_2(N) = \mathbf{F}_2/[N,N]$.

Example 5.1 (The lamplighter $\mathbb{Z}_2 \wr \mathbb{Z} = \langle a, t \mid a^2, [a, t^{-n}at^n], n \in \mathbb{Z} \rangle$) In the lamplighter description of $\mathbb{Z}_2 \wr \mathbb{Z}$, multiplying by t on the right produces a translation of the lamplighter by one unit. Multiplying by a on the right switch the light at the current position of the lamplighter. Let Γ be the subgroup of Γ_2 generated by the images of a and t^2 and note that $\overline{\Gamma}$ is, in fact, isomorphic to $\Gamma_1(N)$. Let $\boldsymbol{\rho} = [a, t^{-1}at] = a^{-1}t^{-1}a^{-1}tat^{-1}at$. In order to apply Lemma 4.5, set $\mathbf{u} = a^{-1}t^{-1}a^{-1}tat^{-1}$, $\mathbf{s} = a$ and $\mathbf{v} = t$ so that $\boldsymbol{\rho} = \mathbf{usv}$. By inspection, $\mathfrak{f}_{\boldsymbol{\rho}}((\bar{u}, \overline{us}, \mathbf{s})) \neq 0$ (condition (1) of Lemma 4.5). Also, because the elements of $\overline{\Gamma}$ can only have lamps on and the lamplighter at even positions, one checks that $\mathfrak{f}_{\boldsymbol{\rho}}((x\bar{u}, x\overline{us}, \mathbf{s})) = 0$ if $x \in \overline{\Gamma}$ (condition (2) of Lemma 4.5). For the same reason, it is clear that $\mathfrak{f}_x((\bar{u}, \overline{us}, \mathbf{s})) = 0$ if $x \in \overline{\Gamma}$, that is, $(\bar{u}, \overline{us}, \mathbf{s}) \notin U$ (condition (3) of Lemma 4.5). By the Magnus embedding and [16, Theorem 1.3], we have

$$\Phi_{\Gamma_2(N)}(n) \geq c\Phi_{\mathbb{Z}^r \wr \Gamma_1(N)}(kn).$$

Applying Lemma 4.5, Proposition 4.14, and the fact that $\overline{\Gamma} \simeq \Gamma_1(N)$, yields

$$\Phi_{\Gamma_2(N)}(kn) \leq C\Phi_{\mathbb{Z} \wr \Gamma_1(N)}(n).$$

The results of [8] gives

$$\Phi_{\mathbb{Z}^r \wr \Gamma_1(N)}(n) \simeq \Phi_{\mathbb{Z} \wr \Gamma_1(N)}(n) \simeq \exp\left(-n/[\log n]^2\right).$$

Hence we conclude that, in the present case where

$$\Gamma_1(N) = \mathbb{Z}_2 \wr \mathbb{Z} = \langle a, t \mid a^2, [a, t^{-n}at^n], n \in \mathbb{Z}\rangle,$$

we have

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n/[\log n]^2\right).$$

This extend immediately to $\mathbb{Z}_q \wr \mathbb{Z} = \langle a, t \mid a^q, [a, t^{-n}at^n], n \in \mathbb{Z}\rangle$. It also extend to similar presentations of $F \wr \mathbb{Z}$ with F finite. See the next class of examples.

Example 5.2 (Examples of the type $K \wr \mathbb{Z}^d$) Let $K = \langle \mathbf{k}_1, \ldots, \mathbf{k}_m \mid N_K\rangle$ be a m generated group. The wreath product $K \wr \mathbb{Z}^d$ admits the presentation $\mathbf{F}_r/N$ with $r = m + d$ generators denoted by

$$\mathbf{k}_1, \ldots, \mathbf{k}_m, \mathbf{t}_1, \ldots, \mathbf{t}_d$$

and relations $[\mathbf{t}_i, \mathbf{t}_j]$, $1 \le i, j \le d$, N_K and

$$[\mathbf{k}', \mathbf{t}^{-1}\mathbf{k}\mathbf{t}], \mathbf{k}, \mathbf{k}' \in \mathbf{F}(\mathbf{k}_1, \ldots, \mathbf{k}_m), \mathbf{t} = \mathbf{t}_1^{x_1} \cdots \mathbf{t}_d^{x_d}, \ (x_1, \ldots, x_d) \neq 0.$$

Without loss of generality, we can assume that the image of $\mathbf{k}_1$ in K is not trivial. Let Γ be the subgroup of $\Gamma_2(N)$ generated by the images of $\mathbf{t}_i^2$, $1 \le i \le d$. Let

$$\boldsymbol{\rho} = [\mathbf{k}_1, \mathbf{t}_1^{-1}\mathbf{k}_1\mathbf{t}_1]$$

and write

$$\boldsymbol{\rho} = \mathbf{usv} \text{ with } \mathbf{u} = \boldsymbol{\rho}\mathbf{t}_1^{-1}\mathbf{k}_1^{-1}, \mathbf{s} = \mathbf{k}_1, \mathbf{v} = \mathbf{t}_1.$$

As in the previous example, $(\Gamma, \boldsymbol{\rho})$ is an exclusive pair and $\overline{\Gamma}$ is in fact isomorphic to $\Gamma_1(N)$. By the same token, it follows that

$$\Phi_{\Gamma_2(N)}(n) \ge c\Phi_{\mathbb{Z}^r \wr \Gamma_1(N)}(kn) \text{ and } \Phi_{\Gamma_2(N)}(kn) \le C\Phi_{\mathbb{Z} \wr \Gamma_1(N)}(n).$$

Now, thanks to the results of [8] concerning wreath products, we obtain

- If K is a non-trivial finite group then

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n/[\log n]^{2/d}\right).$$

- If K is not finite but has polynomial volume growth then

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n\left(\frac{\log\log n}{\log n}\right)^{2/d}\right).$$

- If K is polycyclic with exponential volume growth then

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n/[\log\log n]^{2/(d+1)}\right)$$

In particular, when $\Gamma_1(N) = \mathbb{Z} \wr \mathbb{Z}$ with presentation $\langle a, t \mid [a, t^{-n}at^n], n \in \mathbb{Z}\rangle$ we obtain that

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n\left(\frac{\log\log n}{\log n}\right)^2\right).$$

Example 5.3 (The Baumslag–Solitar group $\mathrm{BS}(1, q)$) Consider the presentation

$$\mathrm{BS}(1, q) = \Gamma_1(N) = \mathbf{F}_2/N = <a, b \mid a^{-1}ba = b^q>$$

with $q > 1$. In order to apply Proposition 4.14, let Γ be the group generated by the image of a^2 and b in $\Gamma_2(N)$. Let $\boldsymbol{\rho} = b^{-q}a^{-1}ba$, $\mathbf{u} = b^{-q}a^{-1}$, $\mathbf{s} = b$, $\mathbf{v} = a$. One checks that $(\Gamma, \boldsymbol{\rho})$ is an exclusive pair and that $\overline{\Gamma} \simeq \mathrm{BS}(1, q^2)$. After some computation, we obtain

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n/[\log n]^2\right).$$

Example 5.4 (Polycyclic groups) Let G be a polycyclic group with polycyclic series $G = G_1 \rhd G_2 \rhd \cdots \rhd G_{r+1} = \{e\}$, $r \geq 2$. For each i, $1 \leq i \leq r$, let a_i be an element in G_i whose projection in G_i/G_{i+1} generates that group. Write $G = \mathbf{F}_r/N$ where $\mathbf{s}_i$ is sent to a_i. This corresponds to the standard polycyclic presentation of G relative to $a_1, \ldots, a_n$ and N contains a word of the form

$$\boldsymbol{\rho} = \mathbf{s}_1^{-1}\mathbf{s}_2\mathbf{s}_1\mathbf{s}_r^{\alpha_r} \cdots \mathbf{s}_2^{\alpha_2}$$

where α_ℓ, $2 \leq \ell \leq r$ are integers. See [21, page 395].

Theorem 5.4 *Let $G = \Gamma_1(N)$ be an infinite polycyclic group equipped with a polycyclic presentation as above with at least two generators.*

- *If G has polynomial volume growth of degree D, then*

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n^{D/(2+D)}[\log n]^{2/(2+D)}\right).$$

- *If G has exponential volume growth then*

$$\Phi_{\Gamma_2(N)}(n) \simeq \exp\left(-n/[\log n]^2\right).$$

Proof Our first step is to construct an exclusive pair $(\Gamma, \boldsymbol{\rho})$ with $\overline{\Gamma} = \bar{\pi}(\Gamma)$ of finite index in $\Gamma_1(N)$.

Assume first that G_1/G_2 is finite. In this case, let $\Gamma = \langle s_2, \ldots, s_r\rangle$. Assume that $x \in \Gamma$ is such that $\mathfrak{f}_{\boldsymbol{\rho}}((\bar{x}\bar{s}_1^{-1}, \bar{x}\bar{s}_1^{-1}\bar{s}_2, \mathbf{s}_2)) \neq 0$. Then there must be a prefix $\mathbf{u}$ of $\boldsymbol{\rho}$ such that $\pi(\mathbf{u}) = \bar{x}\bar{s}_1^{-1}$. Computing modulo $\pi(\Gamma) = G_2$, the only prefixes of $\boldsymbol{\rho}$ that can have this property are $\mathbf{s}_1^{-1}$ and $\mathbf{s}_1^{-1}\mathbf{s}_2$. If $\mathbf{u} = \mathbf{s}_1^{-1}$ then $\bar{x}$ is the identity. If $\mathbf{u} = \mathbf{s}_1^{-1}\mathbf{s}_2$ then $\mathbf{s}_1^{-1}\mathbf{s}_2^2$ is not a prefix of $\boldsymbol{\rho}$ and $\mathfrak{f}_{\boldsymbol{\rho}}((\bar{x}\bar{s}_1^{-1}, \bar{x}\bar{s}_1^{-1}\bar{s}_2, \mathbf{s}_2)) = 0$, a contradiction. It follows that condition 2 of Lemma 4.5 is satisfied. In this case, it is obvious that condition 3 holds as well. Further, $\pi(\Gamma) = G_2$ is a subgroup of finite index in $G = \Gamma_1(N)$.

In the case when $G_1/G_2 \simeq \mathbb{Z}$, set

$$\Gamma = \langle s_1^2, s_2, \ldots, s_r\rangle < \Gamma_2(N).$$

The same argument as used in the case when G_1/G_2 is finite apply to see that condition 2 of Lemma 4.5 is satisfied. To check that condition 3 of Lemma 4.5 is satisfied, observe that, if $\mathfrak{f}_g((\bar{y}, \bar{y}\bar{s}_i, \mathbf{s}_i)) \neq 0$ with $2 \leq i \leq r$ and $g \in \Gamma$ then $\bar{y}$ must belong to $\overline{\Gamma}$. But, by construction $\bar{s}_1^{-1} \notin \overline{\Gamma}$. Therefore $\mathfrak{f}_g(\bar{s}_1^{-1}, \bar{s}_1^{-1}\bar{s}_2, \mathbf{s}_2) = 0$ for every $g \in \Gamma$. Finally, $\overline{\Gamma}$ is obviously of index 2 in $\Gamma_1(N)$.

By the Magnus embedding we have $c\Phi_{\mathbb{Z}^r \wr \Gamma_1(N)}(kn) \leq \Phi_{\Gamma_2(N)}(n)$. By Theorem 4.14 and the existence of the exclusive pair (Γ, ρ) exhibited above, we also have $c\Phi_{\Gamma_2(N)}(kn) \leq \Phi_{\mathbb{Z}\wr\overline{\Gamma}}(n)$ with $\overline{\Gamma}$ of finite index in $\Gamma_1(N)$. Because $\Gamma_1(N)$ is infinite polycyclic, the desired result follows from the known results about wreath products. See [8]. □

6 Iterated comparison and $\mathbf{S}_{d,r}$ with $d > 2$

Let $\mathbf{F}_r/N = \Gamma_1(N)$ be a given presentation. Write $N^{(1)} = [N, N]$ and $N^{(\ell)} = [N^{(\ell-1)}, N^{(\ell-1)}]$, $\ell \geq 2$. The goal of this section is to obtain bounds on the probability of return for random walks on $\Gamma_\ell(N) = \mathbf{F}_r/N^{(\ell-1)}$. Our approach is to iterate the method developed in the previous sections in the study of random walks on $\Gamma_2(N)$.

We need to fix some notation. We will use $* = *_\ell$ to denote convolution in $\Gamma_\ell(N)$. In general, ℓ will be fixed so that there will be no need to distinguish between different $*_\ell$. We will consider several wreath products $A \wr G$ as well as iterated wreath products

$$A \wr (A \wr (\cdots (A \wr G) \cdots))$$

where A and G are given with A abelian (in fact, A will be either $\mathbb{Z}$ or $\mathbb{Z}^r$). Set $W(A, G) = W_1(A, G) = A \wr G$ and $W_k(A, G) = W(A, W_{k-1}(A, G))$. Depending on the context, we will denote convolution in $W_k(A, G)$ by

$$\star_k \text{ or } \star_{W_k} \text{ or } \star_{W_k(A,G)}.$$

Let μ be a probability measure on G and η be a probability measure on A. Note that the measures μ and η can also be viewed, in a natural way, as measures on $W(A, G)$ with η being supported by the copy of A that sits above the identity element of G in $A \wr G$. The associated switch-walk-switch measure on $W = W_1(A, G)$ is the measure

$$q = q_1(\eta, \mu) = \eta \star_1 \mu \star_1 \eta.$$

Iterating this construction, we define the probability measure q_k on $W_k(A, G)$ by the iterative formula

$$q_k = q_k(\eta, \mu) = \eta \star_k q_{k-1} \star_k \eta.$$

We refer to q_k as the iterated switch-walk-switch measure on W_k associated with the initial pair η, μ. We will make repeated use of the following simple lemma (see also [10, Lemma 3.2]).

Lemma 6.1 *Let A, G, H be finitely generated groups with A abelian. Let $\theta : G \to H$ be a group homomorphism. Define $\theta_1 : W_1(A, G) \to W_1(A, H)$ by*

$$\theta_1 : (f, x) \mapsto (\overline{f}, \theta(x)), \text{ where } \overline{f}(h) = \sum_{g:\theta(g)=h} f(g)$$

with the convention that sum over empty set is 0. Then θ_1 is group homomorphism.

Define $\theta_k : W_k(A, G) \to W_k(A, H)$ by iterating the previous construction so

$$\theta_k = (\theta_{k-1})_1 : W_1(A, W_{k-1}(A, G)) \to W_1(A, W_{k-1}(A, H)).$$

Then θ_k is group homomorphism. Moreover, if θ is injective (resp., surjective), then θ_k is also injective (resp., surjective).

Proof The stated conclusions follow by inspection. □

Lemma 6.2 *Let A, G, H be finitely generated groups with A abelian. Let μ and η be a probability measures on G and A, respectively. Let $\theta : G \to H$ be a homomorphism and $\theta_k : W_k(A, G) \to W_k(A, H)$ be as in Lemma 6.1. Let $\theta_k(q_k(\eta, \mu))$ be the push-forward of the iterated switch-walk-switch measure $q_k(\eta, \mu)$ on $W_k(A, G)$ under θ_k. Then we have*

$$\theta_k(q_k(\eta, \mu)) = q_k(\eta, \theta(\mu)).$$

Proof It suffices to check the case $k = 1$ where the desired conclusion reads

$$\theta_1(\eta \star_{A\wr G} \mu \star_{A\wr G} \eta) = \eta \star_{A\wr H} \theta(\mu) \star_{A\wr H} \eta.$$

This equality follows from the three identities

$$\theta_1(\eta \star_{A\wr G} \mu \star_{A\wr G} \eta) = \theta_1(\eta) \star_{A\wr H} \theta_1(\mu) \star_{A\wr G} \theta_1(\eta),$$
$$\theta_1(\mu) = \theta(\mu) \text{ and } \quad \theta_1(\eta) = \eta.$$

The first identity holds because θ_1 is an homomorphism. The other two identities hold by inspection (with some slight abuse of notation). □

6.1 Iterated lower bounds

This section proves lower bounds for the probability of return of symmetric finitely supported random walks on $\Gamma_\ell(N) = \mathbf{F}_r/N^{(\ell-1)}$ for appropriate presentations $\mathbf{F}_r/N$.

Observe that Propositions 3.2–3.3–3.4 (which are based on the results in [4,8]) provide us with good lower bounds for the probability of return on a variety of iterated wreath product. Namely,

- Assume that $A = \mathbb{Z}^b$ with $b \geq 1$ and G has polynomial volume growth of degree D. Then, for $\ell \geq 2$,

$$\Phi_{W_\ell(A,G)}(n) \simeq \exp\left(-n\left(\frac{\log_{[\ell]} n}{\log_{[\ell-1]} n}\right)^{2/D}\right).$$

- Assume that $A = \mathbb{Z}^b$ with $b \geq 1$ and G is polycyclic with exponential volume growth. Then, for $\ell \geq 1$,

$$\Phi_{W_\ell(A,G)}(n) \simeq \exp\left(-n/[\log_{[\ell]} n]^2\right).$$

  This applies, for instance, when G is the Baumslag–Solitar group $\mathrm{BS}(1, q)$, $q > 1$. Further, the same result holds for the wreath product $G = \mathbb{Z}_q \wr \mathbb{Z}$, $q > 1$, (even so it is not polycyclic).

The main other ingredients we use to bound $\Phi_{\Gamma_\ell(N)}$ from below are the following.

- The Dirichlet form comparison techniques of [16], in particular, [16, Theorem 1.3] which states that if H is a finitely generated subgroup of a finitely generated group G then $\Phi_G \lesssim \Phi_H$.
- The Magnus embedding at level $\ell - j + 1$,

$$\bar{\psi}_{\ell-j+1} : \Gamma_{\ell-j+1}(N) \hookrightarrow \mathbb{Z}^r \wr \Gamma_{\ell-j}(N), \quad 0 \leq j \leq \ell - 1.$$

- The natural extension $\widetilde{\psi}_{\ell-j+1}$ of $\bar{\psi}_{\ell-j+1}$,

$$\widetilde{\psi}_{\ell-j+1} : W_{j-1}(\mathbb{Z}^r, \Gamma_{\ell-j+1}(N)) \hookrightarrow W_{j-1}(\mathbb{Z}^r, \mathbb{Z}^r \wr \Gamma_{\ell-j}(N)) = W_j(\mathbb{Z}^r, \Gamma_{\ell-j}(N)$$

  which is provided by Lemma 6.1.

By composing the injective homomorphisms

$$\bar{\psi}_\ell : \Gamma_\ell(N) \hookrightarrow \mathbb{Z}^r \wr \Gamma_{\ell-1}(N) = W_1(\mathbb{Z}^r, \Gamma_{\ell-j})$$

and

$$\widetilde{\psi}_{\ell-j+1} : W_{j-1}(\mathbb{Z}^r, \Gamma_{\ell-j+1}(N)) \hookrightarrow W_j(\mathbb{Z}^r, \Gamma_{\ell-j}(N), \quad 0 \leq 2 \leq \ell - 1,$$

we realize $\Gamma_\ell(N)$ as a subgroup of $W_{\ell-1}(\mathbb{Z}^r, \Gamma_1(N))$. This gives the following general result.

Theorem 6.3 *For any presentation of $G = \mathbf{F}_r/N$ and integer $\ell \geq 2$ we have*

$$\Phi_{\Gamma_\ell(N)} \gtrsim \Phi_{W_{\ell-1}(\mathbb{Z}^r, G)}.$$

Corollary 6.4 *Let $\Gamma_\ell(N) = \mathbf{F}_r/N^{(\ell)}$.*

- *Assume that $\Gamma_1(N)$ has polynomial volume growth of degree D. Then, for $\ell \geq 3$,*

$$\Phi_{\Gamma_\ell(N)}(n) \geq \exp\left(-Cn\left(\frac{\log_{[\ell-1]} n}{\log_{[\ell-2]} n}\right)^{2/D}\right).$$

- *Assume that $\Gamma_1(N)$ is $BS(1, q)$ with $q > 1$, or $\mathbb{Z}_2 \wr \mathbb{Z}$, or polycyclic of exponential volume growth volume growth. Then, for $\ell \geq 2$,*

$$\Phi_{\Gamma_\ell(N)}(n) \geq \exp\left(-Cn/[\log_{[\ell-1]} n]^2\right).$$

- *Assume that $\Gamma_1(N) = K \wr \mathbb{Z}^D$, $D \geq 1$ and K finite. Then, for $\ell \geq 2$,*

$$\Phi_{\Gamma_\ell(N)}(n) \geq \exp\left(-Cn/[\log_{[\ell-1]} n]^{2/D}\right).$$

- *Assume that $\Gamma_1(N) = \mathbb{Z}^a \wr \mathbb{Z}^D$, $a, D \geq 1$. Then, for $\ell \geq 2$,*

$$\Phi_{\Gamma_\ell(N)}(n) \geq \exp\left(-Cn\left(\frac{\log_{[\ell]} n}{\log_{[\ell-1]} n}\right)^{2/D}\right).$$

6.2 Iterated upper bounds

We now present an iterative approach to obtain upper bounds on $\Phi_{\Gamma_\ell(N)}$. Although similar in spirit to the iterated lower bound technique developed in the previous section, the iterative upper bound method is both more difficult and much less flexible. The key idea is to obtain an upper bound for $\Phi_{\Gamma_\ell(N)}$ in term of an appropriate iterated wreath product $W_{\ell-1}(\mathbb{Z}, \Gamma_1')$ where Γ_1' is a finitely generated subgroup of $\Gamma_1(N)$ (which, ideally, should be "similar" to $\Gamma_1(N)$). Once this is done, [8] provides upper bounds on $\Phi_{W_{\ell-1}(\mathbb{Z},G)}$ in terms of the Følner function of base G.

Our first task is to formalize algebraically the content of Proposition 4.14. Recall once more that the Magnus embedding provides an injective homomorphism $\bar{\psi} : \mathbf{F}_r/[N, N] \hookrightarrow \left(\sum_{x \in \mathbf{F}_r/N} \mathbb{Z}_x^r\right) \rtimes \mathbf{F}_r/N$ with $\bar{\psi}(g) = (\bar{a}(g), \bar{\pi}(g))$. Let Γ be a subgroup of $\mathbf{F}_r/[N, N]$ and $\boldsymbol{\rho} \in N \backslash [N, N] \subset \mathbf{F}_r$. Set $\rho = \pi_2(\boldsymbol{\rho})$ and $\overline{\Gamma} = \pi(\Gamma) \subset \mathbf{F}_r/N$.

Assume that $(\Gamma, \boldsymbol{\rho})$ is an exclusive pair as in Definition 4.3. We are going to construct a surjective homomorphism

$$\vartheta : \langle \Gamma, \rho \rangle \to \mathbb{Z} \wr \overline{\Gamma}.$$

Let $g \in \langle \Gamma, \rho \rangle$. Consider two decompositions of g as products

$$g = \gamma_1 \rho^{x_1} \gamma_2 \rho^{x_2} \cdots \gamma_p \rho^{x_p} \gamma_{p+1} = \gamma_1' \rho^{x_1'} \gamma_2' \rho^{x_2'} \cdots \gamma_q' \rho^{x_q'} \gamma_{q+1}'$$

with $\gamma_i \in \Gamma$, $1 \leq i \leq p+1$, $\gamma_i' \in \Gamma$, $1 \leq i \leq q+1$. Set $\sigma_i = \gamma_1 \dots \gamma_i$, $1 \leq i \leq p+1$, and $\sigma_i' = \gamma_1 \dots \gamma_i$, $1 \leq i \leq q+1$. Observe that

$$g = \sigma_1 \rho^{x_1} \sigma_1^{-1} \sigma_2 \rho^{x_2} \sigma_2^{-1} \cdots \sigma_p \rho^{x_p} \sigma_p^{-1} \sigma_{p+1} = \alpha \sigma_{p+1}$$

where

$$\alpha = \sigma_1 \rho^{x_1} \sigma_1^{-1} \sigma_2 \rho^{x_2} \sigma_2^{-1} \cdots \sigma_p \rho^{x_p} \sigma_p^{-1}.$$

Similarly $g = \alpha' \sigma'_{q+1}$ and we have

$$(\alpha')^{-1} \alpha = \sigma'_{q+1} (\sigma_{p+1})^{-1}.$$

By Lemma 4.1, we have

$$\begin{aligned} \bar{a}(g) &= \sum_1^p x_i \tau_{\bar{\sigma}_i} \bar{a}(\rho) + \bar{a}(\sigma_{p+1}) \\ &= \sum_1^q x'_i \tau_{\bar{\sigma}'_i} \bar{a}(\rho) + \bar{a}(\sigma'_{q+1}). \end{aligned}$$

and

$$\sum_1^p x_i \tau_{\bar{\sigma}_i} \bar{a}(\rho) - \sum_1^q x'_i \tau_{\bar{\sigma}'_i} \bar{a}(\rho) = \bar{a}(\sigma'_{q+1} \sigma_{p+1}^{-1}).$$

As $(\Gamma, \boldsymbol{\rho})$ is an exclusive pair, condition (ii) of Definition 4.3 implies that

$$\sum_1^p x_i \tau_{\bar{\sigma}_i} \bar{a}(\rho) - \sum_1^q x'_i \tau_{\bar{\sigma}'_i} \bar{a}(\rho) = \bar{a}(\sigma'_{q+1} \sigma_{p+1}^{-1}) = 0.$$

Hence

$$\sum_1^p x_i \tau_{\bar{\sigma}_i} \bar{a}(\rho) = \sum_1^q x'_i \tau_{\bar{\sigma}'_i} \bar{a}(\rho)$$

in $\sum_{x \in \Gamma_1(N)} \mathbb{Z}_x^r$. This also implies that $a(\sigma_{p+1}) = a(\sigma'_{q+1})$. By construction, we also have $\bar{\pi}(\sigma_{p+1}) = \bar{\pi}(\sigma'_{q+1})$. Hence, $\sigma_{p+1} = \sigma'_{q+1}$ in Γ.

By condition (i) of Definition 4.3 (see Remark 4.4), we can identify

$$\sum_1^p x_i \tau_{\bar{\sigma}_i} \bar{a}(\rho) = \sum_1^q x'_i \tau_{\bar{\sigma}'_i} \bar{a}(\rho)$$

with the element

$$\left(\sum_1^p x_i \mathbf{1}_h(\sigma_i) \right)_{h \in \overline{\Gamma}} \quad \text{of} \quad \sum_{h \in \overline{\Gamma}} \mathbb{Z}_h.$$

This preparatory work allows us to define a map

$$\begin{aligned} &\vartheta : \langle \Gamma, \rho \rangle \to \mathbb{Z} \wr \overline{\Gamma} \\ &g = \gamma_1 \rho^{x_1} \gamma_2 \rho^{x_2} \cdots \gamma_p \rho^{x_p} \gamma_{p+1} \mapsto \left(\left(\sum_1^p x_i \mathbf{1}_h(\sigma_i) \right)_{h \in \overline{\Gamma}}, \bar{\pi}(g) \right). \end{aligned}$$

Lemma 6.5 *The map $\vartheta : \langle \Gamma, \rho \rangle \to \mathbb{Z} \wr \overline{\Gamma}$ is a surjective homomorphism.*

Proof Note that $\vartheta(e)$ is the identity element in $\mathbb{Z} \wr \overline{\Gamma}$. To show that ϑ is an homomorphism, it suffices to check that, for any $g \in \langle \Gamma, \rho \rangle$ and $\gamma \in \Gamma$

$$\vartheta(g\gamma) = \vartheta(g)\vartheta(\gamma), \quad \vartheta(g\rho^{\pm 1}) = \vartheta(g)\vartheta(\rho^{\pm 1}).$$

These identities follow by inspection. One easily check that ϑ is surjective. □

Lemma 6.6 *Let μ be a probability measure supported on Γ and ν be the probability measure defined by $\nu(\rho^{\pm 1}) = 1/2$. Let η be the probability measure on $\mathbb{Z}$ defined by $\eta(\pm 1) = 1/2$. Let $*$ be convolution on $\langle \Gamma, \rho \rangle < \Gamma_2(N)$ and $\star$ be convolution on $\mathbb{Z} \wr \overline{\Gamma}$ Then we have*

$$\vartheta(\nu * \mu * \nu) = \eta \star \bar{\pi}(\mu) \star \eta.$$

Proof This follows from the fact that ϑ is an homomorphism, $\vartheta|_\Gamma = \bar{\pi}$ and $\vartheta(\nu) = \eta$. □

In addition to the canonical projections $\pi_j : \mathbf{F}_r \to \mathbf{F}_r/N^{(j-1)} = \Gamma_j(N)$, for $1 \le j \le i$, we also consider the projection $\pi^i_j : \Gamma_i(N) \to \Gamma_j(N)$.

Definition 6.7 Fix a presentation $\Gamma_1(N) = \mathbf{F}_r/N$ and an integer ℓ. Let Γ_i be a finitely generated subgroup of $\Gamma_i(N)$, $2 \le i \le \ell$. Set

$$\Gamma'_{i-1} = \pi^i_{i-1}(\Gamma_i), \quad 2 \le i \le \ell.$$

Let $\boldsymbol{\rho}_i \in \mathbf{F}_r$, $2 \le i \le \ell$. Set $\rho_\ell = \pi_\ell(\boldsymbol{\rho}_\ell)$. We say that $(\Gamma_i, \boldsymbol{\rho}_i)_2^\ell$ is an exclusive sequence (adapted to $(\Gamma_i(N))_1^\ell$) if the following properties hold:

1. $\Gamma_\ell < \Gamma_\ell(N)$ and $\pi^\ell_{\ell-1}(\rho_\ell)$ is trivial.
2. For $2 \le j \le \ell - 1$, $\Gamma_j < \Gamma'_j$, $\rho_j \in \Gamma'_j$ and $\pi^j_{j-1}(\rho_j)$ is trivial.
3. For each $2 \le i \le \ell$, $(\Gamma_i, \boldsymbol{\rho}_i)$ is an exclusive pair in $\Gamma_2(N^{(i-1)}) = \Gamma_i(N)$.

Theorem 6.8 *Fix a presentation $\Gamma_1(N) = \mathbf{F}_r/N$ and an integer $\ell \ge 2$. Assume that there exists an exclusive sequence $((\Gamma_i, \boldsymbol{\rho}_i))_2^\ell$ adapted to $(\Gamma_j(N))_1^\ell$ such that each Γ_i is finitely generated. Then we have*

$$\Phi_{\Gamma_\ell(N)} \lesssim \Phi_{W_{\ell-1}(\mathbb{Z}, \Gamma'_1)}.$$

where $\Gamma'_1 = \pi^2_1(\Gamma_2) < \Gamma_1(N)$.

Remark 6.9 The technique and results of [8] provides good upper bounds on Φ_G when G is an iterated wreath product such as $W_{\ell-1}(\mathbb{Z}, \Gamma'_1)$ and we have some information on Γ'_1. The real difficulty in applying the theorem above lies in finding an exclusive sequence that terminates with an appropriate Γ'_1.

Proof For convenience, set $\Gamma'_\ell = \Gamma_\ell(N)$ and $W_0(A, G) = G$. For each j, $0 \le j \le \ell - 2$, $\langle \Gamma_{\ell-j}, \rho_{\ell-j} \rangle$ is a finitely generated subgroup of $\Gamma'_{\ell-j}$. By Lemma 6.1, we can realize the group $W_j(\mathbb{Z}, \langle \Gamma_{\ell-j}, \rho_{\ell-j} \rangle)$ as a (finitely generated) subgroup of $W_j(\mathbb{Z}, \Gamma'_{\ell-j})$. Hence ([16, Theorem 1.3])

$$\Phi_{W_j(\mathbb{Z}, \Gamma'_{\ell-j})} \lesssim \Phi_{W_j(\mathbb{Z}, \langle \Gamma_{\ell-j}, \rho_{\ell-j} \rangle)}.$$

Consider the surjective homomorphism

$$\vartheta_{\ell-j} : \langle \Gamma_{\ell-j}, \rho_{\ell-j} \rangle \to \mathbb{Z} \wr \Gamma'_{\ell-j-1}$$

which is provided by Lemma 6.5 and the definition of exclusive sequence adapted to $(\Gamma_i(N))_1^\ell$. By Lemma 6.1, it can be extended to a surjective homomorphism

$$\vartheta_{\ell-j,j} : W_j(\mathbb{Z}, \langle\Gamma_{\ell-j}, \rho_{\ell-j}\rangle) \to W_j(\mathbb{Z}, \mathbb{Z} \wr \Gamma'_{\ell-j-1}) = W_{j+1}(\mathbb{Z}, \Gamma'_{\ell-j-1}).$$

Recall the simple fact that $\Phi_H \gtrsim \Phi_G$ if H is a quotient of the finitely generated group G. It follows that

$$\Phi_{W_j(\mathbb{Z},\Gamma'_{\ell-j})} \lesssim \Phi_{W_{j+1}(\mathbb{Z},\Gamma'_{\ell-j-1})}, 0 \le j \le \ell - 2.$$

It follows that $\Phi_{\Gamma_\ell(N)} \lesssim \Phi_{W_{\ell-1}(\mathbb{Z},\Gamma'_1)}$ which is the desired upper bound. □

6.3 Existence of exclusive sequences

This section describes a sufficient condition for the existence of appropriate exclusive sequences. For this purpose, we will use a result concerning the subgroup of $\Gamma_\ell(N)$ generated by the images of a fix power $\mathbf{s}_i^m$ of the generators $\mathbf{s}_i$, $1 \le i \le r$. Let $\delta_m : \mathbf{F}_r \to \mathbf{F}_r$ be the homomorphism from the free group to itself determined by $\delta_m(\mathbf{s}_i) = \mathbf{s}_i^m$, $1 \le i \le r$.

Lemma 6.10 *Suppose δ_m induces an injective homomorphism $\mathbf{F}_r/N \to \mathbf{F}_r/N$, and $\pi(\mathbf{s}_i^q) \notin \delta_m(\mathbf{F}_r/N)$, $1 \le q \le m-1$, $1 \le i \le r$. Then δ_m induces an injective homomorphism $\mathbf{F}_r/[N,N] \to \mathbf{F}_r/[N,N]$.*

Proof The proof is based on the representation of the elements of $\Gamma_2(N) = \mathbf{F}_r/[N,N]$ using flows on the labeled Cayley graph of $\Gamma_1(N) = \mathbf{F}_r/N$.

Let δ_m denote the induced injective homomorphism on $\Gamma_1(N)$. Let $\mathfrak{f}$ be a flow function defined on edge set $\mathfrak{E}$ of Cayley graph of $\Gamma_1(N)$. Let $\mathfrak{E}_m$ be a subset of $\mathfrak{E}$ given by

$$\mathfrak{E}_m = \{(\delta_m(x)s_i^j, \delta_m(x)s_i^{j+1}, \mathbf{s}_i) : x \in \Gamma_1(N), 0 \le j \le m-1, 1 \le i \le r\}.$$

Let $t_m : \mathfrak{f} \mapsto t_m\mathfrak{f}$ be the map on flows defined by

$$t_m\mathfrak{f}((\delta_m(x)s_i^j, \delta_m(x)s_i^{j+1}, \mathbf{s}_i)) = \mathfrak{f}((x, xs_i, \mathbf{s}_i)), \quad 0 \le j \le m-1,$$

and $t_m\mathfrak{f}$ is 0 on edges not in $\mathfrak{E}_m$. This map is well-defined. Indeed, if two pairs (x, j) and (y, j') in $\Gamma_1(N) \times \{0, \cdots, m-1\}$ correspond to a common edge, that is,

$$(\delta_m(x)s_i^j, \delta_m(x)s_i^{j+1}, \mathbf{s}_i) = (\delta_m(y)s_i^{j'}, \delta_m(y)s_i^{j'+1}, \mathbf{s}_i),$$

then $\delta_m(x)s_i^j = \delta_m(y)s_i^{j'}$, $\delta_m(y^{-1}x) = s_i^{j'-j}$. Since $|j'-j| \le m-1$, from the assumption $\pi(\mathbf{s}_i^q) \notin \delta_m(\mathbf{F}_r/N)$, $1 \le q \le m-1$ it follows that $j' = j$. Then $\delta_m(y^{-1}x) = \bar{e}$ and, since δ_m is injective, we must have $x = y$.

By definition, t_m is additive in the sense that

$$t_m(\mathfrak{f}_1 + \mathfrak{f}_2) = t_m\mathfrak{f}_1 + t_m\mathfrak{f}_2.$$

Also, regarding translations in $\Gamma_1(N)$, we have

$$t_m\tau_y\mathfrak{f} = \tau_{\delta_m(y)}t_m\mathfrak{f}.$$

Therefore the identity $\mathfrak{f}_{\mathbf{uv}} = \mathfrak{f}_{\mathbf{u}} + \tau_{\pi(\mathbf{u})}\mathfrak{f}_{\mathbf{v}}$, of Remark 4.2 yields

$$t_m\mathfrak{f}_{\mathbf{uv}} = t_m\mathfrak{f}_{\mathbf{u}} + \tau_{\delta_m(\pi(\mathbf{u}))}t_m\mathfrak{f}_{\mathbf{v}}.$$

By assumption $\pi(\delta_m(\mathbf{u})) = \delta_m(\pi(\mathbf{u}))$, therefore

$$t_m\mathfrak{f}_{\mathbf{uv}} = t_m\mathfrak{f}_{\mathbf{u}} + \tau_{\pi(\delta_m(\mathbf{u}))}t_m\mathfrak{f}_{\mathbf{v}}.$$

This identity allows us to check that the definition of t_m acting on flows is consistent with $\delta_m : \mathbf{F}_r \to \mathbf{F}_r$. More precisely, for any $\mathbf{g} \in \mathbf{F}_r$, we have

$$\mathfrak{f}_{\delta_m(\mathbf{g})} = t_m \mathfrak{f}_{\mathbf{g}}.$$

To see this, first note that this formula holds true on the generators and their inverses and proceed by induction on the word length of $\mathbf{g} \in \mathbf{F}_r$.

Given $g \in \Gamma_2(N)$, pick a representative $\mathbf{g} \in \mathbf{F}_r$ so that g corresponds to the flow $\mathfrak{f}_{\mathbf{g}}$ on $\Gamma_1(N)$. Define $\widetilde{\delta}_m(g)$ to be the element of $\Gamma_2(N)$ that corresponds to the flow $t_m \mathfrak{f}_{\mathbf{g}} = \mathfrak{f}_{\delta_m(\mathbf{g})}$. This map is well defined and satisfies

$$\widetilde{\delta}_m \circ \pi_2 = \pi_2 \circ \delta_m.$$

This implies that $\widetilde{\delta}_m : \Gamma_2(N) \to \Gamma_2(N)$ is an injective homomorphism. Abusing notation, we will drop the $\widetilde{\ }$ and use the same name, δ_m, for the injective homomorphisms $\Gamma_1(N) \to \Gamma_1(N)$ and $\Gamma_2(N) \to \Gamma_2(N)$ induced by $\delta_m : \mathbf{F}_r \to \mathbf{F}_r$. □

Proposition 6.11 *Let $r \geq 2$ and N be a normal subgroup of $\mathbf{F}_r = \langle \mathbf{s}_1, \dots, \mathbf{s}_r \rangle$. Let $A = \Gamma_1(N)/[\Gamma_1(N), \Gamma_1(N)]$ and $T_q = A/A^q$ with projection map π_{T_q}. Assume that:*

1. *The quotient $\Gamma_1(N) = \mathbf{F}_r/N$ is residually finite and amenable.*
2. *There exist two generators, say $\mathbf{s}_1, \mathbf{s}_2$ such that for any integer p there is a natural integer $q = q(p)$ for which (with a slight abuse of notation)*

$$\pi_{T_q}(\mathbf{s}_1^p) \notin \langle \pi_{T_q}(\mathbf{s}_2) \rangle.$$

3. *The homomorphism δ_m induces an injective homomorphism $\mathbf{F}_r/N \to \mathbf{F}_r/N$, and $\pi(\mathbf{s}_i^q) \notin \delta_m(\mathbf{F}_r/N)$, $1 \leq q \leq m-1$, $1 \leq i \leq r$.*

Then, for each d, there exists an integer M_d and an exclusive sequence $(\Gamma_i, \boldsymbol{\rho}_i)_2^d$ adapted to $(\Gamma_i(N))_1^d$ and such that $\Gamma_1' = \pi_1^2(\Gamma_2)$ equals to $\langle \bar{s}_1^{M_d}, \dots, \bar{s}_r^{M_d} \rangle$ where $\bar{s}_i$ denotes the projection of $\mathbf{s}_i$ in $\Gamma_1(N)$.

Remark 6.12 Note that, by assumption 3, the group Γ_1' is isomorphic to $\Gamma_1(N)$.

Proof Since $\Gamma_1(N)$ is assumed to be residually finite, by the Magnus embedding and [10, Theorem 3.2], it follows that $\Gamma_\ell(N)$ is residually finite as well. Hence the technique developed in Sect. 4.2 apply easily to this situation. We are going to use repeatedly Proposition 4.11.

To start, for each ℓ, we construct an exclusive pair $(H_\ell, \boldsymbol{\sigma}_\ell)$ in $\Gamma_\ell(N)$. Namely, let $\boldsymbol{\sigma}_\ell$ be an element in $(N^{(\ell-2)} \backslash N^{(\ell-1)}) \cap \langle \mathbf{s}_1, \mathbf{s}_2 \rangle$ in reduced form in $\mathbf{F}_r$ and such that it projects to a non-self-intersecting loop in $\Gamma_{\ell-1}(N)$. Without loss of generality, we can assume that be beginning of $\boldsymbol{\sigma}_\ell$ is of the form $\mathbf{s}_1^k \mathbf{s}_2$. Let $s_i, \bar{s}_i$ be the projections of $\mathbf{s}_i$ onto $\Gamma_\ell(N)$ and $\Gamma_{\ell-1}(N)$, respectively. Let $(\bar{s}_1^k, \bar{s}_1^k \bar{s}_2, \mathbf{s}_2)$ be the corresponding edge in $\Gamma_{\ell-1}(N)$. Since $\boldsymbol{\sigma}_\ell$ projects to a simple loop in $\Gamma_{\ell-1}(N)$, we must have

$$\mathfrak{f}_{\boldsymbol{\sigma}_\ell}((\bar{s}_1^k, \bar{s}_1^k \bar{s}_2, \mathbf{s}_2)) \neq 0.$$

Since $\Gamma_{\ell-1}(N)$ is residually finite, there exists a finite index normal subgroup $K_{\boldsymbol{\sigma}_\ell} \lhd \Gamma_{\ell-1}(N)$ as in Lemma 4.7. Let $q = q(k)$ be the natural integer provided by assumption 2 and such that $\pi_{T_q}(\mathbf{s}_1^k) \notin \langle \pi_{T_q}(\mathbf{s}_2) \rangle$. Pick an integer m_ℓ such that

$$[\Gamma_{\ell-1}(N) : K_{\sigma_\ell}] \mid m_\ell \quad \text{and} \quad q \mid m_\ell,$$

and set

$$H_\ell = \left\langle s_i^{m_\ell}, 1 \leq i \leq r \right\rangle < \Gamma_\ell(N).$$

Thinking of $\Gamma_\ell(N)$ and $\Gamma_{\ell-1}(N)$ as $\Gamma_2(N^{(\ell-2)})$ and $\Gamma_1(N^{(\ell-2)})$, respectively, Proposition 4.11 implies that (H_ℓ, σ_ℓ) is an exclusive pair in $\Gamma_\ell(N)$.

Next, assumption 3 and Lemma 6.10 show that, for each integer m and each ℓ, the injective homomorphism $\delta_m : \mathbf{F}_r \to \mathbf{F}_r$ induces on $\Gamma_\ell(N)$ an injective homomorphism still denoted by $\delta_m : \Gamma_\ell(N) \to \Gamma_\ell(N)$. For each $1 \le \ell \le d-1$, set

$$M_1 = 1, \quad M_{d-\ell+1} = m_{\ell+1} \cdots m_d,$$

and, for $2 \le \ell \le d$,

$$\Gamma_\ell = \delta_{M_{d-\ell+1}}(H_\ell) < \Gamma_\ell(N), \quad \boldsymbol{\rho}_\ell = \delta_{M_{d-\ell+1}}(\boldsymbol{\sigma}_\ell).$$

By construction, $((\Gamma_\ell, \rho_\ell))_2^d$ is an exclusive sequence in $(\Gamma_\ell(N))_1^d$ and

$$\Gamma_1' = \pi_1^2(\Gamma_2) = \langle \bar{s_1}^{M_d}, \ldots, \bar{s}_r^{M_d} \rangle < \Gamma_1(N).$$

□

6.4 Free solvable groups and other $\Gamma_d(N)$, $d \ge 3$

In this section, we conclude the proof of Theorem 1.1 by proving that, for $d \ge 3$,

$$\Phi_{\mathbf{S}_{d,r}}(n) \simeq \exp\left(-n\left(\frac{\log_{[d-1]} n}{\log_{[d-2]} n}\right)^{2/r}\right).$$

The lower bound follows from Corollary 6.4. To prove the upper bound, we simply need to check that the given presentation of $\Gamma_1(N)$ satisfies the three assumptions of Proposition 6.11. In the case of $\mathbf{S}_{d,r}$, $\Gamma_1(N) = \mathbb{Z}^r$ and the three assumptions of Proposition 6.11 are satisfied. Theorem 6.8 gives the desired upper bound.

In fact, we are able to deal with a larger class of groups than just $\mathbf{S}_{d,r}$.

Theorem 6.13 *Fix $r \ge 2$ and $d \ge 3$. Let N be a normal subgroup of $\mathbf{F}_r$ such $\Gamma_1(N) = \mathbf{F}_r/N$ is nilpotent with volume growth of degree D. Assume also that, for each m, δ_m induces an injective homomorphism $\mathbf{F}_r/N \to \mathbf{F}_r/N$ (see Sect. 6.3). Then we have*

$$\Phi_{\Gamma_d(N)}(n) \simeq \exp\left(-n\left(\frac{\log_{[d-1]} n}{\log_{[d-2]} n)}\right)^{2/D}\right).$$

Example 6.1 Recall that $\mathbf{S}_{d,r} = \Gamma_d(\gamma_2(\mathbf{F}_r))$. More generally, define

$$\mathbf{S}_{d,r}^c = \Gamma_d(\gamma_{c+1}(\mathbf{F}_r)) = \mathbf{F}_r/(\gamma_{c+1}(\mathbf{F}_r))^{(d)}.$$

Note that $\Gamma_1(\gamma_{c+1}(\mathbf{F}_r)) = \mathbf{F}_r/\gamma_{c+1}(\mathbf{F}_r)$ is the free nilpotent group of nilpotent class c on r generators. Let

$$D(r,c) = \sum_1^c \sum_{k|m} \mu(k) r^{m/k}$$

where μ is the Möbius function. The integer $D(r,c)$ is the degree of polynomial volume growth of the free nilpotent group $\mathbf{F}_r/\gamma_{c+1}(\mathbf{F}_r)$. See [11, Theorem 11.2.2] and [5]. The hypotheses of Theorem 6.13 are clearly satisfied.

Example 6.2 Let $U(r+1)$ be the group of $(r+1)$ by $(r+1)$ upper-triangular matrices with integer entries and ones on the diagonal. Let s_i, $1 \leq i \leq r$be the matrix in $U(r+1)$ with a 1 in position $(i, i+1)$ and zeroes in all other non-diagonal positions. Let $U(r+1) = \mathbf{F}_r/N$ be the corresponding presentation. In this case the degree D is given by $D = \sum_1^r (r+1-i)i$. Again, the hypotheses of Theorem 6.13 are clearly satisfied.

As a final example of a different type, we consider $\mathbb{Z}^a \wr \mathbb{Z}^D$ where $\mathbb{Z}^a, \mathbb{Z}^D$ are equipped with their natural presentations and $\mathbb{Z}^a \wr \mathbb{Z}^D$ is equipped with the presentation described in Example 5.2.

Theorem 6.14 *Let $\mathbb{Z}^a \wr \mathbb{Z}^D = \mathbf{F}_r/N$ with $r = a + D$ as described above. For $d \geq 3$, we have*

$$\Phi_{\Gamma_d(N)}(n) \simeq \exp\left(-n\left(\frac{\log_{[d]} n}{\log_{[d-1]} n}\right)^{2/D}\right).$$

Proof The lower bound is from Corollary 6.4. To obtain the upper bound, we apply Theorem 6.8 and Proposition 6.11. By [10, Theorem 3.2], $\Gamma_1(N)$ is residually finite. To check assumption 2 in Proposition 6.11, we pick $\mathbf{s}_1$ to correspond to a generator of the base $\mathbb{Z}^D$ and $\mathbf{s}_2$ to correspond to a generator of the lamp group $\mathbb{Z}^a$. In the abelianization $\mathbb{Z}^D$, the projection of $\mathbf{s}_2$ is trivial. Assumption 2 follows. To verify assumption 3 of Proposition 6.11, we note that the stretch map δ_m acts on an element $(f, x) \in \mathbb{Z}^a \wr \mathbb{Z}^D$ via the formula $(f, a) \mapsto (f_m, mx)$ with

$$f_m(y) = mf(y/m)\mathbf{1}_{m\mathbb{Z}^d}(y).$$

One can check that this is a homomorphism. It is obviously injective. □

7 Isoperimetric profiles

The L^2-isoperimetric profile of a group G is defined as the $\simeq$-equivalence class Λ_G of the functions $\Lambda_\phi(v)$ associated to any symmetric probability measure ϕ with finite generating support as follows. For $v > 0$, set

$$\Lambda_\phi(v) = \inf\{\lambda_\phi(\Omega) : \Omega \subset G,\ \#\Omega \leq v\}$$

where

$$\lambda_\phi(\Omega) = \inf\{\mathcal{E}_\phi(f, f) : \text{support}(f) \subset \Omega, \|f\|_2 = 1\}.$$

It is well known that, under mild assumptions, the group invariants Φ_G and Λ_G encode essentially the same information. See, e.g., [2,4] and the references cited therein. In particular, the following statement is equivalent to Theorem 1.1 and Theorem 6.13.

Theorem 7.1 *Fix $r \geq 2$ and $d \geq 2$. Let N be a normal subgroup of $\mathbf{F}_r$ such $\Gamma_1(N) = \mathbf{F}_r/N$ is nilpotent with volume growth of degree D and*

$$\Gamma_1(N)/[\Gamma_1(N), \Gamma_1(N)] = \mathbb{Z}^r.$$

Assume also that, for each m, δ_m induces an injective homomorphism $\mathbf{F}_r/N \to \mathbf{F}_r/N$ (see Sect. 6.3). Then

$$\Lambda_{\Gamma_d(N)}(v) \simeq \left(\frac{\log_{[d]} v}{\log_{[d-1]} v}\right)^{2/D}.$$

The L^1-isoperimetric profile is the $\simeq$-equivalence class J_G of the functions $J_{G,S}(v)$ associated to any symmetric finite generating set S and defined by

$$J_{G,S}(v) = \sup \left\{ \frac{\#\Omega}{\#\partial_S \Omega} : \Omega \subset G, \quad \#\Omega \le v \right\}.$$

Here $\partial_S \Omega = \{x \in \Omega : \exists s \in S, xs \notin \Omega\}$. For completeness, let $\text{Føl}_{G,S}$ be the Følner function defined by

$$\text{Føl}_{G,S}(t) = \min\{s : \exists \Omega, \#\Omega = s, \#\partial_S \Omega < s/t\}.$$

The functions $J_{G,S}$ and $\text{Føl}_{G,S}$ are related by

$$\text{Føl}_{G,S}(t) > k \iff J_{G,S}(k) \le t.$$

The functions $G \mapsto J_G$ and $G \mapsto \text{Føl}_G$ (i.e., the $\simeq$-equivalence class of the function $\text{Føl}_{G,S}$) have monotonicity properties with regards to finitely generated subgroups and quotients that are similar to those of the function Φ_G. See, e.g., [9, Lemma 2.3]. It follows that the proofs of Theorems 7.2–6.8 gives the following.

Theorem 7.2 *For any presentation of* $G = \mathbf{F}_r/N$ *and integer* $\ell \ge 2$ *we have*

$$J_{\Gamma_\ell(N)} \gtrsim J_{W_{\ell-1}(\mathbb{Z}^r, G)}.$$

Theorem 7.3 *Fix a presentation* $\Gamma_1(N) = \mathbf{F}_r/N$ *and an integer* $\ell \ge 2$. *Assume that there exists an exclusive sequence* $((\Gamma_i, \boldsymbol{\rho}_i))_2^\ell$ *adapted to* $(\Gamma_j(N))_1^\ell$ *such that each* Γ_i *is finitely generated. Then we have*

$$J_{\Gamma_\ell(N)} \lesssim J_{W_{\ell-1}(\mathbb{Z}, \Gamma_1')}.$$

where $\Gamma_1' = \pi_1^2(\Gamma_2) < \Gamma_1(N)$.

Erschler [8] computes the isoperimetric profiles of any wreath product for which the isoperimetric profiles of the base and lamp groups are known. As a corollary, we obtain the following statement.

Theorem 7.4 *Referring to the setting of Theorem 7.1,*

$$J_{\Gamma_d(N)}(v) \simeq \left(\frac{\log_{[d-1]} v}{\log_{[d]} v} \right)^{1/D}.$$

We note that Theorems 7.1 and 7.4 apply in particular to $\Gamma_d(N) = \mathbf{S}_{d,r}$ ($N = \mathbb{Z}^r$). In this case, $D = r$.

Similarly, for all the examples treated in Sect. 5.3, the L^2-isoperimetric profile Λ can be computed easily from Φ and the isoperimetric profile J can be computed using similar arguments. In all these examples, the final result for J can be expressed in terms of Λ by saying that $J \simeq 1/\sqrt{\Lambda}$. For instance, if $r \ge 2$ and $G = \mathbf{F}_r/N$ is infinite nilpotent with volume growth of degree D (see Theorem 5.1) then

$$\Lambda_{\Gamma_2(N)}(v) \simeq \left(\frac{\log_{[2]} v}{\log_{[1]} v} \right)^{2/D} \quad \text{and } J_{\Gamma_2(N)}(v) \simeq \left(\frac{\log_{[1]} v}{\log_{[2]} v} \right)^{1/D}.$$

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