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ANALYSIS ON COMPACT LIE GROUPS OF LARGE DIMENSION AND ON CONNECTED COMPACT GROUPS

BY

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Abstract. The study of Gaussian convolution semigroups is a subject at the cross-road between abstract and concrete problems in harmonic analysis. This article suggests selected open problems that are in large part motivated by joint work with Alexander Bendikov.

This paper is dedicated to the memory of Andrzej Hulanicki. My first scientific encounter with Andrzej was through his students, which is fitting given the importance they had to him throughout his career. During my Ph.D., I came across the work of Paweł Głowacki (he told me later that he was the referee of my first research paper, published by Studia Mathematica and based on my Ph.D. thesis). Later, I met Waldemar Hebisch, in Boston, in the late nineteen eighties. In 1991, Andrzej invited me (together with my wife, Cathy) to Wrocław for a month. We met for the first time at the Wrocław train station where he picked us up, carrying a mathematical book so that we could recognize him. This turned out to be a beginning of a very enjoyable personal and scientific relation with Andrzej, his many students and collaborators, and with the Mathematical Institute in Wrocław. As many others, I benefited from attending the conferences and schools organized by Andrzej's group in Tuczno and, later, in Zakopane. My first Ph.D. student, Andrzej Żuk, wrote a Master thesis under Andrzej's supervision in Wrocław before following Andrzej's suggestion to work with me in Toulouse (in fact, Żuk prepared his thesis under the joint supervision of Andrzej and myself, spending half his time in Toulouse and the other half in Wrocław, as required by his doctoral fellowship). Throughout his career, Andrzej, in addition to pursuing his own research in mathematics, put a tremendous energy into organizing and supporting mathematics in Poland and in Wrocław. He maintained many long term scientific international relations, including during periods when it was not necessarily an easy thing to do. He, relentlessly,

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brought visitors to Wrocław and offered high quality scientific opportunities to the Mathematical Institute students, many of whom went abroad for their Ph.D. or to take postdoctoral positions. For many mathematicians, including myself, Andrzej's activities created fruitful scientific opportunities through collaborations and other contacts. I feel very fortunate to have shared some good moments and some interests with Andrzej Hulanicki.

1. Introduction. At the beginning of the second half of the twentieth century, it was realized that many classical objects and problems in the related areas of harmonic analysis, stochastic processes with independent stationary increments (Lévy processes) and potential theory could be considered in very general settings. These somewhat abstract developments attracted interest for a while but that interest waned in favor of a return to more concrete problems. It is worth pointing out that the structure underlying the areas of harmonic analysis and Lévy processes—the group structure—is, by itself, a wonderful source of conflicts between abstraction and concrete examples. The definition of a group is certainly one of the highlights of abstract mathematics (my impression is that groups and abstraction play quite an important role in Polish mathematics). On the one hand, it covers examples of central importance, including some relevant to undergraduate mathematics. On the other hand, groups with rather unexpected properties (e.g., groups of intermediate growth) are still being discovered, indicating how incomplete our understanding of this basic structure is. I find it interesting that the theory of algebraic groups (over local fields), of great importance to several areas of mathematics, is not readily accessible to (and is mostly ignored by) the vast majority of mathematicians. It is central to some and very exotic to many.

This article focuses on a particular set of problems where the dialectic between abstraction and concrete classical examples is rich and interesting. The starting point is Brownian motion and the associated convolution semigroup of probability measures having density $(4\pi t)^{-1/2}e^{-|x|^2/4t}$ with respect to Lebesgue measure on the real line. This leads us to various levels of generalization, from Brownian motions on the special orthogonal groups and other Lie groups to Gaussian measures on Hilbert spaces to Gaussian convolution semigroups on general groups. We will mostly discuss Gaussian convolution semigroups on locally compact groups and consider the relations between natural but rather abstract problems concerning these semigroups and concrete problems phrased in terms of the heat kernel on compact Lie groups such as the special orthogonal groups.

2. Problems concerning Gaussian semigroups

2.1. Gaussian semigroups. Let G be a locally compact group with identity element e and Haar measure λ. Let $\mathcal{C}_c(G)$ be the space of continuous functions with compact support. Given two measures μ, ν, we can write (uniquely) $\mu = A_\nu(\mu)+S_\nu(\mu)$ where $A_\nu(\mu)$ is the part of μ which is absolutely continuous with respect to ν and $S_\nu(\mu)$ is the singular part. We write also $A_{\text{Haar}}(\mu) = A_\lambda(\mu)$, $S_{\text{Haar}}(\mu) = S_\lambda(\mu)$.

Convolution operators are fundamental objects in harmonic analysis. Semigroups of convolution operators bring a rich additional structure. In this article, we focus on a particular class of convolution semigroups, those associated with a family $(\mu_t)_{\{t>0\}}$ of probability measures such that:

- $\forall t, s > 0$, $\mu_t * \mu_s = \mu_{t+s}$ (semigroup property).
- $\forall \phi \in \mathcal{C}_c(G)$, $\lim_{t\to 0} \mu_t(\phi) = \phi(e)$ (continuity).
- $\mu_t(A) = \mu_t(A^{-1})$ (symmetry).
- $\lim_{t\to 0} t^{-1}\mu_t(V) = 0$ for any compact set V with $e \notin V$.

The last property defines Gaussian semigroups (i.e., convolution semigroups of probability measures whose Lévy measure is trivial, equal to 0). It implies that the semigroup μ_t lives on the connected component of the identity. For simplicity, we refer to a semigroup $(\mu_t)_{\{t>0\}}$ with these properties as a *Gaussian semigroup*. Note that, for the purpose of this article, this includes the symmetry condition.

Let me mention here that K. Urbanik made important contributions to the definition and the study of Gaussian measures and Gaussian semigroups on groups. See, e.g., [20].

Any Gaussian semigroup on a compact connected group G induces a semigroup of self-adjoint operators defined on $L^2(G)$ by $H_t f = f * \mu_t$. We will denote by $\varDelta$ its infinitesimal generator. The domain of $\varDelta$ contains the set $\mathcal{B}(G)$ of Bruhat test functions as a core. These are functions that are lifted to G from smooth functions on a Lie quotient, and $\varDelta$ can be computed on such functions as a second order differential operator. We will also need to consider the set $\mathcal{B}'(G)$ of Bruhat distributions (the dual of $\mathcal{B}(G)$ equipped with the strong dual topology). Here and throughout this article, we avoid entering into technical details. Details and further references can be found in [2, 13], in the surveys [4, 11, 9] and in [3, 5, 6, 10].

Example 2.1. The compact group $\mathbb{T}^\infty = \mathbb{R}^\infty/\mathbb{Z}^\infty$ can be viewed as a countable product of circle groups. Its Haar measure is the product of the normalized Haar measures on the factors. Its dual group is $\mathbb{Z}^{(\infty)}$, the space of doubly infinite sequences of integers with finitely many non-zero entries.

A measure μ is (symmetric) Gaussian if its Fourier transform

$$\hat{\mu}(\theta) = \int_{\mathbb{T}^\infty} e^{-2i\pi\langle\theta,x\rangle}\,dx, \qquad \theta \in \mathbb{Z}^{(\infty)},$$

is of the form $\hat{\mu}(\theta) = e^{-\langle A\theta,\theta\rangle}$ where $A = (a_{i,j})$ is symmetric and satisfies

$$\langle A\theta,\theta\rangle = \sum_{i,j} a_{i,j}\theta_j\theta_i \geq 0, \qquad \theta = (\theta_i) \in \mathbb{Z}^{(\infty)}.$$

Any Gaussian convolution semigroup $(\mu_t)_{\{t>0\}}$ has Fourier transform

$$\widehat{\mu_t}(\theta) = e^{-t\langle A\theta,\theta\rangle}, \qquad \theta \in \mathbb{Z}^{(\infty)},$$

for some A as above. On the space $\mathcal{D}^\infty(\mathbb{T}^\infty)$ of smooth functions depending only on finitely many coordinates, the associated infinitesimal generator is given by $\Delta f = \sum_{i,j} a_{i,j}\partial_i\partial_j f$.

2.2. Warm-up. As a warm-up for the discussion of a series of difficult open problems, we consider the following problem which is implicit in [10]. For any signed Radon measure ν on G, set $\|\nu\|_{\mathrm{TV}} = \nu_+(G) + \nu_-(G)$ where $\nu = \nu_+ - \nu_-$ is the Hahn decomposition of ν. In particular, $\|\nu\|_{\mathrm{TV}} = \int |f|\,d\lambda = \lambda(|f|)$ if $d\nu = f\,d\lambda$, $f \in L^1(G)$.

Problem 1. Let G be a compact group and $(\mu_t)_{\{t>0\}}$ be a Gaussian semigroup on G. Assume that, for all $f \in \mathcal{C}(G)$, $\lim_{t\to\infty}\mu_t(f) = \lambda(f)$. Find a necessary and sufficient condition for $\lim_{t\to\infty}\|\mu_t - \lambda\|_{\mathrm{TV}} = 0$.

This came up in [10] in the study of the “global hypoellipticity” of the infinitesimal generator Δ of $(\mu_t)_{\{t>0\}}$. Any such infinitesimal generator has an extension to the space $\mathcal{B}'(G)$ of Bruhat distributions and one can ask when any Bruhat distribution U solving $\Delta U = F$ with $F \in \mathcal{C}(G)$ must, itself, be in $\mathcal{C}(G)$. It is easy to see that this is equivalent to $\lim_{t\to\infty}\|\mu_t - \lambda\|_{\mathrm{TV}} = 0$ because, formally, $U = \int_0^\infty F * (\mu_t - \lambda)\,dt$ and

$$\|\mu_t - \lambda\|_{\mathrm{TV}} = \sup\{\|\phi * (\mu_t - \lambda)\|_\infty : \phi \in \mathcal{C}(G),\ \|\phi\|_\infty \leq 1\}.$$

This shows that $t \mapsto \|\mu_t - \lambda\|_{\mathrm{TV}}$ is submultiplicative and thus converges exponentially fast if it converges to 0. See, e.g., [10, Section 5]. The following result solves Problem 1. It follows immediately from [1, Theorem 4.1] which is a general result about general convolution powers, not Gaussian semigroups.

Theorem 2.1. *Let G be a compact group and $(\mu_t)_{\{t>0\}}$ be a Gaussian semigroup on G. Assume that, for all $f \in \mathcal{C}(G)$, $\lim_{t\to\infty}\mu_t(f) = \lambda(f)$. The following properties are equivalent:*

- *There exists $t > 0$ such that $A_{\mathrm{Haar}}(\mu_t) \neq 0$.*
- $\lim_{t\to\infty}\|\mu_t - \lambda\|_{\mathrm{TV}} = 0$.
- *Any $U \in \mathcal{B}'(G)$ such that $\Delta U = F$ with $F \in \mathcal{C}(G)$ can be represented by a continuous function.*

2.3. Some open problems. Gaussian measures can be defined without reference to Gaussian semigroups but we will not insist on that here. A Gaussian measure is a probability measure μ such $\mu = \mu_1$ for some Gaussian semigroup $(\mu_t)_{\{t>0\}}$. A fundamental open problem is the following.

PROBLEM 2. Prove that any two Gaussian measures μ, ν must be either absolutely continuous or singular with respect to each other, or provide an example where $A_\nu(\mu) \not\equiv 0$ and $S_\nu(\mu) \not\equiv 0$.

This type of problem already appears in [19]. A well-known theorem (the Hájek–Feldman dichotomy) asserts that two Gaussian measures on a Hilbert space are either absolutely continuous or singular with respect to each other (this is the solution of the linear version of the problem) but the (infinite-dimensional) group version is open even for abelian compact groups (e.g., $\mathbb{T}^\infty$). In what follows, we will focus on related problems concerning Gaussian semigroups. Let $(\mu_t)_{\{t>0\}}$ be such a semigroup on a locally compact unimodular group G.

PROBLEM 3. Prove that, for any $s, t \in (0, \infty)$, μ_t and μ_s are either absolutely continuous or singular with respect to each other, or provide an example where $A_{\mu_s}(\mu_t) \not\equiv 0$ and $S_{\mu_s}(\mu_t) \not\equiv 0$ for some $s, t \in (0, \infty)$.

PROBLEM 4. Prove that, for any $t \in (0, \infty)$, μ_t is either absolutely continuous or singular with respect to Haar measure, or provide an example where $A_{\text{Haar}}(\mu_t) \not\equiv 0$ and $S_{\text{Haar}}(\mu_t) \not\equiv 0$, for some $t \in (0, \infty)$.

Problem 4 can be formulated in different terms. Observe that if $\mu_t = A_{\text{Haar}}(\mu_t)$ for some t then this property is satisfied for all later times. Set

$$t_A = \inf\{t > 0 : \mu_t = A_{\text{Harr}}(\mu_t)\}.$$

PROBLEM 5 (equivalent to Problem 4). Prove that, for any $t \in (0, t_A)$, $\mu_t = S_{\text{Haar}}(\mu_t)$ and either $A_{\text{Haar}}(\mu_{t_A})$ or $S_{\text{Haar}}(\mu_{t_A})$ vanishes, or provide an example where $A_{\text{Haar}}(\mu_t) \not\equiv 0$ and $S_{\text{Haar}}(\mu_t) \not\equiv \mu_t$), for some $t \in (0, t_A]$.

The following problem introduces a new twist which concerns what happens for $t > t_A$. For such t, let $f_t \in L^1(G)$ be the density of the measure μ_t with respect to Haar measure on G.

PROBLEM 6. Prove or disprove that, for any $t \in (t_A, \infty)$, $f_t \in L^2(G)$.

The optimistic conjecture is that $f_t \in L^p(G)$ for all $p \in [1, \infty)$ when $t > t_A$. Note that it is not hard to check that $f_t \in L^2(G)$ is equivalent to $f_{2t} \in \mathcal{C}(G)$, which makes this problem quite attractive. This leads to our next open problem.

PROBLEM 7. Prove or disprove that if $\mu_t = A_{\text{Haar}}(\mu_t)$ for all $t > 0$ then $f_t \in \mathcal{C}(G)$ for all $t > 0$.

2.4. Discussion of Problems 4–7. An early reference and, indeed, one of the sources of the problems mentioned above is [19]. The first important result directly related to Problems 4–7 above is the fact that

$$A_{\mathrm{Haar}}(\mu) = 0$$

for any Gaussian measure μ on a locally compact connected group G unless G is locally connected and admits a countable basis for its topology (for this result due to Heyer and Siebert, see [7, 13] and the references therein). The next remarkable fact is that on any locally compact connected locally connected metrizable group G there exists a Gaussian semigroup such that $\mu_t = A_{\mathrm{Haar}}(\mu_t)$ and $f_t \in L^\infty(G) \cap \mathcal{C}(G)$ for all $t > 0$. See, again, [7, 13]. Hence we will focus on locally compact connected locally connected metrizable groups. For simplicity, we assume that G is compact since this case already contains the main difficulties.

The problems listed above appear to be rather difficult. In a number of special cases (some of which will be discussed below), it is possible to get a grasp on the relevant properties of Gaussian semigroups. These cases are in support of positive answers to Problems 4–7. But, at the same time, we have no clues how to attack these problems for general Gaussian semigroups. A deep and well-known fact concerning compact connected metrizable groups is that any such group is the projective limit of a sequence of compact Lie groups. See, e.g., [14]. Using this structure, it is easy to classify Gaussian semigroups into the following categories: (a) degenerate, (b) subelliptic, (c) elliptic. Namely, a Gaussian semigroup $(\mu_t)_{\{t>0\}}$ is *degenerate* if there is a Lie quotient of G on which the pushforward of μ_t does not have a positive smooth density with respect to Haar measure. In this case, $\mu_t = S_{\mathrm{Haar}}(\mu_t)$ for all $t > 0$. The Gaussian semigroup $(\mu_t)_{\{t>0\}}$ is *subelliptic* if the pushforward of μ_t on each Lie quotient of G has a positive smooth density. In terms of the infinitesimal generator Δ, the dichotomy between (a) and (b) can be expressed as follows: In case (a), there is a Lie quotient on which the projection of Δ is not subelliptic (i.e., does not satisfy the Hörmander condition when written as a sum of squares of left-invariant vector fields). In case (b), the projection of Δ on each Lie quotient is subelliptic (i.e., does satisfy the Hörmander condition when written as a sum of squares of left-invariant vector fields). Finally, the Gaussian semigroup $(\mu_t)_{\{t>0\}}$ is *elliptic* if the projection of the infinitesimal generator is elliptic on each Lie quotient. Since case (a) is trivial, one is led to the study of subelliptic Gaussian semigroups on Lie groups of dimension growing to infinity. Even in the elliptic case, very little is known about this problem in terms of trying to keep track of the influence of dimension.

EXAMPLE 2.2 (Products). Assume that $G = \prod_{i=1}^\infty \mathfrak{G}_i$ where each $\mathfrak{G}_i$ is a compact connected Lie group and that $\mu_t = \bigotimes_{i=1}^\infty \mu_t^i$ where, for each i,

$(\mu_t^i)_{\{t>0\}}$ is a subelliptic Gaussian semigroup on $\mathfrak{G}_i$. In this (very special) case, Problems 3–5 are solved by a theorem of Kakutani concerning infinite products of measures. Kakutani's theorem asserts in particular that μ_t can only be either singular or absolutely continuous with respect to the Haar measure on G. It also gives a criterion for this dichotomy which can be applied in specific cases to compute the time t_A. See [2, 3, 9, 11] and the references therein. Note, however, that Problems 6–7 are open in the generality of this special case. A positive solution to 6–7 is known when each $\mathfrak{G}_i$ is a circle group. See [2, 5, 9, 11]. It is an interesting question to solve Problem 6–7 when each $\mathfrak{G}_i$ is abelian.

Problem 8 (Special case of Problems 6–7). Assume that $G = \prod_{i=1}^{\infty} \mathfrak{G}_i$ where each $\mathfrak{G}_i$ is a finite-dimensional torus (whose dimension might depend on i) and that $\mu_t = \bigotimes_{i=1}^{\infty} \mu_t^i$ where, for each i, $(\mu_t^i)_{\{t>0\}}$ is an elliptic Gaussian semigroup on $\mathfrak{G}_i$.

- Prove or disprove that $f_t \in \mathcal{C}(G)$ whenever $t > 2t_A$.
- Prove or disprove that "$\mu_t = A_{\text{Haar}}(\mu_t)$ for all $t > 0$" implies "$f_t \in \mathcal{C}(G)$ for all $t > 0$".

Example 2.3 (Central semigroups on semisimple groups). A compact connected group is called *semisimple* if $G = [G, G]$. A Gaussian semigroup $(\mu_t)_{\{t>0\}}$ is *central* if $\mu_t(gAg^{-1}) = \mu_t(A)$ for all $t > 0$, $g \in G$ and Borel subsets A of G. Structure theory asserts that connected compact metrizable semisimple groups are of the form $G = [\prod_{i \in I} \Sigma_i]/H$ where the index set I is finite or countable, the Σ_i's are simple connected compact Lie groups and H is a closed central subgroup of $\prod_{i \in I} \Sigma_i$ (hence, H is a subgroup of a product of finite groups). On $\prod_{i \in I} \Sigma_i$, any central Gaussian semigroup must be of product form and on each Σ_i there is only one (non-degenerate) central Gaussian semigroup, up to a positive constant time change. These facts and assorted analytic results concerning central Gaussian semigroups on simple compact Lie groups yield partial positive answers to Problems 4–6 and solve Problem 7 in this case.

Theorem 2.2 ([8]). *For any central Gaussian semigroup on a compact connected semisimple group the following properties hold.*

- *Either* $A_{\text{Haar}}(\mu_t) = 0$ *for all* $t > 0$ *or there exists a* $t > 0$ *such that* $\mu_t = A_{\text{Haar}}(\mu_t)$.
- *There exists a time* $t_0 \in [0, \infty]$ *such that* $A_{\text{Haar}}(\mu_t) = 0$ *for all* $t < t_0/4$, $\mu_t = A_{\text{Haar}}(\mu_t)$ *for all* $t > 2t_0$ *and* $f_t \in \mathcal{C}(G)$ *for all* $t > 4t_0$.
- *If* $\mu_t = A_{\text{Haar}}(\mu_t)$ *for all* $t > 0$ *then* $f_t \in \mathcal{C}(G)$ *for all* $t > 0$.

This leaves the following version of Problem 6 open.

Problem 9 (Special case of Problem 6). Let G be a connected semisimple group and $(\mu_t)_{\{t>0\}}$ be a central Gaussian semigroup on G. Prove or disprove that, for any $t \in (t_A, \infty)$, $f_t \in L^2(G)$.

3. Analysis on compact Lie groups. In this section, we let $\mathfrak{G}$ be a compact connected Lie group of dimension n. By the Malcev decomposition $\mathfrak{G}$ is isomorphic to $[\mathfrak{Z} \times \mathfrak{G}']/H$ where $\mathfrak{Z}$ is the connected component of the center (a finite-dimensional torus) of $\mathfrak{G}$, $\mathfrak{G}' = [\mathfrak{G}, \mathfrak{G}]$, and $H = \mathfrak{Z} \cap \mathfrak{G}'$. We will not use this here but it shows that, to some extent, one can hope to obtain results for general compact Lie groups from the study of both the abelian and the semisimple cases (see, e.g., [6]).

Given a (non-degenerate) Gaussian semigroup $(\mu_t)_{\{t>0\}}$ on $\mathfrak{G}$ with density f_t with respect to the Haar measure λ, we consider the following parameters.

- The *spectral gap*:

$$\lambda_1 = \inf\{\langle f, \Delta f\rangle / \|f\|_2^2 : f \in \mathcal{C}(G),\ f \perp 1\}.$$

- The *logarithmic Sobolev constant*:

$$\alpha = \inf\{\langle f, \Delta f\rangle / \mathcal{L}(f)\ f \in \mathcal{C}(G)\}$$

  where $\mathcal{L}(f) = \int |f|^2 \log(|f|^2/\|f\|_2^2)\, d\lambda$.
- The L^1-*mixing time*:

$$T = \inf\{t > 0 : \|\mu_t - \lambda\|_{\mathrm{TV}} \le 1/4\}.$$

- The L^2-*mixing time*:

$$\theta = \inf\{t > 0 : \|f_t - 1\|_2 \le 1/4\}.$$

We observe that, always, $1/\lambda_1 \le T \le \theta$ and $\alpha \le \lambda_1/2$.

3.1. A basic example: square tori. Let $\mathbb{R}^n$ be equipped with its canonical Euclidean structure and let

$$\mathbb{T}^n(r) = \mathbb{R}^n / 2\pi r \mathbb{Z}^n$$

be the product of n circles of length r equipped with the heat semigroup

$$\mu_t = \bigotimes_{i=1}^n \mu_t^i, \qquad t > 0,$$

with infinitesimal generator $\Delta = \sum_{i=1}^n (\partial/\partial\theta_i)^2$. Here, each μ_t^i is a copy of the Gauss measure $\nu_t(d\theta) = \phi_t(\theta)d\theta/2\pi r$ on the circle $\mathbb{T}(r) = \mathbb{R}/2\pi r\mathbb{Z}$ where

$$\phi_t(\theta) = \frac{2\pi r}{\sqrt{4\pi t}} \sum_{k\in\mathbb{Z}} e^{-|\theta + 2\pi r k|^2/4t}.$$

Further, we have the spectral formula (use Fourier transform)

$$\phi_t(0) = \sum_{k\in\mathbb{Z}} e^{-tk^2/r^2} = 1 + 2e^{-t/r^2}\Big(1 + \sum_{k=2}^{\infty} e^{-t(k^2-1)/r^2}\Big).$$

As $\sum_{k=2}^{\infty} e^{-t(k^2-1)/r^2} \simeq e^{-3t/r^2}(1 + r/\sqrt{t})$, it follows that

$$1 + 2e^{-t/r^2}\left(1 + c_1 \frac{r}{\sqrt{t}}\, e^{-3t/r^2}\right) \le \phi_t(0) \le 1 + 2e^{-t/r^2}\left(1 + c_2 \frac{r}{\sqrt{t}}\, e^{-3t/r^2}\right).$$

The spectral gap λ_1 and log-Sobolev constant α for $\mathbb{T}^n(r)$ satisfy

$$\lambda_1 = 2\alpha = \frac{1}{r^2}$$

(this is very easy for λ_1 but not so easy for α). Further, as n tends to infinity, $T \sim \theta \sim (r^2/2)\log n = (1/2\lambda_1)\log n$ (see, e.g., [17]). The heat kernel f_t on $\mathbb{T}^n(r)$ (with respect to the normalized Haar measure) satisfies (here, $e = (0,\ldots,0)$)

$$f_t(e) = \phi_t(0)^n = \Big(1 + 2\sum_{k=1}^{\infty} e^{-tk^2/r^2}\Big)^n.$$

Hence $f_t(e)$ is bounded above and below by expressions of the form

$$\left(1 + 2e^{-t/r^2}\left(1 + c\,\frac{r}{\sqrt{t}}\, e^{-3t/r^2}\right)\right)^n.$$

When $t/r^2 \le 1$, this gives

$$\left(1 + \frac{c_1 r}{\sqrt{t}}\right)^n \le f_t(e) \le \left(1 + \frac{c_1 r}{\sqrt{t}}\right)^n.$$

When $t = r^2\log n + s$ with $s > 0$, it yields

$$1 + c_1' e^{-s/r^2} \le f_t(e) \le 1 + c_2' e^{-s/r^2}.$$

In the range $r^2 < t < r^2\log n$, the heat kernel goes from being of size $(1+\epsilon)^n$ to size 1. These descriptions of the size of the heat kernel on $\mathbb{T}^n(r)$ are uniform in the parameters r and n.

3.2. Another example: special orthogonal groups. As a prototype of a simple compact Lie group, consider the special orthogonal group $\mathrm{SO}(n)$. On $\mathrm{SO}(n)$, as on any other simple compact Lie group, there is (up to a constant multiplicative factor) a unique differential operator of order 2 without constant term that commutes with both left and right group multiplication. This operator is the Laplacian associated with the unique bi-invariant Riemannian metric on $\mathrm{SO}(n)$ (up to constant factors). This metric is associated with an Euclidean structure g on the Lie algebra $\mathfrak{so}(n)$ of $\mathrm{SO}(n)$ which can be obtained canonically as

$$g(X,Y) = -\operatorname{Tr}(\operatorname{ad} X \operatorname{ad} Y)$$

where $\operatorname{ad} X(Z) = [X,Z]$ (i.e., the metric g is minus the Killing form). Abusing notation, we also denote by g the associated bi-invariant metric on $\mathrm{SO}(n)$. We denote by Δ the Laplacian on $\mathrm{SO}(n)$ associated with g. Note that this fixes in a canonical way the multiplicative factors alluded to above. Note also that the dimension of $\mathrm{SO}(n)$ is $N = n(n-1)/2$. One faces the following questions:

- What are the parameters $\lambda_1, \alpha, T, \theta$ for $(\mathrm{SO}(n), g)$?
- Can we describe the size of the heat kernel at the identity, $f_t(e)$, uniformly in n?

We now describe what is known about these questions. First, the exact value of the spectral gap is known, namely, $\lambda_1(\mathrm{SO}(n)) = (n-1)/2(n-2)$ (this follows from the extremely well developed representation theory of simple compact Lie groups). Representation theory yields the formula

$$f_t(e) = \sum_\rho \dim(\rho)^2 e^{-t\lambda_\rho}$$

where the sum is over all (equivalence classes of) irreducible representations, together with formulas for $\dim(\rho)$ and λ_ρ. However, these formulas are complicated enough that extracting information concerning the size of the heat kernel $f_t(e)$, uniformly in the dimension parameter n, is a difficult task.

Conjecture 1. There are constants $c_i \in (0,\infty)$, $i = 1, \dots, 4$ such that, for all n, the heat kernel on $\mathrm{SO}(n)$ satisfies

$$\left(1 + c_1 e^{-\lambda_1 t}\left(1 + \frac{e^{-c_2 t}}{\sqrt{t}}\right)\right)^N \le f_t(e) \le \left(1 + c_3 e^{-\lambda_1 t}\left(1 + \frac{e^{-c_4 t}}{\sqrt{t}}\right)\right)^N.$$

Here $\lambda_1 = (n-1)/2(n-2)$ is the spectral gap and $N = n(n-1)/2$ is the dimension.

In fact, we propose this as a conjecture for $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$ as well as $\mathrm{SO}(n)$ and the associated Spin groups. In each case, λ_1 is the (known) spectral gap and N the dimension of the group. In all these cases, $\lambda_1 \sim 1/2$ as $N \to \infty$. Another way to state this conjecture is to say that for this family of groups, $f_t(e)$ behaves in roughly the same way as the similar quantity on $\mathbb{T}^N(1/\sqrt{\lambda_1})$, the square torus of the same dimension with the same spectral gap!

At any fixed dimension, the conjecture matches the well-known behavior of $f_t(e)$, namely

$$f_t(e) - 1 \simeq e^{-\lambda_1 t} \quad \text{as } t \to \infty, \qquad f_t(e) \simeq t^{-N/2} \quad \text{as } t \to 0.$$

In particular, and more precisely, the conjecture states that (uniformly in N) for $t = \lambda_1^{-1} \log N + s$, $s > 0$,

$$1 + c_1 e^{-\lambda_1 s} \leq f_t(e) \leq 1 + c_3 e^{-\lambda_1 s}$$

whereas for $t \in (0, \lambda_1^{-1})$,

$$\left(1 + \frac{c_1'}{\sqrt{t}}\right)^N \leq f_t(e) \leq \left(1 + \frac{c_3'}{\sqrt{t}}\right)^N.$$

A further (and somewhat more subtle) implication of the conjecture is that for any fixed $t_0 \in (0, \infty)$ there are constants $\epsilon_1 = \epsilon_1(t_0)$, $\epsilon_2 = \epsilon_2(t_0) \in (0, \infty)$ such that

$$(1 + \epsilon_1)^N \leq f_{t_0}(e) \leq (1 + \epsilon_2)^N.$$

The most subtle part of the conjecture is the description of $f_t(e)$ in the time interval $(\epsilon, \epsilon^{-1} \log N)$, for any given $\epsilon \in (0, 1)$, uniformly over N.

Remark 3.1. The paper [12] proposes (and claims to prove) a remarkable universal formula for the heat kernel value $f_t(e)$ on any compact semi-simple simply connected Lie group. This formula is close in spirit (and much more precise) than the conjecture made above. Unfortunately, the formula of [12] is clearly wrong and it appears that the computations behind it cannot be salvaged as far as estimates of $f_t(e)$ are concerned. Although some specialists know that the results in [12] are erroneous, there appears to be no record of this error in the literature.

Remark 3.2. The conjectural two-sided inequality in Conjecture 1 implies that $\theta \sim (1/2\lambda_1) \log N$. As noted above, $T \leq \theta$ and $\lambda_1 \sim 1/2$ as $N \to \infty$. Further, it is proved in [17, 18] that T is asymptotically bounded below by $\log N$ as $N \to \infty$. Hence, if valid, Conjecture 1 implies that $T \sim \theta \sim \log N$ as $N \to \infty$. This remark applies to the groups covered by the conjecture, that is, $\mathrm{SO}(n)$ and also $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$.

Remark 3.3. Conjecture 1 cannot be valid as stated for the adjoint groups of $\mathrm{SO}(2l)$ and $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$. Indeed, for those groups, $\lambda_1 \sim 1$ as $N \to \infty$ and the conjectural two-sided inequality in question would yield the conclusion that $\theta \sim \frac{1}{2} \log N$ for these groups. However, it is easy to see from the second term in the spectral expansion of $f_t(e)$ that θ is asymptotically bounded below by $\log N$. See [18]. A reasonable conjecture for all compact simple Lie groups is as follows.

Conjecture 2. In the family of compact simple Lie groups, as the dimension N tends to infinity, we have $T(G) \sim \theta(G) \sim \log N$.

The next theorem states what is actually known concerning T and θ.

Theorem 3.4 ([17, 18]). *For any $\epsilon \in (0, 1)$ there exists $N(\epsilon)$ such that, for any simple compact Lie group G of dimension $N > N(\epsilon)$ equipped with* *its Killing metric, we have*

$$(1-\epsilon)\log N \le T(G) \le \theta(G) \le 2(1+\epsilon)\log N.$$

To link this result with Remark 3.1, Ursula Porod observed years ago that, if true, the erroneous formula of [12] would yield $\theta(G) \sim \frac{3}{2}\log N$ in the simply connected case. It is rather amusing and somewhat intriguing that the erroneous formula of [12] implies an asymptotic that falls squarely in between the known upper and lower bounds!

Remark 3.5. Conjecture 2 cannot be extended as stated to the semisimple case. Indeed, let $G = \mathrm{SU}(n)$ with $n \ge 4$. Let K be the associated adjoint group $K = G/\mathbb{Z}_n$. Fix $n \ge 4$ and consider the families of semisimple groups G^m and K^m. Note that the dimension $N(m)$ of G^m and K^m is $(n-1)(n+1)m$. One can show that $\theta(G^m) \sim \log m \sim \log N(m)$ as $m \to \infty$ whereas $\theta(K^m) \sim \frac{1}{2}\log m \sim \frac{1}{2}\log N(m)$ as $m \to \infty$.

Remark 3.6. A weaker form of Conjecture 1 which covers compact semisimple groups is as follows.

Conjecture 3. There are constants $c_i \in (0,\infty)$, $i = 1,\dots,6$, such that, on any compact semisimple Lie group of dimension N equipped with its Killing metric, the heat kernel satisfies

$$\left(1 + c_1 e^{-c_2 t}\left(1 + \frac{e^{-2c_3 t}}{\sqrt{t}}\right)\right)^N \le f_t(e) \le \left(1 + c_4 e^{-c_5 t}\left(1 + \frac{e^{-c_6 t}}{\sqrt{t}}\right)\right)^N.$$

This is much less precise than Conjecture 1. It does contain the nontrivial statement that, for any fixed t_0, $f_{t_0}(e)$ is exponentially large in N.

3.3. Easy results for products. To connect this section to the problems discussed earlier, consider an arbitrary sequence of compact connected Lie groups $\mathfrak{G}_i$ each equipped with a non-degenerate Gaussian semigroup $(\mu_t^i)_{\{t>0\}}$. Let n_i, λ_1^i, α_i, T_i, θ_i be the corresponding parameters. Let $\mu_t = \bigotimes_{i=1}^\infty \mu_t^i$ be the corresponding product Gaussian semigroup on $G = \prod_{i=1}^\infty \mathfrak{G}_i$.

Proposition 3.7 ([3]). *Referring to the setup described above, we have:*

- $A_{\mathrm{Haar}}(\mu_t) = 0$ *for any* $t < t_2$ *where*

$$t_2 = \inf\Big\{t > 0 : \sum_i e^{-2t\lambda_1^i} < \infty\Big\}.$$

- $A_{\mathrm{Haar}}(\mu_t) = 0$ *for any* $t < t_0 = \limsup_{i\to\infty} T_i$.
- $\mu_t = A_{\mathrm{Haar}}(\mu_t)$ *and* $f_t \in L^2(G)$ *for all* $t > t_1$ *where*

$$t_1 = \inf\Big\{t > 0 : \sum_i e^{-2(t-\theta_i)\lambda_1^i} < \infty\Big\}.$$

- $f_t \in \mathcal{C}(G)$ *for all* $t > 2t_1$.

EXAMPLE 3.1. Assume that all $\mathfrak{G}_i = \mathfrak{G}_0$ and $\mu_t^i = \mu_{a_i t}^0$, $t > 0$, for a sequence $a_i > 0$. If $\liminf_{i\to\infty} a_i < \infty$ then $A_{\text{Haar}}(\mu_t) = 0$ for all $t > 0$. Further, if $\liminf_{i\to\infty} a_i = \infty$, one easily checks that $1/\lambda_1^i \simeq 1/a_i \simeq T_i \simeq \theta_i$. Hence, $t_1 = t_2$ and we can conclude that $A_{\text{Haar}}(\mu_t) = 0$ for any $t < t_2$, $\mu_t = A_{\text{Haar}}(\mu_t)$ and $f_t \in L^2(G)$ for all $t > t_2$, and $f_t \in \mathcal{C}(G)$ for all $t > 2t_2$. In addition, we also have $\alpha_i \simeq a_i$. It follows from the resulting hypercontractivity property that $f_t \in L^p(G)$, $1 \le p < \infty$, for every $t > t_2$. See [3].

PROBLEM 10. Assume that each $\mathfrak{G}_i$ is a compact connected abelian Lie group (a torus) and that $\lim_{i\to\infty} \lambda_1^i = \infty$.

- Prove or disprove that $\limsup_{i\to\infty} T_i = \limsup_{i\to\infty} \theta_i$.
- Prove or disprove that $\lambda_1^i \simeq \alpha_i$.

EXAMPLE 3.2. Assume that each $\mathfrak{G}_i$ is of the form $\mathfrak{G}_i = \mathrm{SO}(n_i)$ and each μ_t^i is of the form $\mu_t^i = \nu_{a_i t}^i$ where $d\nu_t^i = f_t^i \, d\lambda^i$ is the Gaussian semigroup associated with the Killing metric on $\mathrm{SO}(n_i)$ and λ^i is normalized Haar measure. Recall that t_A is the smallest time so that $S_{\text{Haar}}(\mu_t) = 0$ for all $t > t_A$. In the present case, it is known that there are constant c_1, c_2 such that $c_1 t_* \le t_A \le c_2 t_*$ where

$$t_* = \inf\Big\{ t > 0 : \sum_i n_i^2 e^{-t a_i} < \infty \Big\}.$$

See [6, 8, 9, 11].

It is worth emphasizing the relevance of Conjecture 1 for the problem considered in this section. It is not difficult to see that the product semigroup $\mu_t = \bigotimes_{i=1}^{\infty} \nu_{a_i t}^i$ on $G = \prod_{i=1}^{\infty} \mathrm{SO}(n_i)$ admits a continuous density f_t with respect to the Haar measure $\lambda = \bigotimes_{i=1}^{\infty} \lambda^i$ if and only if the product $f_t(e) = \prod_{i=1}^{\infty} f_{a_i t}^i(e)$ converges.

It is obvious that, if true, conjecture 1 shows that this product converges for a given t if and only if (recall that the spectral gap on $\mathrm{SO}(n_i)$ is $(n_i - 1)/2(n_i - 2)$)

$$\sum_{i=1}^{\infty} n_i^2 e^{-\frac{n_i - 1}{n_i - 2} a_i t} < \infty,$$

and gives a two-sided bound on the size of $f_t(e)$. Let

$$t_\# = \inf\Big\{ t : \sum_{i=1}^{\infty} n_i^2 e^{-\frac{n_i - 1}{n_i - 2} a_i t} < \infty \Big\}.$$

If true, Conjecture 1 would also imply that $t_A = \frac{1}{2} t_\#$ and would solve Problems 6 and 7 positively in the special case discussed here.

3.4. Flat tori. Problem 10 in the previous section can be put in a different way to emphasize the fact that it deals with finite-dimensional tori.

PROBLEM 11. Do there exist constants $c, C \in (0,\infty)$ such that on any compact connected abelian Lie group $\mathfrak{G}$ (i.e., finite-dimensional torus) and for any Gaussian semigroup on $\mathfrak{G}$, we have $\theta \le CT$? or $c\lambda_1 \le \alpha$?

REMARK 3.8. Fix an integer m. Assume that $\mathfrak{G} = \prod_{i=1}^k \mathbb{T}_i$ and $\mu_t = \bigotimes \mu_t^i$ where each $\mathbb{T}_i$ is a torus of dimension at most m and $(\mu_t^i)_{\{t>0\}}$ is a Gaussian semigroup on $\mathbb{T}_i$. Then it is known that $(m+1)^{-1}\lambda_1^i \le \alpha_i \le \lambda_1^i/2$. See [15]. Further, there exists $C = C(m)$ such that $T_i \le \theta_i \le C(m)T_i$. Indeed, one can show that $\theta_i \simeq T_i \simeq 1/\lambda_1^i \simeq \mathrm{diam}_i^2$ where the implied constants depend only on m and diam_i is the diameter of $\mathbb{T}_i$ in the invariant metric canonically associated with the infinitesimal generator of $(\mu_t^i)_{\{t>0\}}$ (whenever the semigroup is degenerate, we set $\mathrm{diam}_i = \infty$). See [16, 17].

It is worth emphasizing that Problem 11 deals with very classical objects. A torus $\mathfrak{G}$ equipped with an invariant (i.e., flat) Riemannian metric g can be viewed in two equivalent ways.

(1) Equip $\mathbb{R}^n$ with its canonical Euclidean structure. Then there is a (co-compact) lattice (i.e., discrete subgroup of maximal rank) $\Gamma \subset \mathbb{R}^n$ such that $(\mathfrak{G}, g)$ is isometric to $\mathbb{R}^n/\Gamma$ equipped with its canonical metric. Of course, the choice of Γ is not unique (two lattices related by an orthonormal transformation give isometric tori). Choosing a basis $(u_1, \dots, u_n)$ for Γ, the parallelepiped

$$P(\Gamma) = \Big\{ s = \sum_{i=1}^n s_i u_i : s_i \in [0,1),\ 1 \le i \le n \Big\} \subset \mathbb{R}^n = \Big\{ x = \sum_{i=1}^n x_i e_i \Big\}$$

is a fundamental domain for the action of Γ. It follows that the mixing times T, θ, the spectral gap λ_1 and the log-Sobolev constant α for the Riemannian torus $(\mathfrak{G}, g)$ are exactly those for the Laplacian $\Delta = \sum_{i=1}^n (\partial/\partial x_i)^2$ in $P(\Gamma)$ with periodic boundary condition. We can think of these parameters as functions of the lattice Γ. Obviously, the different but related version of Problem 11 for the Laplacian in $P(\Gamma)$ with the Neumann boundary condition is of great interest as well.

(2) Write $\mathfrak{G} = \mathbb{R}^n/\mathbb{Z}^n$ as a group (and as a manifold). As the metric g is invariant, it is determined by its value (i.e., a symmetric positive definite matrix $(a_{i,j})$) in the tangent space at the identity element (i.e., 0). In the invariant frame given by writing $\mathfrak{G} = \mathbb{R}^n/\mathbb{Z}^n$, the Laplacian is $\sum_{i,j=1}^n a_{i,j} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}$. From this viewpoint, the parameters $T, \theta, \lambda_1, \alpha$ are functions of $(a_{i,j})$ (constants under conjugation by $\mathrm{SL}_n(\mathbb{Z})$).

The evidence for a positive solution of Problem 11 is that the desired inequalities hold true when the Riemannian torus $\mathfrak{G}$ is “rectangular”. Here, in the language of (1) above, rectangular means that $\mathfrak{G} = \mathbb{R}^n/\Gamma$ with $\Gamma = \{\gamma = \sum_{i=1}^n \gamma_i e_i\}$ where $(e_i)_{i=1}^n$ is the canonical basis of $\mathbb{R}^n$. Taking the alternative viewpoint (2), rectangular means that $(a_{i,j})$ is $SL_n(\mathbb{Z})$-

conjugate to a diagonal matrix. More generally, for any m, there are constants $c(m), C(m)$ such that the inequalities $\theta \leq C(m)T$, $c(m)\lambda_1 \leq \alpha$ hold true whenever the Riemannian torus $\mathfrak{G}$ is a product of Riemannian tori of dimension bounded by m.

3.5. Analysis on compact irreducible homogeneous spaces. As we have seen in Section 3, simple Lie groups form an interesting class of manifolds each of which carries a canonical Gaussian semigroup (up to a constant time scaling factor). In fact, the natural definition in that direction is that of compact isotropy irreducible Riemannian manifold. A compact Riemannian manifold (M, g) is said to be *isotropy irreducible* if for each point $p \in M$ the isotropy group at p (i.e., the group of all isometries fixing p) acts irreducibly on the tangent space at p via its isotropy representation. See [21]. A direct consequence of this definition is that the metric g is unique (up to multiplication by a constant) among all metrics with the same isometry group and is an Einstein metric. As the group of isometries of M must act transitively, M is a homogeneous space. In fact, this class of compact manifolds is also the class of connected effective homogeneous spaces G/H with G and H compact and Ad_H acting irreducibly on $\mathfrak{g}/\mathfrak{h}$ (the quotient of the associated Lie algebras).

Let (M, g) be a compact isotropy irreducible manifold. Let Δ be the Laplace–Beltrami operator on M. Since, in general, M is not a group we need to alter our notation a bit. Namely, we let $f_t(x, y)$ be the kernel of the heat semigroup $e^{t\Delta}$ with respect to the normalized volume measure λ on M. As (M, g) is a Riemannian homogeneous space, the quantities

$$\int_M |f_t(x,y) - 1|\, d\lambda(y) \quad \text{and} \quad \int_M |f_t(x,y) - 1|^2\, d\lambda(y)$$

are independent of x and we can define the parameters $T = T(M, g)$ and $\theta = \theta(M, g)$ by

$$T = \inf\{t > 0 : \|f_t(x, \cdot) - 1\|_1 \leq 1/4\},$$
$$\theta = \inf\{t > 0 : \|f_t(x, \cdot) - 1\|_2 \leq 1/4\}.$$

Of course, we can also define the parameters $\lambda_1 = \lambda_1(M, g)$ (smallest positive eigenvalue) and $\alpha = \alpha(M, g)$ (log-Sobolev constant). We also let $N = N(M)$ be the topological dimension of M. Since g is Einstein, we also introduce its Einstein constant $\rho = \rho(M, g) \geq 0$ which is such that $\mathrm{Ric} = \rho g$ where Ric denotes the Ricci tensor of g.

As explained in [21], if $\rho = 0$ then $M = \mathfrak{G}$ is an isotropy irreducible torus. Otherwise, $\rho > 0$ and $M = G/H$ with G compact semisimple (of course, not all such homogeneous spaces are isotropy irreducible, see [21]).

As noted earlier, the following inequalities always hold: $\alpha \leq \lambda_1/2$, $T \leq \theta$.

Moreover, $\lambda_1 \geq N\rho/(N-1)$, $\alpha \geq N\rho/[2(N-1)]$ and, in fact (see [15]),

$$\alpha \geq \frac{N(N-1)}{2(N+1)^2}\,\rho + \frac{2N}{(N+1)^2}\,\lambda_1.$$

As discussed above, the metric g is well defined up to a positive constant factor. Hence it is useful to recall that if g is changed to $g^\kappa = \kappa g$ with $\kappa > 0$ (a positive real) then ρ, α and λ_1 are changed to $\rho^\kappa = \kappa^{-1}\rho$, $\alpha^\kappa = \kappa^{-1}\alpha$ and $\lambda_1^\kappa = \kappa^{-1}\lambda_1$ whereas T and θ are changed to $T^\kappa = \kappa T$, $\theta^\kappa = \kappa\theta$.

In the case $\rho > 0$, one problem is to decide whether or not there is a constant c such that $\lambda_1 \leq c\rho$ (uniformly over all isotropy irreducible compact manifolds with $\rho > 0$, or perhaps over a smaller class). Because of the extremely well developed theory of representations of compact simple Lie groups and the fact that there exists a classification of all isotropy irreducible manifolds (see the discussion in [21]), one can try to attack this question by inspection. This however is a gigantic task and one would like to find some kind of more global approach (see [18] for the case of compact simple Lie groups). Good upper bounds on T, θ involving ρ and the dimension N are given in [17]. A lower bound on T involving λ_1 and N is given in [18].

In the case when $\rho = 0$, i.e., for isotropy irreducible tori, we know very little (except for square tori). For instance, we do not know if Rothaus' inequality $\alpha \geq \frac{2N}{(N+1)^2}\lambda_1$ (see [15]) can be improved to $\alpha \geq c\lambda_1$ with $c > 0$ independent of the dimension.

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