

# Pseudo-Poincaré Inequalities and Applications to Sobolev Inequalities

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**Abstract** Most smoothing procedures are via averaging. Pseudo-Poincaré inequalities give a basic  $L^p$ -norm control of such smoothing procedures in terms of the gradient of the function involved. When available, pseudo-Poincaré inequalities are an efficient way to prove Sobolev type inequalities. We review this technique and its applications in various geometric setups.

## 1 Introduction

This paper is concerned with the question of proving the Sobolev inequality

$$\forall f \in \mathcal{C}_c^\infty(M), \quad \|f\|_q \leq S(M, p, q) \|\nabla f\|_p \quad (1.1)$$

when  $M = (M, g)$  is a Riemannian manifold, perhaps with boundary  $\partial M$ , and  $\mathcal{C}_c^\infty(M)$  is the space of smooth compactly supported functions on  $M$  (if  $M$  is a manifold with boundary  $\partial M$ , then points on  $\partial M$  are interior points in  $M$  and functions in  $\mathcal{C}_c(M)$  do not have to vanish at such points). We say that  $(M, g)$  is *complete* when  $M$  equipped with the Riemannian distance is a complete metric space.

In (1.1),  $p, q \in [1, \infty)$  and  $q > p$ . The norms  $\|\cdot\|_p$  and  $\|\cdot\|_q$  are computed with respect to some fixed reference measure, perhaps the Riemannian measure  $dv$  on  $M$  or, more generally, a measure  $d\mu$  on  $M$  of the form  $d\mu = \sigma dv$ , where  $\sigma$  is a smooth positive function on  $M$ . We set  $V(x, r) = \mu(B(x, r))$ , where  $B(x, r)$  is the geodesic ball of center  $x \in M$  and radius  $r \geq 0$ . The gradient  $\nabla f$  of  $f \in \mathcal{C}^\infty(M)$  at  $x$  is the tangent vector at  $x$  defined by

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$$g_x(\nabla f(x), u) = df|_x(u)$$

for any tangent vector  $u \in T_x$ . Its length  $|\nabla f|$  is given by  $|\nabla f|^2 = g(\nabla f, \nabla f)$ .

We will not be concerned here with the (interesting) problem of finding the best constant  $S(M, p, q)$  but only with the validity of the Sobolev inequality (1.1), for some constant  $S(M, p, q)$ .

In  $\mathbb{R}^n$ , equipped with the Lebesgue measure  $dx$ , (1.1) holds for any  $p \in [1, n)$  with  $q = np/(n - p)$ . The two simplest contexts where the question of the validity of (1.1) is meaningful is when  $M = \Omega$  is a subset of  $\mathbb{R}^n$ , or when  $\mathbb{R}^n$  is equipped with a measure  $\mu(dx) = \sigma(x)dx$ . In the former case, it is natural to relax our basic assumption and allow domains with nonsmooth boundary. It then becomes important to pay more attention to the exact domain of validity of (1.1) as approximation by functions that are smooth up to the boundary may not be available (cf., for example, [13, 14]).

The fundamental importance of the inequality (1.1) in analysis and geometry is well established. It is beautifully illustrated in the work of V. Maz'ya. One of the fundamental references on Sobolev inequalities is Maz'ya's treatise "Sobolev Spaces" [13] which discuss (1.1) and its many variants in  $\mathbb{R}^n$  and in domains in  $\mathbb{R}^n$  (cf. also [1, 3, 14] and the references therein). Maz'ya's treatise anticipates on many later works including [2]. More specialized works that discuss (1.1) in the context of Riemannian manifolds and Lie groups include [11, 19, 23] among many other possible references.

The aim of this article is to discuss a particular approach to (1.1) that is based on the notion of pseudo-Poincaré inequality. This technique is elementary in nature and quite versatile. It seems it has its origin in [4, 7, 17, 18] and was really emphasized first in [7, 18], and in [2]. To put things in some perspective, recall that the most obvious approach to (1.1) is via some "representation formula" that allows us to "recover"  $f$  from its gradient through an integral transform. One is then led to study the mapping properties of the integral transform in question.

However, this natural approach is not well suited to many interesting geometric setups because the needed properties of the relevant integral transforms might be difficult to establish or might even not hold true. For instance, it seems hard to use this approach to prove the following three (well-known) fundamental results.

**Theorem 1.1.** *Assume that  $(M, g)$  is a Riemannian manifold of dimension  $n$  equipped with its Riemannian measure and which is of one of the following three types:*

1. *A connected simply connected noncompact unimodular Lie group equipped with a left-invariant Riemannian structure.*
2. *A complete simply connected Riemannian manifold without boundary with nonpositive sectional curvature (i.e., a Cartan-Hadamard manifold).*
3. *A complete Riemannian manifold without boundary with nonnegative Ricci curvature and maximal volume growth.*

Then for any  $p \in [1, n)$  the Sobolev inequality (1.1) holds on  $M$  with  $q = np/(n - p)$  for some constant  $S(M, p, q) < \infty$ .

One remarkable thing about this theorem is the conflicting nature of the curvature assumptions made in the different cases. Connected Lie groups almost always have curvature that varies in sign, whereas the second and third cases we make opposite curvature assumptions. Not surprisingly, the original proofs of these different results have rather distinct flavors.

The result concerning unimodular Lie groups is due to Varopoulos and more is true in this case (cf. [22, 23]).

The result concerning Cartan–Hadamard manifolds is a consequence of a more general result due to Michael and Simon [15] and Hoffmann and Spruck [12]. A more direct prove was given by Croke [9] (cf. also [11, Section 8.1 and 8.2] for a discussion and further references).

The result concerning manifolds with nonnegative Ricci curvature and maximal volume growth (i.e.,  $V(x, r) \geq cr^n$  for some  $c > 0$  and all  $x \in M, r > 0$ ) was first obtained as a consequence of the Li–Yau heat kernel estimate using the line of reasoning in [22].

One of the aims of this paper is to describe proofs of these three results that are based on a common unifying idea, namely, the use of what we call pseudo-Poincaré inequalities. Our focus will be on how to prove the desired pseudo-Poincaré inequalities in the different contexts covered by this theorem. For relevant background on geodesic coordinates and Riemannian geometry see [5, 6, 10].

## 2 Sobolev Inequality and Volume Growth

There are many necessary conditions for (1.1) to hold and some are discussed in Maz'ya's treaty [13] in the context of Euclidean domains. For instance, if (1.1) holds for some fixed  $p = p_0 \in [1, \infty)$  and  $q = q_0 > p_0$  and we define  $m$  by  $1/q_0 = 1/p_0 - 1/m$ , then (1.1) also holds for all  $p \in [p_0, m)$  with  $q$  given by  $1/q = 1/p - 1/m$  (this easily follows by applying the  $p_0, q_0$  inequality to  $|f|^\alpha$  with a properly chosen  $\alpha > 1$  and using the Hölder inequality). More importantly to us here is the following result (cf., for example, [2] or [19, Corollary 3.2.8]).

**Theorem 2.1.** *Let  $(M, g)$  be a complete Riemannian manifold equipped with a measure  $d\mu = \sigma dv$ ,  $0 < \sigma \in C^\infty(M)$ . Assume that (1.1) holds for some  $1 \leq p < q < \infty$  and set  $1/q = 1/p - 1/m$ . Then for any  $r \in (m, \infty)$  and any bounded open set  $U \subset M$*

$$\forall f \in \mathcal{C}_c^\infty(U), \quad \|f\|_\infty \leq C_r \mu(U)^{1/m-1/r} \|\nabla f\|_r. \quad (2.1)$$

**Corollary 2.1.** *If the complete Riemannian manifold  $(M, g)$  equipped with a measure  $d\mu = \sigma dv$  satisfies (1.1) for some  $1 \leq p < q < \infty$ , then*

$$\inf\{s^{-m}V(x, s) : x \in M, s > 0\} > 0$$

with  $1/q = 1/p - 1/m$ .

*Proof.* Fix  $r > m$  and apply (2.1) to the function

$$\phi_{x,s}(y) = y \mapsto (s - \rho(x, y))_+ = \max\{(s - \rho(x, y), 0)\},$$

where  $\rho$  is the Riemannian distance on  $(M, g)$ . Because  $(M, \rho)$  is complete, this function is compactly supported and can be approximated by smooth compactly supported functions in the norm  $\|f\|_\infty + \|\nabla f\|_r$ , justifying the use of (2.1). Moreover,  $|\nabla \phi_{x,s}| \leq 1$  a.e. so that  $\|\nabla \phi_{x,s}\|_r \leq V(x, r)^{1/r}$ . This yields  $s \leq C_r V(x, s)^{1/m-1/r} V(x, s)^{1/r} = C_r V(x, s)^{1/m}$  as desired.  $\square$

*Remark 2.1.* Let  $\Omega$  be an unbounded Euclidean domain.

(a) If we assume that (1.1) holds but only for all traces  $f|_\Omega$  of functions  $f \in C_c^\infty(\mathbb{R}^n)$ , then we can conclude that (2.1) holds for such functions. Applying (2.1) to  $\psi_{x,s}(y) = (s - \|x - y\|)_+$ ,  $x \in \Omega$ ,  $s > 0$ , yields

$$|\{z \in \Omega : \|x - z\| < s\}| \geq cs^m.$$

(b) If, instead, we consider the intrinsic geodesic distance  $\rho = \rho_\Omega$  in  $\Omega$  and assume that (1.1) holds for all  $\rho$ -Lipschitz functions vanishing outside some  $\rho$ -ball, then the same argument, properly adapted, yields  $V(x, s) \geq cs^m$ , where  $V(x, s)$  is the Lebesgue measure of the  $\rho$ -ball of radius  $s$  around  $x$  in  $\Omega$ .

For domains with rough boundary, the hypotheses made respectively in (a) and (b) may be very different.

### 3 The Pseudo-Poincaré Approach to Sobolev Inequalities

Our aim is to illustrate the following result which provide one of the most elementary and versatile ways to prove a Sobolev inequality in a variety of contexts (cf., for example, [2, Theorem 9.1]). The two main hypotheses in the following statement concern a family of linear operators  $A_r$  acting, say, on smooth compactly supported functions. The first hypothesis captures the idea that  $A_r$  is smoothing. The sup-norm of  $A_r f$  is controlled in terms of the  $L^p$ -norm of  $f$  only and tends to 0 as  $r$  tends to infinity. The second hypothesis implies, in particular, that  $A_r f$  is close to  $f$  if  $|\nabla f|$  is in  $L^p$  and  $r$  is small.

**Theorem 3.1.** Fix  $m, p \geq 1$ . Assume that for each  $r > 0$  there is a linear map  $A_r : \mathcal{C}_c^\infty(M) \rightarrow L^\infty(M)$  such that

- $\forall f \in \mathcal{C}_c^\infty(M), r > 0, \|A_r f\|_\infty \leq C_1 r^{-m/p} \|f\|_p.$
- $\forall f \in \mathcal{C}_c^\infty(M), r > 0, \|f - A_r f\|_p \leq C_2 r \|\nabla f\|_p.$

Then, if  $p \in [1, m)$  and  $q = mp/(m - p)$ , there exists a finite constant  $S(M, p, q) = C(p, q)C_2C_1^{1/m}$  such that the Sobolev inequality (1.1) holds on  $M$ .

*Outline of the proof.* The proof is entirely elementary and is given in [2]. For illustrative purpose and completeness, we explain the first step. Consider the distribution function of  $|f|$ ,  $F(s) = \mu(\{x : |f(x)| > s\})$ . Then

$$F(s) \leq \mu(\{|f - A_r f| > s/2\}) + \mu(\{|A_r f| > s/2\}).$$

By hypothesis, if  $s = 2C_1 r^{-m/p} \|f\|_p$ , then  $\mu(\{|A_r f| > s/2\}) = 0$  and

$$F(s) \leq \mu(\{|f - A_r f| > s/2\}) \leq 2^p C_2^p r^p s^{-p} \|\nabla f\|_p^p.$$

This gives

$$s^{p(1+1/m)} F(s) \leq 2^{p(1+1/m)} C_1^{p/m} C_2^p \|\nabla f\|_p^p \|f\|_p^{p/m}.$$

This is a weak form of the desired Sobolev inequality (1.1). But, as is already apparent in [13], such a weak form of (1.1) actually imply (1.1) (cf. also [2, 19]).  $\square$

*Remark 3.1.* When  $p = 1$  and  $\mu = v$  is the Riemannian volume, we get

$$s^{1+1/m} v(\{|f| > s\}) \leq 2^{1+1/m} C_1^{1/m} C_2 \|\nabla f\|_1 \|f\|_1^{1/m}.$$

For any bounded open set  $\Omega$  with smooth boundary  $\partial\Omega$  we can find a sequence of functions  $f_n \in \mathcal{C}_c^\infty(M)$  such that  $f_n \rightarrow \mathbf{1}_{\Omega_n}$  and  $\|\nabla f_n\|_1 \rightarrow v_{n-1}(\partial\Omega)$ . This yields the isoperimetric inequality

$$v(\Omega)^{1-1/m} \leq 2^{1+1/m} C_1^{1/m} C_2 v_{n-1}(\partial\Omega).$$

Of course, as was observed long ago by Maz'ya and others, the classical co-area formula and the above inequality imply

$$\forall f \in \mathcal{C}_c^\infty(M), \|f\|_{m/(m-1)} \leq 2^{1+1/m} C_1^{1/m} C_2 \|\nabla f\|_1.$$

There are many situations where one does not expect (1.1) to hold, but where one of the local versions

$$\forall f \in \mathcal{C}_c^\infty(M), \|f\|_q \leq S(M, p, q)(\|\nabla f\|_p + \|f\|_p), \quad (3.1)$$

or ( $\Subset$  indicates an open relatively compact inclusion)

$$\forall \Omega \Subset M, \forall f \in \mathcal{C}_c^\infty(\Omega), \|f\|_q \leq S(\Omega, p, q)(\|\nabla f\|_p + \|f\|_p) \quad (3.2)$$

may hold. This is handled by the following local version of Theorem 3.1 (cf. [2] and [19, Section 3.3.2]).

**Theorem 3.2.** *Fix an open subset  $\Omega \subset M$ . Assume that for each  $r \in (0, R)$  there is a linear map  $A_r : \mathcal{C}_c^\infty(\Omega) \rightarrow L^\infty(M)$  such that*

- $\forall f \in \mathcal{C}_c^\infty(\Omega), r \in (0, R), \|A_r f\|_\infty \leq C_1 r^{-m/p} \|f\|_p.$
- $\forall f \in \mathcal{C}_c^\infty(\Omega), r \in (0, R), \|f - A_r f\|_p \leq C_2 r \|\nabla f\|_p.$

*Then, if  $p \in [1, m)$  and  $q = mp/(m - p)$ , there exists a finite constant  $S = S(p, q)$  such that*

$$\forall f \in \mathcal{C}_c^\infty(\Omega), \|f\|_q \leq SC_1^{1/m}(C_2 \|\nabla f\|_p + R^{-1} \|f\|_p). \quad (3.3)$$

Another useful version is as follows. For any open set  $\Omega$  we let  $W^{1,p}(\Omega)$  be the space of those functions in  $L^p(\Omega)$  whose first order partial derivatives in the sense of distributions (in any local chart) can be represented by a locally integrable function and such that

$$\int_\Omega |\nabla f|^p dv < \infty.$$

We write  $\|f\|_{\Omega,p}$  for the  $L^p$ -norm of  $f$  over  $\Omega$ . Note that  $\mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega)$  is dense in  $W^{1,p}(\Omega)$  for  $1 \leq p < \infty$  (cf., for example, [1, 3, 13]).

**Theorem 3.3.** *Fix an open subset  $\Omega \subset M$ . Assume that for each  $r \in (0, R)$  there is a linear map  $A_r : \mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega) \rightarrow L^\infty(M)$  such that*

- $\forall f \in \mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega), r \in (0, R), \|A_r f\|_\infty \leq C_1 r^{-m/p} \|f\|_p.$
- $\forall f \in \mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega), r \in (0, R), \|f - A_r f\|_p \leq C_2 r \|\nabla f\|_p.$

*Then, if  $p \in [1, m)$  and  $q = mp/(m - p)$ , there exists a finite constant  $S = S(p, q)$  such that*

$$\forall f \in W^{1,p}(\Omega), \|f\|_q \leq SC_1^{1/m}(C_2 \|\nabla f\|_p + R^{-1} \|f\|_p). \quad (3.4)$$

## 4 Pseudo-Poincaré Inequalities

The term Poincaré inequality (say, with respect to a bounded domain  $\Omega \subset M$ ) is used with at least two distinct meanings:

- The Neumann type  $L^p$ -Poincaré inequality for a bounded domain  $\Omega \subset M$  is the inequality

$$\forall f \in W^{1,p}(\Omega), \quad \inf_{\xi \in \mathbb{R}} \int_{\Omega} |f - \xi|^p dv \leq P_N(\Omega) \int_{\Omega} |\nabla f|^p dv.$$

- The Dirichlet type  $L^p$ -Poincaré inequality for a bounded domain  $\Omega \subset M$  is the inequality

$$\forall f \in C_c^\infty(\Omega), \quad \int_{\Omega} |f|^p dv \leq P_D(\Omega) \int_{\Omega} |\nabla f|^p dv.$$

When  $p = 2$  and the boundary is smooth, the first (respectively, the second) inequality is equivalent to the statement that the lowest nonzero eigenvalue  $\lambda_N(\Omega)$  (respectively,  $\lambda_D(\Omega)$ ) of the Laplacian with the Neumann boundary condition (respectively, the Dirichlet boundary condition) is bounded below by  $1/P_N(\Omega)$  (respectively,  $1/P_D(\Omega)$ ). Note that if  $M = \mathbb{S}^n$  is the unit sphere in  $\mathbb{R}^{n+1}$  and  $\Omega = B(o, r)$ ,  $r < 2\pi$ , is a geodesic ball, then  $P_N(\Omega) \rightarrow 1/(n+1)$  and  $P_D(\Omega) \rightarrow \infty$  as  $r$  tends to  $2\pi$ .

Here, we will use the term Poincaré inequality for the collection of the Neumann type Poincaré inequalities on metric balls. More precisely, we say that the  $L^p$ -Poincaré inequality holds on the manifold  $M$  if there exists  $P \in (0, \infty)$  such that

$$\forall B = B(x, r), \quad \forall f \in W^{1,p}(B), \quad \inf_{\xi \in \mathbb{R}} \int_B |f - \xi|^p dv \leq Pr^p \int_B |\nabla f|^p dv. \quad (4.1)$$

The notion of pseudo-Poincaré inequality was introduced in [7, 18] to describe the inequality

$$\forall f \in C_c^\infty(M), \quad \|f - f_r\|_p \leq Cr \|\nabla f\|_p, \quad (4.2)$$

where

$$f_r(x) = V(x, r)^{-1} \int_{B(x, r)} f dv.$$

Although this looks like a version of the previous Poincaré inequality, it is quite different in several respects. The most important difference is the global nature of each of the members of the pseudo-Poincaré inequality family: in (4.2) all integrals are over the whole space.

We say the *doubling volume condition* holds on  $M$  if there exists  $D \in (0, \infty)$  such that

$$\forall x \in M, \quad r > 0, \quad V(x, 2r) \leq DV(x, r). \quad (4.3)$$

The only known strong relation between (4.1) and (4.2) is the following result from [8, 18].

**Theorem 4.1.** *If a complete manifold  $M$  equipped with a measure  $d\mu = \sigma dv$  satisfies the conjunction of (4.3) and (4.1), then the pseudo-Poincaré inequality (4.2) holds on  $M$ .*

The most compelling reason for introducing the notion of pseudo-Poincaré inequality is that unimodular Lie groups always satisfy (4.2) with  $C = 1$  (cf. [22] and the development in [7]). The proof is extremely simple and the result slightly stronger.

**Theorem 4.2.** *Let  $G$  be a connected unimodular Lie group equipped with a left-invariant Riemannian distance and Haar measure. For any group element  $y$  at distance  $r(y)$  from the identity element  $e$*

$$\forall f \in \mathcal{C}_c(G), \quad \|f - f_y\|_p \leq r(y) \|\nabla f\|_p,$$

where  $f_y(x) = f(xy)$ .

*Proof.* Indeed, let  $\gamma_y : [0, r(y)] \rightarrow G$  be a (unit speed) geodesic joining  $e$  to  $y$ . Thus,

$$|f(x) - f(xy)|^p \leq r(y)^{p-1} \int_0^{r(y)} |\nabla f(x\gamma_y(s))|^p ds.$$

Integrating over  $x \in G$  yields the desired result.  $\square$

With this simple observation and Theorem 3.1, we immediately find that any simply connected noncompact unimodular Lie group  $M$  of dimension  $n$  satisfies the Sobolev inequality

$$\|f\|_{np/(n-p)} \leq S(M, p) \|\nabla f\|_p.$$

This is because the volume growth function  $V(x, r) = V(r)$  is always faster than  $cr^n$  (cf. [23] and the references therein). In fact, for  $r \in (0, 1)$ , we obviously have  $V(r) \simeq r^n$  and, for  $r > 1$ , either  $V(r) \simeq r^N$  for some integer  $N \geq n$  or  $V(r)$  grows exponentially fast. This line of reasoning yields the following improved result (due to Varopoulos [22], with a different proof).

**Theorem 4.3.** *Let  $G$  be a connected unimodular Lie group equipped with a left-invariant Riemannian structure and Haar measure. If the volume  $V(r)$  of the balls of radius  $r$  in  $G$  satisfies  $V(r) \geq cr^m$  for some  $m > 0$  and all  $r > 0$ , then (1.1) holds on  $G$  for all  $p \in [1, m]$  and  $q = mp/(m - p)$ .*

In this article, we think of a pseudo-Poincaré inequality as an inequality of the more general form

$$\forall f \in \mathcal{C}_c^\infty(M), \quad \|f - A_r f\|_p \leq Cr \|\nabla f\|_p, \quad (4.4)$$

where  $A_r : \mathcal{C}_c^\infty(M) \rightarrow L^\infty(M)$  is a linear operator. It is indeed very useful to replace the ball averages

$$f_r = V(x, r)^{-1} \int_{B(x, r)} f d\mu$$

by other types of averaging procedures. One interesting case is the following instance.



**Theorem 4.4.** *Let  $(M, g)$  be a Riemannian manifold, and let  $\Delta$  be the Friedrichs extension of the Laplacian defined on smooth compactly supported functions on  $M$ . Let  $H_t = e^{t\Delta}$  be the associated semigroup of selfadjoint operator on  $L^2(M, dv)$  (the minimal heat semigroup on  $M$ ). Then*

$$\forall f \in C_c^\infty(M), \quad \|f - H_t f\|_2 \leq \sqrt{t} \|\nabla f\|_2. \quad (4.5)$$

Consequently, if there are constants  $C \in (0, \infty)$ ,  $T \in (0, \infty]$  and  $m > 2$  such that

$$\forall t \in (0, T), \quad \|H_t f\|_\infty \leq C t^{-m/4} \|f\|_2, \quad (4.6)$$

then there exists a constant  $S = S(C, m) \in (0, \infty)$  such that the Sobolev inequality

$$\forall f \in C_c^\infty(M), \quad \|f\|_{2m/(m-2)} \leq S(\|\nabla f\|_2 + T^{-1} \|f\|_2) \quad (4.7)$$

holds on  $M$ .

*Proof.* In order to apply Theorem 3.2 with  $A_r = H_{r^2}$ , it suffices to prove (4.5). But

$$H_t f - f = \int_0^t \partial_s H_s f ds$$

and

$$\langle \partial_s H_s f, H_\tau f \rangle = \langle \Delta H_s f, H_\tau f \rangle = -\|H_{(s+\tau)/2}(-\Delta)^{1/2} f\|_2^2 \geq -\|\nabla f\|_2^2.$$

Hence  $\|H_t f - f\|_2^2 \leq t \|\nabla f\|_2^2$  as desired.  $\square$

*Remark 4.1.* One can show that (4.7) and (4.6) are, in fact, equivalent properties. This very important result was first proved by Varopoulos [21]. This equivalence holds in a much greater generality (cf. also [23]). When  $m \in (0, 2)$ , one can replace (4.7) by the Nash inequality

$$\forall f \in C_c^\infty(M), \quad \|f\|_2^{2(1+2/m)} \leq N(\|\nabla f\|_2 + T^{-1} \|f\|_2) \|f\|_1^{4/m}$$

which is equivalent to (4.6) (for any fixed  $m > 2$ ). See, for example, [2, 4, 19, 23] and the references therein.

## 5 Pseudo-Poincaré Inequalities and the Liouville Measure

Given a complete Riemannian manifold  $M = (M, g)$  of dimension  $n$  (without boundary), we let  $T_x M$  be the tangent space at  $x$ ,  $\mathbb{S}_x \subset T_x M$  the unit sphere, and  $SM$  the unit tangent bundle equipped with the Liouville measure defined by

$$\int_{SM} f d\mu = \int_M \int_{\mathbb{S}_x} f(x, u) d_{\mathbb{S}_x} u dv(x)$$

where we write  $\xi = (x, u) \in SM$  and  $d_{\mathbb{S}_x} u$  is the normalized measure on the unit sphere. We denote by  $\Phi_t$  the geodesic flow on  $M$  (with phase space  $SM$ ). For any  $t$ ,  $\Phi_t : SM \rightarrow SM$  is a diffeomorphism and the Liouville measure is invariant under  $\Phi_t$ . By definition, for any  $x \in M$ ,  $u \in \mathbb{S}_x \subset T_x M$ , we have  $\Phi_t(x, u) = (\gamma_{x,u}(t), \dot{\gamma}_{x,u}(t))$ , where  $\gamma_{x,u} : [0, \infty) \rightarrow M$  is the (unit speed) geodesic starting at  $x$  with tangent unit vector  $u$  and  $\dot{\gamma}_{x,u}(t)$  is the unit tangent vector to  $\gamma_{x,u}$  at  $\gamma_{x,u}(t)$  in the forward  $t$  direction.

If  $f : SM \rightarrow \mathbb{R}$  is a function on  $SM$  that depends on  $\xi = (x, u) \in SM$  only through  $x \in M$ , we have (for any fixed  $t > 0$ , and with a slight abuse of notation, namely  $f(\xi) = f(x)$ )

$$\int_M f dv = \int_{SM} f d\mu = \int_{SM} f \circ \Phi_t d\mu = \int_{SM} f(\gamma_{x,u}(t)) d_{\mathbb{S}_x} u dv(x). \quad (5.1)$$

For any  $(x, u) \in SM$ , let  $r(x, u)$  be the distance from  $x$  to the cutlocus in the direction of  $u$ . Namely,

$$r(x, u) = \inf\{t > 0 : d(x, \Phi_t(x, u)) < t\}.$$

The function  $r$  defined on  $SM$  is always upper semicontinuous and continuous when  $M$  is complete without boundary. Now, let  $\psi : (x, u, s) \mapsto \psi(x, u, s) = \psi_{x,u}(s) \in L_{\text{loc}}^{1,+}(SM \times [0, \infty))$  with  $\psi_{x,u}(t) = 0$  if  $t \geq r(x, u)$ . Call such a function *admissible*. For  $f \in C_c^\infty(M)$  we set

$$A_r f(x) = w(x, r)^{-1} \int_0^r \int_{\mathbb{S}_x} f(\gamma_{x,u}(t)) \psi_{x,u}(t) dt d_{\mathbb{S}_x} u,$$

where

$$w(x, r) = \int_0^r \int_{\mathbb{S}_x} \psi_{x,u}(t) dt d_{\mathbb{S}_x} u.$$

In words,  $A_r f$  is a weighted geodesic average of  $f$  over scales at most  $r$ . Note that, according to our definition, these averages never look past the cutlocus.

*Example 5.1.* Let  $\psi_{x,u}(t) = J(x, u, t)$  be the density of the volume element  $dv$  in geodesic polar coordinate around  $x$  so that

$$dv(y) = J(x, u, t) dt d_{\mathbb{S}_x} u, \quad y = \gamma_{x,u}(t) = \Phi_t(x, u), \quad t < r(x, u).$$

By definition, we set  $J(x, u, t) = 0$  for  $t \geq r(x, u)$ . Then  $w(x, r) = V(x, r)$  and  $A_r f(x) = f_r(x)$  is the mean of  $f$  in  $B(x, r)$ .

**Theorem 5.1.** *On any complete manifold without boundary and for any choice of admissible  $\psi \in L_{\text{loc}}^{1,+}(SM \times [0, \infty))$ , we have*

$$\int_M |f - A_r f|^p dv \leq D(r) \int_M |\nabla f|^p dv$$

with

$$D_p(r) = \sup_{(x,u) \in SM} \left\{ r \int_0^r \frac{\psi_{x,u}(t)t^{p-1}}{w(x,r)} dt \right\}.$$

*Proof.* Write

$$\begin{aligned} & \int_M |f - A_r f|^p dv \\ & \leq \int_M \frac{1}{w(x,r)} \int_0^r \int_{\mathbb{S}_x} |f(x) - f(\gamma_{x,u}(t))|^p \psi_{x,u}(t) dt d_{\mathbb{S}_x} u dv(x) \\ & \leq \int_M \frac{1}{w(x,r)} \int_0^r \int_{\mathbb{S}_x} \left( \int_0^t |\nabla f|(\gamma_{x,u}(s)) ds \right)^p \psi_{x,u}(t) dt d_{\mathbb{S}_x} u dv(x) \\ & \leq \int_M \frac{1}{w(x,r)} \int_0^r \int_{\mathbb{S}_x} \int_0^t |\nabla f|^p(\gamma_{x,u}(s)) ds \psi_{x,u}(t) t^{p-1} dt d_{\mathbb{S}_x} u dv(x) \\ & = \int_0^r \int_M \int_{\mathbb{S}_x} |\nabla f|^p(\gamma_{x,u}(s)) \left( \int_s^r \frac{\psi_{x,u}(t)t^{p-1}}{w(x,r)} dt \right) d_{\mathbb{S}_x} u dv(x) ds \\ & \leq \left( \sup_{(x,u) \in SM} \left\{ \int_0^r \frac{\psi_{x,u}(t)t^{p-1}}{w(x,r)} dt \right\} \right) \int_0^r \int_M \int_{\mathbb{S}_x} |\nabla f|^p(\gamma_{x,u}(s)) d_{\mathbb{S}_x} u dv(x) ds \\ & \leq D(r) \int_M |\nabla f|^p dv, \end{aligned}$$

where  $D_p(r)$  is as defined in the theorem. Note the crucial use of (5.1) at the last step.  $\square$

**Corollary 5.1.** *Let  $(M, g)$  be an isotropic Riemannian manifold. Then, for any  $p \in [1, \infty]$ , the pseudo-Poincaré inequality*

$$\forall f \in \mathcal{C}_c^\infty(M), \quad \|f - f_r\|_p \leq r \|\nabla f\|_p$$

*is satisfied.*

*Proof.* Use the previous theorem with  $\psi_{x,u}(t) = J(x, u, t)$ , in which case  $A_r f = f_r$ ,  $w(x, r) = V(x, r)$ . Observe that  $J(x, u, t)$  is independent of  $(x, u)$  because  $M$  is isotropic. It follows that

$$\int_0^r J(x, u, t) dt = \int_{\mathbb{S}_x} \int_0^r J(x, u, t) dt du = V(x, r).$$

Hence  $D(r) \leq r^p$ .  $\square$

Note that isotropic Riemannian manifolds are the same as two-point homogeneous Riemannian manifolds. They must be either  $\mathbb{R}^n$  or a rank one symmetric space.

**Corollary 5.2.** *Assume that  $M$  is a simply connected complete  $n$ -dimensional manifold without boundary and with nonpositive sectional curvature (i.e., a Cartan–Hadamard manifold). Set*

$$A_r f(x) = nr^{-n} \int_0^r \int_{\mathbb{S}_x} f(\gamma_{x,u}(t)) t^{n-1} dt d_{\mathbb{S}_x} u.$$

Then, for any  $p \in [1, \infty]$ , the inequalities

$$\forall f \in \mathcal{C}_c^\infty(M), \quad \|f - A_r f\|_p \leq r \|\nabla f\|_p, \quad \|A_r f\|_\infty \leq (\Omega_n r^n)^{-1/p} \|f\|_p.$$

are satisfied ( $\Omega_n$  is the volume of the  $n$ -dimensional Euclidean unit ball).

*Proof.* Apply the theorem with  $\psi_{x,u}(t) = t^{n-1} \mathbf{1}_{[0, r(x,u))}(t)$ . This gives  $D(r) \leq r^p$  on any manifold (i.e., we have not use nonpositive curvature yet). Now, since  $M$  has nonpositive sectional curvature and is simply connected, we have  $r(x, u) = \infty$ , and the classical comparison theorem gives  $\omega_{n-1} t^{n-1} \leq J(x, u, t)$  ( $\omega_{n-1}$  the volume of the unit sphere in  $\mathbb{R}^n$ ). Hence

$$\begin{aligned} |A_r f(x)| &\leq \frac{1}{(\omega_{n-1}/n)r^n} \int_0^r \int_{\mathbb{S}_x} |f(\gamma_{x,u}(t))| J(x, u, t) dt d_{\mathbb{S}_x} u \\ &\leq \frac{1}{(\omega_{n-1}/n)r^n} \int_M |f| dv. \end{aligned}$$

The proof is complete.  $\square$

This and Theorem 3.1 yield the following classical result (case (2) of Theorem 1.1).

**Corollary 5.3.** *Assume that  $M$  is a simply connected complete  $n$ -dimensional manifold without boundary and with nonpositive sectional curvature (i.e., a Cartan–Hadamard manifold). Then the Sobolev inequality (1.1) holds on  $M$  with  $q = np/(n-p)$  for any  $p \in [1, n)$ .*

This argument allow us to obtain a generalized version of this important result. Namely, for any  $x \in M$ , let

$$\mathcal{R}_x = \{u \in \mathbb{S}_x : r(x, u) = \infty\}.$$

In words,  $\mathcal{R}_x$  is the set of unit tangent vectors  $u \in T_x M$  associated with rays starting at  $x$  (a ray is a semiinfinite distance minimizing geodesic starting at  $x$ ). Let

$$|\mathcal{R}_x| = \int_{\mathcal{R}_x} d_{\mathbb{S}_x} u$$

be the normalized volume of  $\mathcal{R}_x$  as a subset of  $\mathbb{S}_x$ . Now, set

$$A_r^* f(x) = \frac{n}{|\mathcal{R}_x| r^n} \int_{\mathcal{R}_x} \int_0^r f(\gamma_{x,u}(t)) t^{n-1} dt d_{\mathbb{S}_x} u.$$

Obviously, Theorem 5.1 yields

$$\|f - A_r^* f\|_p \leq r^p \|\nabla f\|_p.$$

Further, if  $M$  has nonpositive sectional curvature along all rays,

$$|A_r^* f(x)| \leq \frac{n}{\omega_{n-1} |\mathcal{R}_x| r^n} \int_M |f| dv.$$

Hence we obtain the following statement.

**Theorem 5.2.** *Assume that  $M$  is a complete Riemannian  $n$ -manifold without boundary and with nonpositive curvature and such that  $\rho = \min_x \{|\mathcal{R}_x|\} > 0$ . Then  $M$  satisfies (1.1) with  $q = np/(n - p)$ .*

The simplest example of application of this result is to the surface of revolution known as the catenoid (it looks essentially like two planes connected through a compact cylinder) which is a celebrated example of minimal surface in  $\mathbb{R}^3$ . The theorem applies for  $p \in [1, 2)$  and yields, for instance, the Sobolev inequality  $\|f\|_2 \leq S \|\nabla f\|_1$ .

## 6 Homogeneous Spaces

In this section, we revisit the pseudo-Poincaré inequality on unimodular Lie groups to extend it to a class of homogeneous spaces. The argument we will use contains similarities as well as serious differences with the argument based on the invariance of the Liouville measure that was described in Section 5. We present it in the context of sub-Riemannian geometry. For an introduction to sub-Riemannian geometry (cf. [16]).

Let  $G$  be a unimodular Lie group, and let  $K$  be a compact subgroup. Let  $M = G/K$  be the associated homogeneous space equipped with its  $G$  invariant measure  $d\mu$ . Let  $\tau_g : M \rightarrow M$  be the action of  $G$  on  $M$ , and let  $\tau_g f(x) = f(\tau_g x)$ ,  $f \in \mathcal{C}_c(M)$ .

Assume that  $M$  is equipped with a (constant rank) sub-Riemannian structure, i.e., a vector subbundle  $H \subset TM$  equipped with a fiber inner product  $\langle \cdot, \cdot \rangle_H$  such that any local frame  $(X_1, \dots, X_k)$  for  $H$  is bracket generating (i.e., satisfies the Hörmander condition). For any function  $f \in \mathcal{C}_c^\infty(M)$ , let  $\nabla_H f(x)$  be the vector in  $H_x$  such that  $df|_x(u) = \langle \nabla_H f, u \rangle_{H_x}$  for any  $u \in H_x$ .

Assume further that  $(H, \langle \cdot, \cdot \rangle_H)$  is  $G$  invariant, i.e., for all  $g \in G$ ,  $x \in M$ ,  $X, Y \in H_x$  we have  $d\tau_g(X) \in H_{\tau_g x}$  and

$$\langle d\tau_g(X), d\tau_g(Y) \rangle_{H_{\tau_g x}} = \langle X, Y \rangle_{H_x}.$$

The space  $H$  is called the *horizontal space*. Under the Hörmander condition, any two points can be joined by absolutely continuous curves in  $M$  that stay tangent to  $H$  almost surely. For any such  $c : [0, T] \rightarrow M$  with  $\dot{c}(t) \in H_{c(t)}$  we set

$$\ell_H(c) = \int_0^T \langle \dot{c}, \dot{c} \rangle_{H_{c(t)}}^{1/2} dt.$$

This is the horizontal length of  $c$ . By definition, for any two points  $x, y \in M$ ,  $d_H(x, y)$  is the infimum of the horizontal length of horizontal curves joining  $x$  to  $y$ . It is not hard to check that  $d_H(x, y)$  is also equal to the infimum of all  $T$  such that there exists an absolutely continuous horizontal curve  $c : [0, T] \rightarrow M$  with  $\langle \dot{c}, \dot{c} \rangle_H \leq 1$  joining  $x$  to  $y$ . Since the action of  $G$  preserves horizontal length, it also preserves the distance  $d_H$ . We let  $B_H(x, r) = \{y \in M : d_H(x, y) < r\}$  and  $V_H(r) = \mu(B_H(x, r))$  which is, indeed, independent of  $x$ . Our aim is to prove the following result.

**Theorem 6.1.** *Let  $M = G/K$  with  $G$  unimodular and  $K$  compact be an homogeneous manifold. Assume that  $M$  is equipped with a sub-Riemannian structure  $(H, \langle \cdot, \cdot \rangle_H)$  satisfying the Hörmander condition and preserved by the action of  $G$ . Then, for any  $1 \leq p \leq \infty$ ,*

$$\forall f \in \mathcal{C}_c(M), \quad \|f - f_r\|_p \leq r \|\nabla_H f\|_p.$$

*Proof.* We can choose the Haar measure on  $G$  and the  $G$  invariant measure  $\mu$  on  $M$  so that for any  $F \in \mathcal{C}_c(G)$

$$\int_G F(g) dg = \int_M \int_K F(g_x k) dk d\mu(x),$$

where  $dk$  is the normalized Haar measure on  $K$ . Here,  $g_x$  stands for any element of  $G$  such that  $x = g_x K$ . Note that

$$x \rightarrow \int_K F(g_x k) dk \in \mathcal{C}_c(M)$$

is indeed independent of the choice  $g_x$ ,  $x \in M$ . We need to observe that there is such an integration formula for any choice of an origin in  $M$ . Namely, for any  $z \in M$  there is a compact subgroup  $K_z$  in  $G$  that fixes  $z$  and we can write

$$\int_G F(g) dg = \int_M \int_{K_z} F(g_x k) d_{K_z} k d\mu(x) \quad (6.1)$$

for some choice of Haar measure on  $K_z$ . Because  $\mu$  is invariant under the action of  $G$  and  $G$  is unimodular, the Haar measure on  $K_z$  must be taken to be the normalized Haar measure (cf., for example, [20]).

Now, for any  $y \in g_y K \in M$  we pick an horizontal path  $c : [0, T] \rightarrow M$  of horizontal length  $l$  with  $\langle \dot{c}, \dot{c} \rangle_H \leq 1$  and joining  $o = eK$  to  $y$ . For any  $g \in G$

and  $f \in \mathcal{C}_c(M)$

$$|f(\tau_g o) - f(\tau_g y)|^p \leq T^{p-1} \int_0^T |\nabla_H f(\tau_g c(t))|^p dt.$$

Hence

$$\int_G |f(\tau_g o) - f(\tau_g y)|^p dg \leq T^{p-1} \int_0^T \int_G |\nabla_H f(\tau_g c(t))|^p dg dt.$$

We now use (6.1) with  $z = c(t)$  and  $F(g) = |\nabla f(\tau_g c(t))|^p$  to compute

$$\begin{aligned} \int_G |\nabla_H f(g c(t))|^p dg &= \int_M \int_{K_{c(t)}} |\nabla_H f(\tau_{g_x k} c(t))|^p dK_{c(t)} k d\mu(x) \\ &= \int_M |\nabla_H f(x)|^p d\mu(x). \end{aligned}$$

Hence

$$\int_G |f(\tau_g o) - f(\tau_g y)|^p dg \leq T^p \|\nabla_H f\|_p^p.$$

Optimizing over the value of  $T$  yields

$$\int_G |f(\tau_g o) - f(\tau_g y)|^p dg \leq d(o, y) \|\nabla_H f\|_p^p.$$

Next, for any  $g \in G$

$$V_H(r)^{-1} \int_{B_H(o, r)} f(\tau_g y) d\mu(y) = V_H(r)^{-1} \int_{B_H(\tau_g o, r)} f(y) d\mu(y).$$

Hence, if we set

$$f_r(x) = V_H(r)^{-1} \int_{B_H(x, r)} f d\mu,$$

we have

$$\begin{aligned} \|f - f_r\|_p^p &= \int_M \left| f(x) - V_H(r)^{-1} \int_{B_H(x, r)} f d\mu \right|^p d\mu(x) \\ &= \int_G \left| f(\tau_g o) - V_H(r)^{-1} \int_{B_H(\tau_g o, r)} f d\mu \right|^p dg \\ &= \int_G \left| f(\tau_g o) - V_H(r)^{-1} \int_{B_H(o, r)} f(\tau_g y) d\mu(y) \right|^p dg \end{aligned}$$

$$\begin{aligned}
&\leq V_H(r)^{-1} \int_{B_H(o,r)} \int_G |f(\tau_g o) - f(\tau_g y)|^p dg d\mu(y) \\
&\leq r^p \|\nabla_H f\|_p^p.
\end{aligned}$$

This finishes the proof of Theorem 6.1.  $\square$

**Corollary 6.1.** *In the context of Theorem 6.1, if  $V_H(r) \geq cr^m$  for all  $r > 0$ , then the Sobolev inequality*

$$\forall f \in \mathcal{C}_c(M), \quad \|f\|_q \leq S(M, H, p, q) \|\nabla_H f\|_p$$

*holds on  $M$  for all  $p \in [1, m)$  with  $q = mp/(m - p)$  and a finite constant  $S(M, H, c, p, q)$ .*

This covers the case of unimodular Lie groups ( $M = G$ ,  $K = \{e\}$ ) equipped with a family of left-invariant vector fields  $\{X_1, \dots, X_k\}$  that generates the Lie algebra (in this case,  $|\nabla_H f|^2 = \sum_1^k |X_i f|^2$ ). It also covers the case of noncompact symmetric spaces  $M = G/K$ ,  $G = NAK$  semisimple, equipped with their canonical Riemannian structure. Note that when  $M$  is a noncompact symmetric space, the inequality  $\|f\|_p \leq C(M, p) \|\nabla f\|_p$  holds as well for different reasons.

## 7 Ricci Curvature Bounded Below

This section offers variations on results from [19]. Let  $(M, g)$  be a Riemannian manifold of dimension  $n$  (without boundary) with Ricci curvature tensor  $\text{Ric}$ . Fix  $K, R \geq 0$ , an open set  $\Omega \subset M$ . We assume throughout that  $\Omega_R = \{y \in M : d(x, y) \leq R\}$  is compact and that the Ricci tensor is bounded by  $\text{Ric} \geq -Kg$  over  $\Omega_R$ . It follows that for any  $x \in \Omega$  and  $r \in (0, R)$  almost every point  $y$  in  $B(x, r)$  can be joined to  $x$  by a unique minimizing geodesic  $\gamma_{x,y} : [0, d(x, y)] \rightarrow \Omega_R$ . Note that we use somewhat conflicting notation by letting  $\gamma_{x,u}$  denote the unit speed geodesic starting at  $x$  in the direction  $u \in \mathbb{S}_x$  and letting  $\gamma_{x,y}$  denote the minimizing unit speed geodesic from  $x$  to  $y$  (when it exists). Note also that for any  $x \in \Omega$  and  $u \in \mathcal{S}_x$  and  $r \in (0, R)$ , the unit speed geodesic  $\gamma_{x,u}$  (not necessarily minimizing) is defined at least on the interval  $[0, r]$  because  $\Omega_R$  is compact in  $M$ .

In this context, the Bishop–Gromov comparison theorem yields the following properties:

- For all  $x \in \Omega$  and  $0 < s < r < R$

$$V(x, r) \leq V(x, s) (r/s)^n e^{\sqrt{(n-1)K}r}.$$



- For any  $x \in \Omega$ ,  $u \in \mathbb{S}_x$ , and  $0 < s < r < R$  such that  $\gamma_{x,u}$  is minimizing on  $[0, r]$

$$J(x, u, r) \leq J(x, u, s)(r/s)^{n-1}e^{\sqrt{(n-1)K}r}.$$

We will use these properties to prove the following result.

**Theorem 7.1.** *Referring to the above setup concerning  $(M, g)$  and  $\Omega, K, R$ , we have*

$$\forall r \in (0, R), \quad \forall f \in \mathcal{C}_c^\infty(\Omega), \quad \|f - f_r\|_p^p \leq 8^n e^{3\sqrt{(n-1)K}r} r^p \|\nabla f\|_p^p.$$

*Proof.* For simplicity, we write  $dx$  for the Riemannian measure  $v(dx)$ . It suffices to show that, for any  $f \in \mathcal{C}_c^\infty(\Omega)$  and  $r \in (0, R)$

$$\int_{\Omega} \int_{\Omega} |f(x) - f(y)|^p \frac{\mathbf{1}_{B(x,r)}(y)}{V(x,r)} dx dy \leq 8^n e^{3\sqrt{(n-1)K}r} r^p \|\nabla f\|_p^p.$$

By the Bishop–Gromov volume inequality, for  $x, y \in \Omega$

$$\frac{\mathbf{1}_{B(x,r)}(y)}{V(x,r)} \leq 2^n e^{\sqrt{(n-1)K}r} \frac{\mathbf{1}_{B(x,r)}(y)}{\sqrt{V(x,r)V(y,r)}} \quad (7.1)$$

and it suffices to bound

$$I = \int_{\Omega} \int_{\Omega} |f(x) - f(y)|^p \frac{\mathbf{1}_{B(x,r)}(y)}{\sqrt{V(x,r)V(y,r)}} dx dy.$$

Let  $W$  be the (symmetric) subset of  $\Omega \times \Omega$  of all  $(x, y)$  with  $d(x, y) < r$  such that there exists a unique minimizing geodesic  $\gamma_{x,y} : [0, d(x, y)] \rightarrow M$  joining  $x$  to  $y$ . As was noted above, for any  $x \in \Omega$ , almost all  $y \in B(x, r)$  have this property and the image of  $\gamma_{x,y}$  is contained in  $\Omega_R$ . Hence

$$\begin{aligned} I &= \int_W |f(x) - f(y)|^p \frac{\mathbf{1}_{B(x,r)}(y)}{\sqrt{V(x,r)V(y,r)}} dx dy \\ &\leq \int_W \int_0^{d(x,y)} \frac{d(x,y)^{p-1} |\nabla f(\gamma_{x,y}(s))|^p \mathbf{1}_{B(x,r)}(y)}{\sqrt{V(x,r)V(y,r)}} ds dx dy. \end{aligned}$$

The following step is essential to the proof. By symmetry between  $x$  and  $y$  and since  $\gamma_{x,y}(s) = \gamma_{y,x}(d(x, y) - s)$ , we have

$$\begin{aligned} &\int_W \int_0^{d(x,y)/2} \frac{d(x,y)^{p-1} |\nabla f(\gamma_{x,y}(s))|^p \mathbf{1}_{B(x,r)}(y)}{\sqrt{V(x,r)V(y,r)}} ds dx dy = \\ &\int_W \int_{d(x,y)/2}^{d(x,y)} \frac{d(x,y)^{p-1} |\nabla f(\gamma_{x,y}(s))|^p \mathbf{1}_{B(x,r)}(y)}{\sqrt{V(x,r)V(y,r)}} ds dx dy. \end{aligned}$$

Hence

$$I \leq 2r^{p-1} \int_W \int_{d(x,y)/2}^{d(x,y)} \frac{|\nabla f(\gamma_{x,y}(s))|^p \mathbf{1}_{B(x,r)}(y)}{\sqrt{V(x,r)V(y,r)}} ds dx dy.$$

By the Bishop–Gromov comparison theorem, we have

$$\forall x, y \in M, 0 < s < r, \quad \frac{\mathbf{1}_{B(x,r)}(y)}{\sqrt{V(x,r)V(y,r)}} \leq 2^n e^{\sqrt{(n-1)K}r} \frac{\mathbf{1}_{B(x,r)}(\gamma_{x,y}(s))}{V(\gamma_{x,y}(s), r)}.$$

Moreover, again by the Bishop–Gromov comparison theorem, for all  $(x, y) \in W$  and  $d(x, y)/2 < s < d(x, y) \leq r$ , the Jacobian  $J(\gamma_{x,y}(s))$  of the map  $y \mapsto z = \phi(y) = \gamma_{x,y}(s)$  is bounded below by  $2^{-n+1}e^{-\sqrt{(n-1)K}r}$ . Note that we use here the fact that the image of the whole  $\gamma_{x,y}$  lies in  $\Omega_R$ , where the Ricci lower bound is satisfied.

For each  $s \in [0, r]$  we set

$$W_s = \{(x, y) \in W : s \leq d(x, y)\}.$$

Using the two observations above in the previous upper bound for  $I$  and setting  $C(r) = 4^n r^{p-1} e^{2\sqrt{(n-1)K}r}$ , we obtain

$$\begin{aligned} I &\leq C(r) \int_W \int_{d(x,y)/2}^{d(x,y)} \frac{|\nabla f(\gamma_{x,y}(s))|^p J(\gamma_{x,y}(s)) \mathbf{1}_{B(x,r)}(\gamma_{x,y}(s))}{V(\gamma_{x,y}(s), r)} ds dx dy \\ &\leq C(r) \int_0^r \int_{W_s} \frac{|\nabla f(\gamma_{x,y}(s))|^p J(\gamma_{x,y}(s)) \mathbf{1}_{B(\gamma_{x,y}(s), r)}(x)}{V(\gamma_{x,y}(s), r)} dx dy ds \\ &\leq C(r) \int_0^r \int_{M \times M} \frac{|\nabla f(z)|^p \mathbf{1}_{B(z,r)}(x)}{V(z, r)} dx dz ds \\ &= C(r)r \int_M |\nabla f(z)|^p dz. \end{aligned}$$

Taking (7.1) into account, we obtain the desired result.  $\square$

As corollaries of Theorems 3.1 and 7.1, we obtain the following three well-known results.

**Theorem 7.2.** *For any relatively compact set  $\Omega$  in a Riemannian manifold  $M$  (without boundary) of dimension  $n$ , the Sobolev inequality*

$$\forall f \in \mathcal{C}_c^\infty(\Omega), \quad \|f\|_q \leq S(\Omega, p, q) (\|\nabla f\|_p + \|f\|_p)$$

*holds for any  $p \in [1, n)$  and  $q = pn/(n-p)$ .*

For the proof of the next result, in addition to Theorems 3.1 and 7.1, one uses the Bishop–Gromov comparison theorem in the form of the volume lower bound

$$\forall x \in M, \forall s \in (0, r), \quad V(x, s) \geq c(x, r)s^n$$

with  $c(x, r) = e^{\sqrt{(n-1)K}r} V(x, r)$  which is valid as long as the closed ball  $\overline{B(x, 2r)}$  is compact and  $K \geq 0$  is such that the Ricci curvature tensor is bounded below by  $\text{Ric} \geq -Kg$  on  $B(x, 2r)$ .

**Theorem 7.3.** *On any Riemannian manifold  $M$  of dimension  $n$  (without boundary), the Sobolev inequality*

$$\forall f \in \mathcal{C}_c^\infty(B(x, r)), \quad \|f\|_q \leq \frac{C(p, n, Kr^2)r}{V(x, r)^{1/n}} (\|\nabla f\|_p + r^{-1}\|f\|_p)$$

holds for any  $p \in [1, n)$  and  $q = pn/(n-p)$  as long as the closed ball  $\overline{B(x, 2r)}$  is compact and  $K \geq 0$  is such that the Ricci curvature tensor is bounded below by  $\text{Ric} \geq -Kg$  on  $B(x, 2r)$ .

If one follows the constants in the proof of Theorem 7.3, one finds that

$$C(p, n, Kr^2) \leq C_1(n, p)e^{C_2(n, p)\sqrt{Kr^2}}$$

for some finite constants  $C_1(n, p)$  and  $C_2(n, p)$ .

The next result can be obtained from the previous theorem by letting  $r$  tend to infinity (which is possible when  $K = 0$  since  $Kr^2 = 0$  for all  $r > 0$ ).

**Theorem 7.4.** *For any complete Riemannian manifold  $M$  of dimension  $n$  with nonnegative Ricci curvature and maximum volume growth (i.e., there exists  $c > 0$  such that  $V(x, r) \geq cr^n$  for all  $x \in M$ ,  $r > 0$ ) the Sobolev inequality*

$$\forall f \in \mathcal{C}_c^\infty(M), \quad \|f\|_q \leq S(c, n, p)\|\nabla f\|_p$$

holds for any  $p \in [1, n)$  and  $q = pn/(n-p)$ .

## 8 Domains with the Interior Cone Property

In this final section, we illustrate yet a slightly different use of the pseudo-Poincaré inequality. Let  $(M, g)$  be a complete Riemannian manifold without boundary.

Fix  $\delta \in (0, 1]$  and  $r > 0$ . A  $(\delta, r)$ -cone at  $x$  is a set of the form

$$\mathfrak{C}(x, \omega_x, r) = \{y = \gamma_{x,u}(s) : u \in \omega_x, 0 \leq s < r\},$$

where  $\omega_x$  is an open subset of  $\mathbb{S}_x$  with the property that  $r(x, u) > r$  for all  $u \in \omega_x$  and  $|\omega_x| \geq \delta$ . Here,  $|\omega_x|$  denotes the measure of  $\omega_x$  with respect to the normalized measure on the sphere  $\mathbb{S}_x$ . We always assume further that for any continuous function  $f$  the function

$$x \mapsto \int_{\mathfrak{C}(x, \omega_x, r)} f(y) dv(y)$$

is measurable. In particular,  $x \mapsto v(\mathfrak{C}(x, \omega_x, r))$  is measurable.

Note that the existence of a  $(\delta, r)$  cone at  $x$  is a non trivial assumption. A domain  $\Omega$  which contains an  $(\delta, r)$  cone at  $x$  for any  $x \in \Omega$  is said to satisfy the  $(\delta, r)$  *interior cone condition*.

In the Euclidean space context, the interior cone condition is perhaps the most classical condition for the validity of various Sobolev embedding theorems (cf. [1, 13]). In the geometric context of complete Riemannian manifolds, we offer two results based on the interior cone conditions. The Sobolev inequalities stated in the following two theorems are of a different nature than those discussed earlier in this paper because the functions involved need not vanish at the boundary of  $\Omega$ . To obtain these inequalities, we use Theorem 3.3.

**Theorem 8.1.** *Let  $\Omega$  be a domain in an  $n$ -dimensional complete Riemannian manifold  $(M, g)$  (without boundary). Fix  $K, R \geq 0$  and assume that*

- *The Ricci curvature is bounded by  $\text{Ric} \geq -Kg$  on  $\Omega$ .*
- *There exists  $\delta \in (0, 1)$  such that for any  $x \in \Omega$ , there is a  $(\delta, R)$ -cone  $\mathfrak{C}(x, \omega_x, R)$  at  $x$  contained in  $\Omega$  with the additional property that  $v(\mathfrak{C}(x, \omega_x, r)) \geq cr^n$  for any  $r \in (0, R)$ .*

For any  $f \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$  we set

$$\mathfrak{A}_r f(x) = \frac{1}{v(\mathfrak{C}(x, \omega_x, r))} \int_{\mathfrak{C}(x, \omega_x, r)} f dv.$$

Then for all  $f \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$  the inequality

$$\forall r \in (0, R), \quad \|f - \mathfrak{A}_r f\|_{\Omega, p} \leq (\omega_{n-1}/cn) e^{2\sqrt{(n-1)K}r} r \|\nabla f\|_{\Omega, p}$$

holds. Further, for  $p \in [1, n)$  and  $q = np/(n-p)$  the Sobolev inequality

$$\forall f \in C^\infty(\Omega) \cap W^{1,p}(\Omega), \quad \|f\|_{\Omega, q} \leq S(c, p, n, Kr^2) (\|\nabla f\|_{\Omega, p} + R^{-1} \|f\|_{\Omega, p})$$

is satisfied.

*Proof.* Let  $f \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$ . For any  $x \in \Omega$  we write

$$v(\mathfrak{C}(x, r)) |f(x) - \mathfrak{A}_r f(x)| \leq \int_{\omega_x} \int_0^r \int_0^s |\nabla f(\gamma_{x,u}(\tau))| d\tau |J(x, u, s)| ds d\mathbb{S}_x u.$$

By the Bishop–Gromov comparison theorem, for all  $0 < \tau < s < r < r(x, u)$

$$J(x, u, \tau) (s/\tau)^{n-1} e^{\sqrt{(n-1)K}s} \geq J(x, u, s).$$

Hence

$$\begin{aligned}
& v(\mathfrak{C}(x, r)) |f(x) - \mathfrak{A}_r f(x)| \\
& \leq e^{\sqrt{(n-1)K} r} \int_{\omega_x} \int_0^r \int_0^s \tau^{1-n} |\nabla f(\gamma_{x,u}(\tau))| J(x, u, \tau) d\tau s^{n-1} ds d_{\mathbb{S}_x} u \\
& \leq \frac{e^{\sqrt{(n-1)K} r} r^n}{n} \int_{\Omega \cap B(x, r)} \frac{|\nabla f(z)|}{d(x, z)^{n-1}} dz.
\end{aligned}$$

Using the hypothesis  $v(\mathfrak{C}(x, r)) \geq cr^n$ , we obtain

$$\begin{aligned}
& |f(x) - \mathfrak{A}_r f(x)|^p \\
& \leq \frac{e^{p\sqrt{(n-1)K} r}}{(cn)^p} \left( \int_{\Omega \cap B(x, r)} \frac{dz}{d(x, z)^{n-1}} \right)^{p-1} \int_{\Omega \cap B(x, r)} \frac{|\nabla f(z)|^p}{d(x, z)^{n-1}} dz.
\end{aligned}$$

The Bishop comparison theorem yields

$$\int_{\Omega \cap B(z, r)} \frac{dx}{d(z, x)^{n-1}} \leq \omega_{n-1} e^{\sqrt{(n-1)K} r} r.$$

The inequality

$$\|f - \mathfrak{A}_r f\|_{\Omega, p} \leq (\omega_{n-1}/cn) e^{2\sqrt{(n-1)K} r} r \|\nabla f\|_{\Omega, p}$$

follows, and Theorem 3.3 gives the desired Sobolev inequality.  $\square$

**Theorem 8.2.** Fix  $R > 0$  and  $\delta \in (0, 1)$ . Let  $\Omega$  be a domain in an  $n$ -dimensional complete simply connected Riemannian manifold  $(M, g)$  without boundary and with nonpositive sectional curvature (i.e., a Cartan–Hadamard manifold). Assume that  $\Omega$  as the  $(\delta, R)$  interior cone property. Namely, for any  $x \in \Omega$  there is a  $(\delta, R)$ -cone  $\{y = \gamma_{x,u}(s) : u \in \omega_x, 0 \leq s < R, \}$  at  $x$  contained in  $\Omega$ . For any  $f \in \mathcal{C}^\infty(\Omega)$  we set

$$A_r f(x) = \frac{n}{|\omega_x| r^n} \int_{\omega_x} \int_0^r f(\gamma_{x,u}(s)) s^{n-1} ds d_{\mathbb{S}_x} u.$$

Then for all  $f \in \mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega)$  the inequality

$$\forall r \in (0, R), \quad \|f - A_r f\|_{\Omega, p} \leq \delta^{-1/p} r \|\nabla f\|_{\Omega, p}$$

holds. Further, for  $p \in [1, n)$  and  $q = np/(n-p)$ , the Sobolev inequality

$$\forall f \in \mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega), \quad \|f\|_{\Omega, q} \leq S(\delta, p, n) (\|\nabla f\|_{\Omega, p} + R^{-1} \|f\|_{\Omega, p})$$

is satisfied.

*Proof.* Let  $f \in \mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega)$ . For any  $x \in \Omega$  we write

$$|f(x) - A_r f(x)| \leq \frac{n}{\delta r^n} \int_{\omega_x} \int_0^r \int_0^s |\nabla f(\gamma_{x,u}(\tau))| d\tau s^{n-1} ds d_{\mathbb{S}_x} u.$$

Hence, using (5.1) for the last step,

$$\begin{aligned} \|f - A_r f\|_{\Omega,p}^p &\leq \frac{n}{\delta r^n} \int_0^r \int_{\Omega} \int_{\omega_x} \int_{\tau}^r |\nabla f(\gamma_{x,u}(\tau))|^p d_{\mathbb{S}_x} u s^{n+p-2} ds d\tau dx \\ &\leq \frac{n r^{p-1}}{(n+p-1)\delta} \int_0^r \int_{\Omega} \int_{\omega_x} |\nabla f(\gamma_{x,u}(\tau))|^p d_{\mathbb{S}_x} u dx d\tau \\ &= \frac{r^{p-1}}{\delta} \int_0^r \int_{\Omega} \int_{\omega_x} |\nabla f(\gamma_{x,u}(\tau))|^p \mathbf{1}_{\Omega}(\gamma_{x,u}(\tau)) d_{\mathbb{S}_x} u dx d\tau \\ &\leq \frac{r^{p-1}}{\delta} \int_0^r \int_M \int_{\mathbb{S}_x} |\nabla f(\gamma_{x,u}(\tau))|^p \mathbf{1}_{\Omega}(\gamma_{x,u}(\tau)) d_{\mathbb{S}_x} u dx d\tau \\ &= \frac{r^p}{\delta} \|\nabla f\|_{\Omega,p}^p. \end{aligned}$$

This yields the desired pseudo-Poincaré inequality. To obtain the stated Sobolev inequality, we simply observe that

$$|A_r f(x)| \leq \frac{n}{\delta r^n} \int_{\omega_x} \int_0^r |f(\gamma_{x,u}(s))| s^{n-1} ds d_{\mathbb{S}_x} u \leq \frac{n}{\delta r^n} \int_{\Omega} |f| dv$$

because, on any Cartan–Hadamard manifold,  $J(x, u, s) \geq s^{n-1}$  for all  $u$ . It then suffices to apply Theorem 3.3.  $\square$

*Example 8.1.* Let  $M$  be an  $n$ -dimensional Cartan–Hadamard manifold. Let  $C$  be a closed geodesically convex set, and let  $\Omega = M \setminus C$ . We claim that  $\Omega$  has the  $(1/2, \infty)$  interior cone property. Indeed, for any  $x \in \Omega$ , let  $\omega_x$  be the subset of those unit tangent vectors  $v$  at  $x$  such that  $C \cap \{y = \gamma_{x,v}(s) : s \geq 0\} = \emptyset$ . Because  $C$  is geodesically convex and  $x \in \Omega$ , for any  $u \in \mathbb{S}_x$ , either  $u$  or  $-u$  belongs to  $\omega_x$ . This implies that  $|\omega_x| \geq 1/2$ . Applying Theorem 8.2 with  $R = \infty$ , we obtain the Sobolev inequality

$$\forall f \in \mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega), \quad \|f\|_{\Omega,q} \leq S(p, n) \|\nabla f\|_{\Omega,p}, \quad q = np/(n-p).$$

Note that the functions  $f \in \mathcal{C}^\infty(\Omega) \cap W^{1,p}(\Omega)$  do not necessarily vanish along the boundary of  $\Omega$ . Balls and sublevel sets of Busemann functions provide examples of geodesically convex sets in Cartan–Hadamard manifolds.

*Example 8.2.* Consider a geodesic ball of radius  $\rho \geq 1$  in the hyperbolic plane. It is not hard to see that it has the  $(\delta, 1)$  interior cone property. One may ask if, uniformly in  $\rho \geq 1$ , these balls have the  $(\delta, a\rho)$  interior cone property for some  $\delta, a \in (0, 1)$ . The answer is no. If one wants to fit cones of length  $a\rho$  in a ball of radius  $\rho$ , then the aperture  $\alpha(a, \rho)$  has to tend to 0 with  $\rho$ .

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