GAFA, Geom. funct. anal.
Vol. 11 (2001) 1139 – 1191
1016-443X/01/0601139-53 $ 1.50+0.20/0

GAFA Geometric And Functional Analysis

HARNACK INEQUALITY AND HYPERBOLICITY FOR SUBELLIPTIC p-LAPLACIANS WITH APPLICATIONS TO PICARD TYPE THEOREMS

T. Coulhon, I. Holopainen and L. Saloff-Coste

Contents

1 Introduction

Let M be a complete non-compact Riemannian manifold. For $p \in (1, +\infty)$, let Δ_p be the p-Laplace operator on M. One says that M is p-hyperbolic if there exists a Green function for Δ_p (see [Ho1,2]); otherwise, M is said to be p-parabolic. It is well known that one can give sufficient conditions for p-parabolicity in terms of the volume growth of M and sufficient conditions for p-hyperbolicity in terms of its isoperimetric profile. These results are due to Ahlfors, Cheng–Yau, Varopoulos, Grigor'yan and Fernandez for $p = 2$, and to Zorich–Keselman and Troyanov for general p (precise references are given below). The two main new points we want to make in this connection are the following: we deduce the parabolicity criterion from the p-version of an inequality by Cheng–Yau on supersolutions of Δ_p, and

T.C.'s research partially supported by the European Commission (European TMR Network "Harmonic Analysis," 1998–2001, Contract ERBFMRX-CT97-0159). I.H.'s research partially supported by the Academy of Finland, projects 6355 and 44333. L.S.-C.'s research supported in part by NSF grant DMS 9802855.

in the hyperbolicity criterion we replace the 1-isoperimetric profile by a p-isoperimetric profile, which is more accurate.

We shall also give sufficient conditions for an elliptic p-Harnack inequality, therefore for the (strong) p-Liouville property, to hold on M, first in terms of doubling property and Poincaré inequalities, second, using the Cheng–Yau type inequality, under an assumption of quadratic volume growth. Similar results already appeared in the literature, see section 4 below.

Let us point out that these methods are not limited to the Riemannian setting. For the most part, we present them in a natural sub-Riemannian framework.

Criteria for p-parabolicity and p-Harnack inequality have applications to Picard type theorems, i.e. theorems saying that, if M and N are two n-dimensional Riemannian manifolds, and if in a certain sense M is small enough and N big enough, then there does not exist non-trivial quasiregular mappings between M and N, or between M and $N \setminus \{x_0\}$. Indeed, the n-parabolicity as well as the n-Liouville property, if understood in a strong enough sense, are preserved by non-constant qr maps. This recasts several results by Gromov, Varopoulos, Holopainen and Holopainen–Rickman (for precise references, see section 5 below) into a more general and more precise picture.

Some readers might notice that in some papers of the second author, the present paper is quoted (with a somewhat different title) as a paper of the two other authors; this has a simple explanation: this work started long ago in collaboration between Coulhon and Saloff-Coste, a preprint circulated, and more recently Holopainen contributed to the work and joined the team of an improved and enlarged paper.

2 Preliminaries

Let M be a connected $\mathcal{C}^\infty$ manifold. Let Δ be a second order differential operator with real $\mathcal{C}^\infty$ coefficients such that $\Delta 1 = 0$ (i.e. Δ has no zero order term). We assume that there exists a positive $\mathcal{C}^\infty$ measure v on M such that

$$\langle \Delta\varphi, \psi\rangle = \langle \varphi, \Delta\psi\rangle\,, \quad \langle \Delta\varphi, \varphi\rangle \geq 0\,,$$

for all $\varphi, \psi \in \mathcal{C}_0^\infty(M)$, where $\langle \varphi, \psi\rangle = \int_M \varphi\bar{\psi}\, dv$.

In what follows we consider the pair (M, Δ) as our main object of study. Implicitly, the measure v is also part of our data. One reason it is somewhat natural not to mention v is that, unless Δ is very degenerate, v is essentially determined by Δ, up to a multiplicative constant. Still the measure v plays a crucial rôle in our analysis. As explained below, Δ induces a certain "geometric structure" on M and by (M,Δ) we mean in fact the manifold M equipped with the measure v and the geometric structure induced by Δ. In different words, our basic structure is M equipped with the form $\langle \Delta\varphi, \psi\rangle$, $\varphi,\psi \in \mathcal{C}_0^\infty(M)$, on $L^2(M,dv)$.

2.1 The gradient. There is a notion of "length of the gradient" canonically associated with any second order differential operator Δ as above. Consider the expression

$$\Gamma(\varphi,\psi) = \Gamma_\Delta(\varphi,\psi) = -\tfrac{1}{2}\big[\Delta(\varphi\psi) - \varphi\Delta\psi - \psi\Delta\varphi\big]\,.$$

A computation in local coordinates shows that, for every $x \in M$, $\Gamma(\varphi,\psi)(x)$ is a non-negative quadratic form. Define the "length of the gradient" of a smooth function φ by setting

$$|\nabla\varphi| = \Gamma(\varphi,\varphi)^{1/2}.$$

($\Gamma(\varphi,\varphi)$ is also called the "carré du champ"). Note that one has the Cauchy-Schwarz inequality

$$\big|\Gamma(\varphi,\psi)\big| \le |\nabla\varphi||\nabla\psi|$$

as well as the rules

$$\Gamma(\varphi_1\varphi_2,\psi) = \varphi_1\Gamma(\varphi_2,\psi) + \varphi_2\Gamma(\varphi_1,\psi)$$

and

$$\Gamma(\theta\circ\varphi,\psi) = (\theta'\circ\varphi)\Gamma(\varphi,\psi)\,.$$

There is a less canonical but more explicit way to define $|\nabla\varphi|$ which also allows $\nabla\varphi$ itself to make sense. Equip M with an arbitrary Riemannian metric g; denote by $d\mu$ the associated Riemannian volume, by grad the gradient induced by the Riemannian metric and by div the Riemannian divergence. Write $dv = m d\mu$ where m is a positive smooth function on M. The operator Δ determines a smooth section $\mathcal{A}$ of the bundle of symmetric endomorphisms (i.e. for each $x \in M$, $\mathcal{A}_x$ is an endomorphism of the tangent space T_xM at x which is symmetric and positive with respect to the Euclidean structure g_x) such that

$$\Delta\varphi = -m^{-1}\mathrm{div}(m\mathcal{A}\,\mathrm{grad}\,\varphi)\,. \tag{2.1}$$

Note that $\mathcal{A}_x$ is nothing but the principal symbol of Δ in an orthonormal frame at x. Of course, if Δ is the Laplace-Beltrami operator of (M,g), then $m \equiv 1$ and $\mathcal{A} \equiv \mathrm{Id}$ where Id denotes the identity. In general, $\mathcal{A}$ may be highly degenerate since we have not yet ruled out the case $\Delta \equiv 0$. In any case, with this notation, we have

$$\Gamma(\varphi, \psi) = g(\mathcal{A} \operatorname{grad} \varphi, \operatorname{grad} \psi) .$$

In particular,

$$|\nabla \varphi|^2 = g(\mathcal{A} \operatorname{grad} \varphi,\ \operatorname{grad} \varphi) .$$

The natural definition of $\nabla \varphi$ is then

$$\nabla \varphi = \nabla_\Delta \varphi = \mathcal{A} \operatorname{grad} \varphi .$$

To check that this is indeed the correct definition, consider the special case where $\mathcal{A}$ is everywhere invertible. Define a new Riemannian metric g_0 by setting

$$g_0(X, Y) = g(\mathcal{A}^{-1} X, Y) .$$

This metric is canonically attached to Δ (i.e. does not depend of the arbitrary choice of g) and we have

$$\Delta \varphi = -m_0^{-1} \operatorname{div}_0 (m_0 \operatorname{grad}_0 \varphi)$$

where m_0 is such that $dv = m_0 d\mu_0$. We claim that

$$\nabla \varphi = \operatorname{grad}_0 \varphi ; \quad |\nabla \varphi|^2 = g_0(\operatorname{grad}_0 \varphi, \operatorname{grad}_0 \varphi) .$$

Indeed, on the one hand, $\operatorname{grad}_0 \varphi$ is the unique smooth vector field such that for any smooth vector field X, $g_0(\operatorname{grad}_0 \varphi, X) = d\varphi(X)$. On the other hand we have

$$g_0(\nabla \varphi, X) = g(\mathcal{A}^{-1} \mathcal{A} \operatorname{grad} \varphi, X) = d\varphi(X) .$$

This proves the claim.

2.2 The distance. There is a "distance" $\rho = \rho_\Delta$ canonically associated with Δ and defined by

$$\rho(x, y) = \sup \left\{ \varphi(x) - \varphi(y) : \varphi \in \mathcal{C}_0^\infty(M) ,\ |\nabla \varphi| \leq 1 \right\} .$$

This ρ has all the properties of a distance except that it might well take the value $+\infty$. We make the following basic hypotheses which will be in force throughout the paper:

1. $\forall\, x, y \in M$, $\rho(x, y) < +\infty$.
2. The distance ρ is continuous and the topology induced by it is the same as the topology of M as a manifold.
3. (M, ρ) is a complete metric space.

These are natural hypotheses and the two first conditions are the crucial ones. Of course, these conditions are satisfied when M is equipped with a Riemannian metric g such that (M, g) is a complete Riemannian manifold,

$v = md\mu$ where μ is the Riemannian volume and m is a smooth positive function on M, and

$$\Delta\varphi = -m^{-1}\mathrm{div}(m\,\mathrm{grad}\,\varphi)\,.$$

In this case, ρ is simply the Riemannian distance associated with g.

Subelliptic operators in a domain of $\mathbb{R}^n$ satisfy the two first conditions above. See section 2.7 below for a precise statement and references.

Under these hypotheses, one shows that ρ can also be defined in the following alternative way. Let us say that a tangent vector X at x is a subunit vector if it satisfies

$$|d\varphi(X)| \le |\nabla\varphi|(x)$$

for any smooth function φ. An absolutely continuous path $\gamma: [0,t] \to M$ is a subunit path if $\dot\gamma(s)$ is a subunit vector for all $s \in [0,t]$. Then, $\rho(x,y)$ is equal to the infimum of all $t > 0$ such that there exists a subunit path $\gamma : [0,t] \to M$ satisfying $\gamma(0) = x$, $\gamma(t) = y$.

We will use two important properties of the distance ρ which follow from the above basic hypotheses. The first property is that, for each $x \in M$, the function $\rho_x(y) = \rho(x,y)$ satisfies

$$|\nabla\rho_x| \le 1 \tag{2.2}$$

almost everywhere. The second property is that ρ is a geodesic distance in the sense that, given $x, y \in M$, there exists a subunit path $\gamma: [0,t] \to M$ joining x to y and such that $\rho(x,y) = t$.

2.3 The doubling property. Denote by $B_x(r) = \{y \in M : \rho(x,y) < r\}$ the ball of radius $r > 0$ centered at $x \in M$ and set $V_x(r) = v(B_x(r))$.

For each fixed $R > 0$, consider the classical doubling property [D(R)] which may or may not be satisfied by (M,Δ):

$$\exists\, D > 0 \text{ s.t. } \forall\, r \in (0,R)\,,\ \forall\, x \in M\,,\quad V_x(2r) \le DV_x(r)\,. \qquad [\mathrm{D}(R)]$$

Observe that [D(R)] implies

$$\forall\, r,s\,,\ 0 < r < s < 2R\,,\quad V_x(s) \le D(s/r)^{\nu_0} V_x(r)\,,$$

with $\nu_0 = \log D/\log 2$. The property [D(R)] plays a crucial role in the sequel.

We shall say that (M,Δ) satisfies [D] if it satisfies [D(R)] uniformly in R:

$$\exists\, D > 0 \text{ s.t. } \forall\, r > 0\,,\ \forall\, x \in M\,,\quad V_x(2r) \le DV_x(r)\,. \qquad [\mathrm{D}]$$

2.4 The Poincaré inequalities. For $B = B_x(r)$ and any $\alpha > 0$, set $\alpha B = B_x(\alpha r)$. For any function f, denote by f_B the mean of f over B given by

$$f_B = \frac{1}{V_x(r)} \int_B f \, dv \, .$$

Fix $\kappa \geq 1$ and $R > 0$. Consider the family [P(p,R)] of Poincaré inequalities which may or may not be satisfied by (M, Δ):

$$\exists \, P_p > 0 \text{ s.t. } \forall \, r \in (0, R] \, , \ x \in M \, , \qquad \int_B |f - f_B|^p dv \leq P_p r^p \int_{\kappa B} |\nabla f|^p \, dv \, , \qquad [\mathrm{P}(p,R)]$$

for all $f \in \mathcal{C}^\infty(B_x(\kappa r))$. Here, $|\nabla f|$ is the length of the gradient associated with Δ as in section 2.1.

We shall say that (M, Δ) satisfies [P(p)] if it satisfies [P(p,R)] uniformly in R:

$$\exists \, P_p > 0 \text{ s.t. } \forall \, r > 0 \, , \ x \in M \, , \int_B |f - f_B|^p dv \leq P_p r^p \int_{\kappa B} |\nabla f|^p \, dv \, , \ [\mathrm{P}(p)]$$

for all $f \in \mathcal{C}^\infty(B_x(\kappa r))$.

Remark. In [J] (see also [HK2], [MS]), it is shown that [D(R)] and [P(p,R)] imply the stronger Poincaré inequality

$$\int_B |f - f_B|^p dv \leq P'_p r^p \int_B |\nabla f|^p \, dv$$

for all $f \in \mathcal{C}^\infty(B_x(r))$, $x \in M$, $0 < r \leq R$. This however is not a trivial fact. What is relatively easy to see is that, assuming that [D(R)] holds true, [P(p,R)] for some fixed $\kappa > 1$ implies [P(p,R)] for any $\kappa > 1$. Only this easier fact will be used in the sequel.

2.5 The p-Laplacian. In Riemannian geometry, the p-Laplacian, $1 < p < +\infty$, is the operator defined by

$$\Delta_p \varphi = -\mathrm{div}\big(|\mathrm{grad}\, \varphi|^{p-2} \mathrm{grad}\, \varphi\big) \, .$$

Clearly $\Delta_p \varphi$ can be characterized by

$$\langle \Delta_p \varphi, \psi \rangle = \int_M g(\mathrm{grad}\, \psi, \mathrm{grad}\, \varphi) |\mathrm{grad}\, \varphi|^{p-2} \, d\mu$$

for all $\psi \in \mathcal{C}_0^\infty(M)$ where g is the Riemannian metric and μ the Riemannian measure. This operator plays the rôle of the Laplace-Beltrami operator when one replaces the energy functional $\int |\mathrm{grad}\, \varphi|^2 \, d\mu$ by its L^p version $\int |\mathrm{grad}\, \varphi|^p \, d\mu$.

In the more general setting introduced above, we define $\Delta_p\varphi$, $\varphi \in \mathcal{C}_0^\infty(M)$, by setting

$$\langle \Delta_p\varphi, \psi\rangle = \int_M \Gamma(\psi,\varphi)|\nabla\varphi|^{p-2}\,dv$$

for all $\psi \in \mathcal{C}_0^\infty(M)$. If we introduce on M an arbitrary Riemannian metric as in (2.1) then

$$\Delta_p\varphi = -m^{-1}\mathrm{div}\big(m|\nabla\varphi|^{p-2}\mathcal{A}\,\mathrm{grad}\,\varphi\big)\,.$$

2.6 The non-smooth case. Since in section 5 we shall have to deal with pull-backs of n-Laplacians by quasi-regular maps, we have to extend slightly the class of equations under consideration. Start with a smooth manifold M equipped with an arbitrary (and irrelevant) Riemannian metric g with Riemannian volume $d\mu$.

Let $\mathcal{B}$ be a measurable section of the bundle of symmetric endomorphisms and let m be a positive measurable function on M. Equip M with the measure

$$dv = m d\mu$$

and the bilinear form

$$\forall\,\varphi,\psi \in \mathcal{C}^\infty(M)\,,\quad \Gamma(\varphi,\psi) = g(\mathcal{B}\,\mathrm{grad}\,\varphi, \mathrm{grad}\,\psi)$$

which is well defined. This formally relates to the construction given in the smooth coefficients case by considering the second order differential operator

$$\Delta\varphi = -m^{-1}\mathrm{div}\,(m\mathcal{B}\,\mathrm{grad}\,\varphi)$$

which should however now be interpreted in the appropriate weak sense. We will refer to this situation as before by using the notation (M,Δ). Indeed, the data above yield the notion of "length of the gradient"

$$|\nabla\varphi|^2 = g(\mathcal{B}\,\mathrm{grad}\,\varphi, \mathrm{grad}\,\varphi)\,,$$

as well as a distance ρ on M defined as in section 2.2. The doubling property [D(R)] and the Poincaré inequality [P(p,R)] also generalizes without changes. We will always assume that conditions 1–3 of section 2.2 are satisfied.

Formally, the corresponding p-Laplacian is given by

$$\Delta_p\varphi = -m^{-1}\mathrm{div}\big(m|\nabla\varphi|^{p-2}\,\mathcal{B}\,\mathrm{grad}\,\varphi\big) \tag{2.3}$$

which must be interpreted in a weak sense as we now explain in some detail.

For $1 < p < +\infty$, define $W_0^{1,p}(M,\Delta)$ as the completion of $\mathcal{C}_0^\infty(M)$ with respect to the norm $\big(\int_M |\varphi|^p\,dv + \int_M |\nabla\varphi|^p\,dv\big)^{1/p}$. If $u \in W_0^{1,p}(M,\Delta)$ and $\psi \in \mathcal{C}_0^\infty(M)$, the expression $\Gamma(\psi,u)$ is well defined by polarization, and $\Gamma(\psi,u)|\nabla u|^{p-2}$ is in $L^1(M,dv)$ by Hölder. By density of $\mathcal{C}_0^\infty(M)$ in $W_0^{1,p}(M,\Delta)$, the linear form

$$\psi \to \int_M \Gamma(\psi,u)|\nabla u|^{p-2}\,dv$$

is then defined for $\psi \in W_0^{1,p}(M,\Delta)$. For Ω a measurable subset of M, let $W_{loc}^{1,p}(\Omega)$ be the set of functions u on Ω such that ψu belongs to $W_0^{1,p}(M,\Delta)$ for all $\psi \in \mathcal{C}_0^\infty(\Omega)$. We say that a function u on $\Omega \subset M$ is Δ_p-harmonic, in short p-harmonic, if $u \in W_{loc}^{1,p}(\Omega)$ and

$$\int_\Omega \Gamma(\psi,u)|\nabla u|^{p-2}\,dv = 0$$

for all $\psi \in W_0^{1,p}(M,\Delta)$ compactly supported in Ω. We say that a function u defined on $\Omega \subset M$ is a Δ_p-supersolution (or p-supersolution, or even supersolution when there is no ambiguity) if $u \in W_{loc}^{1,p}(\Omega)$ and

$$\int_\Omega \Gamma(\psi,u)|\nabla u|^{p-2}\,dv \geq 0$$

for all non-negative $\psi \in W_0^{1,p}(M,\Delta)$ compactly supported in Ω. We say that u is a subsolution if $-u$ is a supersolution.

Remark. As explained above, there is no real difficulty in dealing with non-smooth structures, as long as one is ready to assume that the basic hypotheses 1–3 concerning the distance ρ are satisfied. It should be emphasized however that verifying the basic hypotheses 1–3 of ρ is a non-trivial problem. For instance, in domains of $\mathbb{R}^n$, there exists no satisfactory subelliptic theory of operator with measurable coefficients whereas there exists a reasonably good theory for subelliptic operator with $\mathcal{C}^\infty$ coefficients, see [FP]. Similarly, deciding whether or not conditions [D(R)] and/or [P(p,R)] hold is a difficult question. A natural context where one can work with operators having measurable coefficients as above is when there exist a smooth positive function $\underline{m}$ and a smooth section $\mathcal{A}$ of the bundle of symmetric endomorphisms such that

$$C_0^{-1}\underline{m} \leq m \leq C_0\underline{m}$$

and

$$\forall\, x \in M\,,\ \forall\, h \in T_xM\,,\quad C_0^{-2}g_x(\mathcal{A}_xh,h) \leq g_x(\mathcal{B}_xh,h) \leq C_0^2 g_x(\mathcal{A}_xh,h)\,. \tag{2.4}$$

Then it is easy to check that conditions 1–3 of section 2.2 are satisfied by (M,Δ) if and only if they are by $(M,\underline{\Delta})$ where

$$\underline{\Delta}\varphi = -\underline{m}^{-1}\mathrm{div}(\underline{m}\mathcal{A}\,\mathrm{grad}\,\varphi)\,.$$

Similarly, conditions [D(R)] and [P(p,R)] are satisfied by (M,Δ) if and only if they are by $(M,\underline{\Delta})$.

2.7 Examples.

Riemannian manifolds. Let (M,g) be a complete Riemannian manifold of dimension n. Let Δ be the Laplace-Beltrami operator on (M,g) and v be the Riemannian volume. Let Ω be an open precompact subset of M and let $K=K(\Omega)$ be a non-negative real number such that

$$\forall\, x\in\Omega\,,\ \forall\, h\in T_x(M)\,,\quad \mathrm{Ric}_x(h,h)\geq -Kg_x(h,h)\,.$$

Here, Ric is the Ricci curvature of (M,g) which is a symmetric two-tensor.

Let $B=B_x(r)$ be a metric ball contained in Ω. Then the well-known Bishop–Gromov comparison theorem asserts that

$$\forall\, 0<s<r\,,\quad V_x(r)\leq (r/s)^n e^{\sqrt{(n-1)K}r}V_x(s)\,.$$

Moreover, P. Buser proved in [Bu] that there exists a constant $C=C(n)$ such that

$$\int_B |f-f_B|dv\leq Cre^{C\sqrt{K}r}\int_B|\nabla f|dv\,.$$

In particular, if (M,g) has nonnegative Ricci curvature, it satisfies the doubling property [D] of section 2.3 and the Poincaré inequality [P(p)] of section 2.4 for any $1\leq p<+\infty$.

Riemannian coverings. Let (N,g) be a compact Riemannian manifold. Let $\overline{N}$ be its universal cover, and let $\pi_1(N)$ be the fundamental group of N. The fundamental group acts on $\overline{N}$ with quotient N. If H is a normal subgroup, we can consider the Riemannian manifold $\overline{M}=\overline{N}/H$. The deck transformation group G of the covering of N by $\overline{M}$ is $G=\pi_1(N)/H$. Finally, let K be a subgroup of G, not necessarily normal, and consider the Riemannian manifold $M=\overline{M}/K$. The group $G=\pi_1(N)/H$ is finitely generated. Using a finite symmetric set of generators, we can define the usual word metric on G and consider the cardinality $\gamma(r)$ of the ball of radius r in G. We say that G has polynomial growth if there exist C and A such that $\gamma(r)\leq Cr^A$ for all $r>0$.

PROPOSITION 2.1. *Referring to the above notation, assume that the group G has polynomial growth. Then the Laplace operator on M satisfies* [D] *and* [P(p)] *for all* $1\leq p<+\infty$.

We refer the reader to [S2], [CoS2] for the proof and further results.

Invariant operators on Lie groups. Let G be a connected, unimodular Lie group equipped with its Haar measure $d\mu$. Let $\{X_1,\dots,X_k\}$ be a family of smooth left invariant vector fields on G and set

$$\Delta = -\sum_1^k X_i^2\,.$$

Here $|\nabla_\Delta\varphi|^2 = \sum_1^k |X_i\varphi|^2$. In this case, one can define $\nabla_\Delta\varphi$ as the vector $(X_1\varphi,\dots,X_k\varphi)$ and the p-Laplacian takes the form

$$\Delta_p\varphi = -\sum_{i=1}^k X_i\big(|\nabla\varphi|^{p-2}X_i\varphi\big)\,.$$

The conditions 1–3 of section 2.2 are satisfied if and only if $\{X_1,\dots,X_k\}$ generates the Lie algebra of G, that is, if these vectors and their brackets of all orders span the tangent space at the identity. See [Hör], [VaSC]. If this condition (often called Hörmander condition) is satisfied, then for each $R>0$, the doubling condition [D(R)] and the Poincaré inequality [P(p,R)], $1\le p<+\infty$, are satisfied. If G is nilpotent, more generally if G has polynomial growth, then the global doubling condition [D] and the global Poincaré inequality [P(p)] are satisfied. See [Va3], [VaSC].

Subelliptic operators. Let us consider now the case where $M=\mathbb{R}^n$. Then, we can write

$$\Delta\psi(x) = -m(x)^{-1}\sum_1^n \partial_i\big(m(x)a_{i,j}(x)\partial_j\psi(x)\big)$$

where $0<m(x)<+\infty$ and the matrix $(a_{i,j}(x))$ is symmetric positive semidefinite. We assume that m and $(a_{i,j})$ are $\mathcal{C}^\infty$. The operator Δ is self-adjoint on $L^2(\mathbb{R}^n,v)$ where $dv(x)=m(x)dx$. In this case, the length of the gradient associated with Δ is

$$|\nabla_\Delta\psi| = \sum_1^n a_{i,j}\partial_i\psi\partial_j\psi$$

and the definition of a subunit vector $\zeta=(\zeta_i)_1^n$ at x reads

$$\forall\,\xi\in\mathbb{R}^n\,,\qquad \Big|\sum_1^n \zeta_i\xi_i\Big|^2 \le \sum_1^n a_{i,j}(x)\xi_i\xi_j\,.$$

Following standard notation, we say that Δ is uniformly subelliptic if

$$\big\|(I+\Delta_0)^{2\epsilon}\psi\big\|_2 \le C\big(\|\Delta\psi\|_2+\|\psi\|_2\big)\,,\quad \psi\in\mathcal{C}_0^\infty(\mathbb{R}^n)\,, \tag{2.5}$$

and $m,m^{-1}\le C$, for some $C,\epsilon>0$. Here, $\Delta_0=-\sum_1^n\partial_i^2$ is the standard Laplacian on $\mathbb{R}^n$.

Basic references on subelliptic operators are [OlR], [FP], [FS] and the survey [JS]. There is also a large literature, starting with Hörmander's paper [Hör], on the special case where Δ can be written as a sum of squares of smooth vector fields. See e.g. [NSW].

Theorem 2.2. *Assume that m, m^{-1}, $a_{i,j}$ and their derivatives of any order are bounded in $\mathbb{R}^n$. Assume further that Δ satisfies* (2.5). *Then the distance ρ associated with Δ satisfies the conditions 1–3 of section 2.2. Moreover Δ satisfies* [D] *and* [P(p)] *for all $1 \le p < +\infty$.*

The references [FP], [FS], [JS] treat the local aspect of this theorem, which is the heart of the matter. The fact that [P(p)] holds globally is proved in [CoS2].

3 p-parabolicity and p-hyperbolicity

3.1 An inequality for supersolutions. This section is organized around the following result which is adapted from an inequality of Cheng and Yau ([ChenY, Theorem 1, p. 335]). For similar and related results in the discrete setting, see [S4,5]. Henceforth, we shall consider a pair (M, Δ) as in section 2.6. We assume throughout that conditions 1–3 of section 2.2 are satisfied.

Theorem 3.1. *Fix $1 < p < +\infty$. Let u be a positive p-supersolution in a fixed ball $B = B_o(R)$. For any sequence $r = r_0 < r_1 < \cdots < r_\ell = R$, we have*

$$(p-1)\int_{B_o(r)} |\nabla \log u|^p dv \le \left(\sum_0^{\ell-1} \frac{(r_{i+1}-r_i)^{p/(p-1)}}{(V_o(r_{i+1}) - V_o(r_i))^{1/(p-1)}}\right)^{1-p}. \quad (3.1)$$

Proof. For any function $f \in W^{1,p}_{loc}(B)$, $f \ge \varepsilon$, we have

$$\Delta_p \log f - \frac{\Delta_p f}{f^{p-1}} = (p-1)|\nabla \log f|^p$$

in the weak sense. That is, for any non-negative function $\psi \in W^{1,p}_0(M)$ with compact support in B,

$$\int \Gamma(\psi, \log f)|\nabla \log f|^{p-2} dv - \int \Gamma\left(f, \frac{\psi}{f^{p-1}}\right)|\nabla f|^{p-2} dv = (p-1)\int \psi |\nabla \log f|^p dv\,.$$

Thus, if $f = u$ is a positive supersolution,

$$(p-1)\int \psi|\nabla \log u|^p dv \le \int \Gamma(\psi, \log u)|\nabla \log u|^{p-2} dv\,.$$

Taking

$$\psi(y) = \psi_{t,\delta}(y) = \begin{cases} 1 & \text{if } x \in B_o(t) \\ 1 - \delta^{-1}[\rho(o,y) - t] & \text{if } x \in B_o(t+\delta) \setminus B_o(t) \\ 0 & \text{otherwise} \end{cases}$$

yields

$$(p-1)\int_{B_o(t)} |\nabla \log f|^p dv \le \frac{1}{\delta}\int_{B_o(t+\delta)\setminus B_o(t)} |\nabla \log f|^{p-1} dv\,.$$

The right-hand side can be bounded by

$$\left[\frac{V_o(t+\delta) - V_o(t)}{\delta}\right]^{1/p} \left[\frac{1}{\delta}\int_{B_o(t+\delta)\setminus B_o(t)} |\nabla \log f|^p dv\right]^{1/q}$$

where $1/p + 1/q = 1$.

Assume now that $s \to V_o(s)$ is locally absolutely continuous (i.e. it is almost everywhere differentiable and it is the integral of its derivative). Then, letting δ tend to zero yields

$$H(t)^q \le \frac{1}{p-1} V'(t)^{q/p} H'(t) \tag{3.2}$$

where we have set $V(t) = V_o(t)$ and $H(t) = (p-1)\int_{B_o(t)} |\nabla \log f|^p\, dv$. Write (3.2) as

$$\frac{qH'(t)}{pH(t)^q} \ge \frac{1}{V'(t)^{q/p}}\,.$$

For any $0 < s < t \le R$, this yields

$$\frac{1}{H(s)^{q/p}} - \frac{1}{H(t)^{q/p}} \ge \int_s^t \frac{d\sigma}{V'(\sigma)^{q/p}} \tag{3.3}$$

where the identity $(q/p) + 1 = q$ has been used. The right-hand side can be bounded from below by

$$\left[\frac{(t-s)^p}{V(t) - V(s)}\right]^{1/(p-1)}$$

because

$$\begin{aligned}(t-s)^p = \left(\int_s^t d\sigma\right)^p &\le \left(\int_s^t V'(\sigma) d\sigma\right)\left(\int_s^t \frac{1}{V'(\sigma)^{q/p}} d\sigma\right)^{p/q} \\ &= \big(V(t) - V(s)\big)\left(\int_s^t \frac{1}{V'(\sigma)^{q/p}} d\sigma\right)^{p/q}.\end{aligned}$$

Hence (3.3) implies

$$\frac{1}{H(r)^{q/p}} - \frac{1}{H(R)^{q/p}} \ge \sum_0^{\ell-1} \left[\frac{(r_{i+1} - r_i)^p}{V(r_{i+1}) - V(r_i)}\right]^{1/(p-1)}$$

for any sequence $r = r_0 < \cdots < r_\ell = R$. This clearly gives the desired result.

The additional hypothesis that $s \to V(s)$ is locally absolutely continuous appears to be quite natural and harmless. There are many cases (e.g. the Riemannian setting, see [Ch, p. 116]) where it is satisfied. However, there is no reason why it should hold in full generality under the conditions introduced in section 2.2. This is not a problem because the differential inequality (3.2) can be replaced by the difference inequality obtained by letting δ be a fixed parameter. Instead of (3.3) one then obtains

$$\frac{1}{H(s)^{q/p}} - \frac{1}{H(t)^{q/p}} \geq \sum_{i=n}^{m-1} \frac{1}{[V(\sigma_{i+1}) - V(\sigma_i)]^{q/p}}$$

where $s = r + \delta n$, $t = R + \delta m$ with $m \geq n$ and $\sigma_i = r + \delta i$. The argument used above also shows that

$$(t-s)^p \leq \big(V(t) - V(s)\big)\bigg(\sum_{i=n}^{m-1} \frac{1}{[V(\sigma_{i+1}) - V(\sigma_i)]^{q/p}}\bigg)^{1/q}.$$

The desired conclusion follows.

Remark. Define the (p, Δ)-capacity (in short p-capacity) of a pair (U, C), with $U \subset M$ open and $C \subset U$ compact by

$$\mathrm{Cap}_p(U, C) = \inf_u \int_U |\nabla u|^p dv\,,$$

where the infimum is taken over all functions $u \in \mathcal{C}_0^\infty(U)$, with $u \geq 1$ in C. Then an alternative way to prove Theorem 3.1 (though maybe not with the optimal constant $(p-1)$) is to combine a so-called logarithmic Caccioppoli inequality

$$\int_{B_o(r)} |\nabla \log u|^p dv \leq C \mathrm{Cap}_p\big(B_o(R), \bar{B}_o(r)\big) \tag{3.4}$$

(see [HeKM, Theorem 3.53]) with the capacity estimate

$$\mathrm{Cap}_p\big(B_o(R), \bar{B}_o(r)\big) \leq \bigg(\sum_0^{\ell-1} \frac{(r_{i+1} - r_i)^{p/(p-1)}}{(V_o(r_{i+1}) - V_o(r_i))^{1/(p-1)}}\bigg)^{1-p}.$$

The latter follows from a standard estimate

$$\mathrm{Cap}_p\big(B_o(r_{i+1}), \bar{B}_o(r_i)\big) \leq \frac{V_o(r_{i+1}) - V_o(r_i)}{(r_{i+1} - r_i)^p} \tag{3.5}$$

by using the property

$$\mathrm{Cap}_p\big(B_o(R), \bar{B}_o(r)\big) \leq \bigg(\sum_0^{\ell-1} \mathrm{Cap}_p\big(B_o(r_{i+1}), \bar{B}_o(r_i)\big)^{1/(1-p)}\bigg)^{1-p}, \tag{3.6}$$

see [HeKM, Theorem 2.6].

3.2 Volume growth and p-parabolicity Define $D_p(f) = \int_M |\nabla f|^p \, dv$ and

$$d_p = \inf \left\{ D_p(f) : f \in \mathcal{C}_0^\infty(M); \frac{1}{|U|} \int_U |f| \, dv = 1 \right\}$$

where U is some fixed relatively compact open subset of M and $|U| = v(U)$ its volume. Denote by $\mathbf{D}_p^0$ the closure of $\mathcal{C}_0^\infty(M)$ for the norm $\left(\frac{1}{|U|}\int_U |f|^p dv + D_p(f)\right)^{1/p}$. Denote by $\mathbf{D}_p$ the Banach space of all functions f such that $\frac{1}{|U|}\int_U |f|^p \, dv + D_p(f) < +\infty$.

From now on we shall make a mild additional assumption on the local geometry of M, namely that for every open relatively compact set Ω, the following local Poincaré inequality holds:

$$\int_\Omega |f - f_\Omega|^p dv \le C_\Omega \int_M |\nabla f|^p dv \,. \tag{3.7}$$

This property is satisfied by all the examples considered in section 2.7.

One says that (M, Δ) is p-parabolic if every positive p-supersolution on M is constant, and that (M, Δ) is p-hyperbolic if it is not p-parabolic. Equivalent properties to p-parabolicity are

1. The constant function $x \to 1$ belongs to $\mathbf{D}_p^0$.
2. The two spaces $\mathbf{D}_p^0$ and $\mathbf{D}_p$ are equal.
3. $d_p = 0$.
4. $\mathrm{Cap}_p(M, \bar{U}) = 0$.

In particular properties 1, 2, 3 and 4 do not depend on the choice of U. For the equivalence between 1, 3 and 4, see [Tr2, Proposition 1 and Theorem 3], and also [GoT2, Theorem 3.1]. For the equivalence between 1 and 2, see [Y, Theorem 3.2]. The equivalence between p-parabolicity and 4 is proved in [Ho1, Theorem 5.2], as well as in [K] in the Riemannian setting. In our general case, we may use the logarithmic Cacciopoli inequality (3.4) to show that 4 implies p-parabolicity. For the converse, assume that U is a relatively compact open subset of M, with $\mathrm{Cap}_p(M, \bar{U}) > 0$. Then it is possible to find, by usual methods of variational calculus, a positive p-supersolution $u \in \mathbf{D}_p^0$, with $u \equiv 1$ in $\bar{U}$ and $\mathrm{Cap}_p(M, \bar{U}) = D_p(u)$ (so that u is not constant).

Note that if (M, Δ) is p-parabolic and satisfies some mild local assumptions (called $(P)_{loc}$ and $(DV)_{loc}$ in [CoS2]; for example for Riemannian manifolds, Ricci curvature bounded from below is enough, no assumption on the injectivity radius is required) then it is q-parabolic for $q > p$. This is clear by discretization (apply Hölder to the obvious p-version of Proposition 6.9 in [CoS2]); as a consequence one can consider a notion of parabolic dimension (see [Co2]). This fact is proved in [Tr1], Theorem 5.2 for Riemannian manifolds with Ricci curvature bounded from below and positive injectivity radius. Note that it can be false if M is not complete ([Tr1, Cor. 3.1]).

It is well known, especially for $p = 2$, that p-hyperbolicity is only possible if the volume growth is large enough. There are at least two traditional points of view to prove this. The first one uses characterization 4 of p-parabolicity. Then a sufficient condition of p-parabolicity is available as soon as one has a bound on the p-capacity of, say, a ball of radius one in terms of the volume growth of the larger concentric balls. This is the route followed, for $p = 2$, in [Gri1], [BiM1], [Ca1], [Os], [O], [St]. The relevant p-capacity estimates are available in [Maz] (in a Euclidean setting, but the methods have a much wider range, see [Gri3]). The second point of view is very similar, but uses characterization 3 of p-parabolicity and the estimate of the volume to build test functions that yield $d_p = 0$. This is the approach of [Va1] for $p = 2$. We shall explain how this second approach can be dealt with for $p \neq 2$ at the end of this section.

Here we would like to propose a third point of view, that works directly with the definition of p-parabolicity and uses the Cheng–Yau type estimate on p-supersolutions. The following statement is a straightforward consequence of Theorem 3.1 and of inequality (3.3).

Corollary 3.2. *Fix $1 < p < +\infty$. The pair (M, Δ) is p-parabolic as soon as there exists an increasing sequence $(r_i)_0^{+\infty}$ going to infinity and a point o such that*

$$\sum_0^{+\infty} \left[\frac{(r_{i+1} - r_i)^p}{V_o(r_{i+1}) - V_o(r_i)}\right]^{1/(p-1)} = +\infty\,.$$

If this condition holds we shall say that (M, Δ) satisfies condition [V(p)]. *If V_o is locally absolutely continuous,* [V(p)] *can be weakened to*

$$\int_1^{+\infty} \frac{dr}{V_o'(r)^{1/(p-1)}} = +\infty\,. \tag{3.8}$$

Both of these conditions are implied by the stronger one

$$\int_1^{+\infty} \left(\frac{r}{V_o(r)}\right)^{1/(p-1)} dr = +\infty\,. \tag{3.9}$$

For instance, p-parabolicity is implied by $\liminf_{+\infty} r^{-p} V_o(r) < +\infty$, or by $V_o(r) \le c r^p [\log r]^{p-1}$, $r \ge 1$.

REMARKS. (1) Condition [V(p)] can be formulated as

$$\int_1^{+\infty} \frac{(dr)^p}{V_o(dr)} = +\infty$$

where for a non-negative, left continuous, non-decreasing function V,

$$\int_1^{+\infty} \frac{(dr)^p}{V(dr)}$$

is the so-called Hellinger integral defined by

$$\sup\left\{ \sum_0^{+\infty} \left[\frac{(r_{i+1}-r_i)^p}{V(r_{i+1})-V(r_i)}\right]^{1/(p-1)} ;\ k\in\mathbb{N},\ 1\le r_0<r_1<\cdots<r_k \right\}.$$

It is shown in [St] (for $p=2$ but the proof easily extends to $p\neq 2$) that one has

$$c_p \int_1^{+\infty} \left(\frac{r}{V(r)}\right)^{1/(p-1)} dr \le \int_1^{+\infty} \frac{(dr)^p}{V(dr)} \le \int_1^{+\infty} f \frac{dr}{V'(r)^{1/(p-1)}} \,,$$

where V' is the Radon–Nikodym derivative of the absolutely continuous part of V. For the inequality

$$\int_1^{+\infty} \frac{dr}{V'(r)^{1/(p-1)}} \ge c_p \int_1^{+\infty} \left(\frac{r}{V(r)}\right)^{1/(p-1)} dr,$$

see also [Va1], [ZK]. The converse is false in general, but true if V is convex ([Gri2, §7]).

(2) An example due to Greene (see [Va1]) in the case $p=2$ shows that condition (3.8) is not necessary for M to be p-parabolic. If M is rotationally invariant around o, (3.8) is equivalent to the p-parabolicity ([Gri2, Cor. 5.6] for $p=2$, and [GoT1, §5.2] for the general case), whereas (3.9) remains strictly stronger (see [FeR], or [Gri2, §7]). We shall meet in Proposition 3.4 below a situation where (3.9) is necessary and sufficient.

(3) Let us outline another more direct proof of Corollary 3.2 inspired by [Va1]. For simplicity we shall work in the Riemannian setting and prove that condition (3.8) is incompatible with

$$\left(\int_U |f| dv\right)^p \le C\, D_p(f)\,,$$

where U is the geodesic ball of radius one and center o.

The same proof can be adapted to treat the discrete condition in Corollary 3.2 and our general setting.

Take

$$\Lambda(R) = \int_1^R \frac{dr}{V_o'(r)^{1/(p-1)}}\,, \quad R\ge 1$$

$$g(r) = 1 - \frac{\Lambda(r)}{\Lambda(R)} \text{ if } 1 \le r \le R\,,\ 1 \text{ if } 0 \le r \le 1\,,\ 0 \text{ if } r \ge R\,.$$

Then

$$|g'(r)| = \frac{1}{\Lambda(R)(V_o'(r))^{1/p-1}}\,, \qquad \text{if } 1 \le r \le R\,,\ 0 \text{ otherwise}\,,$$

and, for $f(x) = g(d(x,o))$,

$$D_p(f) = \int_1^{+\infty} |g'(r)|^p V_o'(r)\,dr = \frac{1}{\Lambda^p(R)} \int_1^R \frac{dr}{V_o'(r)^{1/(p-1)}} = \frac{1}{\Lambda^{p-1}(R)}\,.$$

Since $\int_U |f| dv = |U| > 0$, the claim follows.

Except maybe for the first sufficient condition in the case $p \neq 2$, Corollary 3.2 is essentially well known, at least in the Riemannian case. The fact that (3.8) implies p-parabolicity has been proved in [A] for M a Riemann surface and $p = 2$ and extended in [Va1] to Riemannian manifolds and $p = 2$ (see also [Gri1] and [Ca2]). Of course, the particular case of quadratic growth was already in [ChenY]. For a complete account and further references in the case $p = 2$, see [Gri2, §7], and [Li]. The case where p is the topological dimension of the Riemannian manifold has been treated in [ZK]; a similar result in a sub-Riemannian setting has been obtained in [Z]. The general case of arbitrary p appears in [Tr1]; in [Ho4] it was shown that (3.9) implies p-parabolicity. As far as condition [V(p)] is concerned, it is shown in [St], in a general Dirichlet forms setting, that for $p = 2$ it implies 2-parabolicity. The sufficiency of [V(p)] can also be seen by using the classical upper estimate for the capacity of a ball of radius r (estimate (3.5) above) and property (3.6) (see [Ho3, Thm. 4.8]). The analogue of the above results for graphs has been obtained in [S4,5] (a partial result in this direction appears in [So]). Finally, the sufficiency of condition (3.9) for p-parabolicity was recently shown in a general setting of metric measure spaces in [HoK].

3.3 *p*-isoperimetric profile and *p*-hyperbolicity. Assume in this section that M has infinite volume ($v(M) = +\infty$), otherwise it is certainly p-parabolic (let us mention here the following interesting statement in [ZK], that a Riemannian n-manifold is n-parabolic if and only if there is a conformal change of metric that transforms it into a complete manifold of finite volume; see [Z] for the corresponding result in sub-Riemannian setting).

To start, consider the case where M is a Riemannian manifold of dimension n. In [Gri1] and in [Fe], a sufficient condition for the 2-hyperbolicity of a non-compact Riemannian manifold is given in terms of its isoperimetric profile. Namely it is shown that if one defines

$$\psi(t) = \inf\left\{|\partial\Omega|\,;\ \Omega \in \mathcal{K}(M)\,,\ |\Omega| \geq t\right\},$$

where $\mathcal{K}(M)$ is the set of smooth relatively compact domains in M, $|\Omega|$ is the Riemannian n-volume of Ω and $|\partial\Omega|$ the Riemannian $(n-1)$-volume of the smooth hypersurface $\partial\Omega$, then

$$\int^{+\infty} \frac{dt}{\left(\psi(t)\right)^2} < +\infty \tag{3.10}$$

implies 2-hyperbolicity. See also [Gri2, §8], and [Ca2] for a nice proof using rearrangements of functions and co-area formula. In [Tr1], it is shown for all $p \in (1, +\infty)$ that

$$\int^{+\infty} \frac{dt}{\left(\psi(t)\right)^{\frac{p}{p-1}}} < +\infty$$

implies p-hyperbolicity.

Again, this is traditionally connected to the notion of p-capacity: proving $d_p > 0$ amounts to having a lower bound on capacity of sets, which can be obtained in terms of the isoperimetric profile (see [Maz] and [Gri3]).

Here we are going to work with a slightly different isoperimetric profile, namely

$$\varphi_1(t) = \sup\left\{\frac{|\Omega|}{|\partial\Omega|}\,;\ \Omega \in \mathcal{K}(M)\,,\ |\Omega| \leq t\right\},$$

and show that

$$\int^{+\infty} \left(\frac{\varphi_1(t)}{t}\right)^{p/(p-1)} dt < +\infty$$

implies p-hyperbolicity.

This result does not really compare with the previous one. Roughly speaking, ψ is the greatest non-decreasing function such that

$$\psi(|\Omega|) \leq |\partial\Omega|\,, \quad \forall\, \Omega \in \mathcal{K}(M)\,,$$

whereas φ_1 is the smallest non-decreasing function such that

$$\frac{|\Omega|}{\varphi_1(|\Omega|)} \leq |\partial\Omega|\,, \quad \forall\, \Omega \in \mathcal{K}(M)\,.$$

However, if one assumes that, for some $c > 0$,

$$\varphi_1(t) \leq c \sup\left\{\frac{|\Omega|}{|\partial\Omega|}\,;\ \Omega \in \mathcal{K}(M)\,,\ ct \leq |\Omega| \leq t\right\},$$

then $\varphi_1(t)/t$ is dominated by $c/\psi(ct)$ and our result is stronger.

Our formulation has the advantage that one can replace the 1-isoperimetric profile φ_1 by a p-isoperimetric profile, which leads to a more accurate and natural condition, even in the case $p = 2$. Indeed, note that, by the co-area formula,

$$\varphi_1(t) = \sup\left\{ \frac{\|f\|_1}{\||\nabla f|\|_1}\,;\ f \in \ \mathrm{Lip}_0(\Omega)\setminus\{0\}\,,\ \Omega \in \mathcal{K}(M)\,,\ |\Omega| = t \right\},$$

where $\mathrm{Lip}_0(\Omega)$ denotes the space of Lipschitz functions with support in Ω (this space can be replaced by $\mathcal{C}_0^\infty(\Omega)$ without affecting the function φ_1).

Returning to our general setting of a pair (M,Δ) as in section 2.6, for $1 \le p < +\infty$, define φ_p by

$$\varphi_p(t) = \sup\left\{ \frac{\|f\|_p}{\||\nabla f|\|_p}\,;\ f \in \mathcal{C}_0^\infty(\Omega)\setminus\{0\}\,,\ \Omega \in \mathcal{K}(M)\,,\ |\Omega| = t \right\} \tag{3.11}$$

where $|\Omega| = v(\Omega)$. We shall prove the following result.

Theorem 3.3. *Let (M,Δ) be as in section 2.6. Assume that (M,Δ) satisfies* (3.7). *Then, if*

$$\int^{+\infty} \left(\frac{\varphi_p(t)}{t}\right)^{p/(p-1)} dt < +\infty\,, \tag{3.12}$$

(M,Δ) is p-hyperbolic.

Remarks. (1) The case $p = 2$ of this theorem, in the Riemannian setting, is nothing but [Gri1, Theorem 2.3], the proof of the latter result uses the heat kernel, which is not available for $p \neq 2$.

(2) Let $1 \le q < p$ and $f \in \mathrm{Lip}_0(\Omega)\setminus\{0\}$. Set $g = |f|^{p/q}$; then $\|g\|_q^q = \|f\|_p^p$ and

$$\||\nabla g|\|_q^q \le \tfrac{p}{q}\big\||f|^{\frac{p}{q}-1}|\nabla f|\big\|_q^q \le \tfrac{p}{q}\|f\|_p^{p-q}\||\nabla f|\|_p^q\,,$$

thus

$$\frac{\|g\|_q^q}{\||\nabla g|\|_q^q} \ge \frac{q}{p}\frac{\|f\|_p^q}{\||\nabla f|\|_p^q}\,.$$

One therefore concludes that $\varphi_q(t) \ge (q/p)^{1/q}\varphi_p(t)$ for $1 \le q < p$. It follows that the sufficient condition 3.12 for p-hyperbolicity is weaker than

$$\int^{+\infty} \left(\frac{\varphi_q(t)}{t}\right)^{p/(p-1)} dt < +\infty\,, \tag{3.13}$$

for $1 \le q < p$. Now [CoL] provides an example where the rate of growth of $\varphi_2(t)$ is strictly slower than the one of $\varphi_1(t)$. This example is improved in [Ca1], where it is shown that one can have $\varphi_2(t) \le Ct^{1/D}$ and, for large t, $\varphi_1(t) \le C't^{2/D}$, but for any $\varepsilon > 0$ there is a sequence t_n going to $+\infty$ such that $t_n^{-\frac{2}{D}+\varepsilon}\varphi_1(t_n)$ is not bounded. For $D/2 \le p < D$, one could not tell that such a manifold is p-hyperbolic by using criterion (3.10) or (3.13) for $q=1$ instead of (3.12) or [Gri1, Theorem 2.3].

(3) Say that M satisfies the Faber–Krahn type inequality (F^p_φ), where $\varphi:\mathbb{R}_+^*\to\mathbb{R}_+^*$ is non-decreasing (see [Co2], [BaCLS]) if, for every Ω compact domain of M with smooth boundary,

$$\|f\|_p\le\varphi(|\Omega|)\||\nabla f|\|_p\,,\quad\forall\, f\in \mathrm{Lip}_0(\Omega)\,.$$

For example, if $\varphi(t)=Ct^{1/D}$, $D>p$, (F^p_φ) is equivalent to the Sobolev inequality

$$\|f\|_{pD/(D-p)}\le C'\||\nabla f|\|_p\,,\quad\forall f\in\mathcal{C}_0^\infty(M)\,. \tag{S^p_D}$$

For more information, see [Co1,2], [BaCLS]. It is easy to see that (S^p_D), $p>D$, implies the p-hyperbolicity of M (in fact a localization at infinity of this inequality, $(S^p_D)(\infty)$, in the terminology of [CoS2], see also [Co2], is enough). More generally, (F^p_φ) implies the p-hyperbolicity as soon as $\varphi(t)\le Ct^{1/D}$, for some $D>p$ and every $t\ge1$. The above theorem shows that it is enough for $\varphi(t)$ to be just below $t^{1/p}$ for large t. Indeed, M always satisfies (F^p_φ) with $\varphi=\varphi_p$, and conversely, if M satisfies (F^p_φ), then $\varphi_p\le\varphi$.

Proof of Theorem 3.3. Take $f\in\mathcal{C}_0^\infty(M)$, non-negative. Let U be a relatively compact open set in M, where f is non-identically zero. Set

$$a_0=v\big(\{x;f(x)\ge\tfrac{f_U}{2}\}\big)>0\,,$$

where $f_U=\frac{1}{U}\int_U f\,dv$.

Either

$$v\big(\{x;f(x)>0\}\big)\le a_0\,,$$

or there exists λ_1, $0<\lambda_1<\lambda_0=f_U/2$ such that

$$a_0\le v\big(\{x;f(x)\ge\lambda_1\}\big)\le 2a_0\,.$$

Here we use the fact that $\lambda\to v(\{x;f(x)\ge\lambda\})$ is non-increasing and left continuous. In the first case, we have

$$\|f\|_p\le\varphi_p(a_0)\||\nabla f|\|_p\,,$$

therefore

$$\lambda_0 a_0^{1/p}\le\varphi_p(a_0)\||\nabla f|\|_p\,,$$

and we stop the construction there.

In the second case, set $a_1=2a_0$ and $f_1=(f-\lambda_1)_+\wedge(\lambda_0-\lambda_1)$. We have then

$$(\lambda_0-\lambda_1)a_0^{1/p}\le\|f_1\|_p\le\varphi_p(a_1)\||\nabla f_1|\|_p\,.$$

Now set $a_k=2^k a_0$, $k\in\mathbb{N}^*$. Suppose that λ_ℓ and $f_\ell=(f-\lambda_\ell)_+\wedge(\lambda_{\ell-1}-\lambda_\ell)$ have been constructed for $\ell=1,\dots,k$ such that

$$a_{\ell-1}\le v\big(\{x;f(x)\ge\lambda_\ell\}\big)\le a_\ell\,.$$

Either

$$v\big(\{x; f(x) \geq 0\}\big) \leq a_k\,,$$

or there exists λ_{k+1}, $0 < \lambda_{k+1} < \lambda_k$ such that

$$a_k \leq v\big(\{x; f(x) \geq \lambda_{k+1}\}\big) \leq a_{k+1}\,.$$

In the first case, set $\lambda_{k+1} = 0$ and $f_{k+1} = f \wedge \lambda_k$, and stop the construction there (i.e. set $\lambda_\ell = 0$, $f_\ell = 0$, for $\ell \geq k+2$). In the second case, set $f_{k+1} = (f - \lambda_{k+1})_+ \wedge (\lambda_k - \lambda_{k+1})$. Now apply $(F^p_{\varphi_p})$ to f_{k+1}. In the first case, one obtains

$$\lambda_k a_{k-1}^{1/p} \leq \varphi_p(a_k)\||\nabla f_{k+1}|\|_p\,,$$

and in the second one

$$(\lambda_k - \lambda_{k+1})a_{k-1}^{1/p} \leq \varphi_p(a_{k+1})\||\nabla f_{k+1}|\|_p\,.$$

Finally in all cases and for all $k \in \mathbb{N}$, we obtain

$$2^{-1/p}(\lambda_k - \lambda_{k+1})a_k^{1/p} \leq \varphi_p(a_{k+1})\||\nabla f_{k+1}|\|_p\,.$$

Note that the construction eventually falls into the first case since f has compact support.

We can therefore write

$$\lambda_0 = \sum_{k=0}^{+\infty}(\lambda_k - \lambda_{k+1}) \leq 2^{1/p}\sum_{k=0}^{+\infty}\varphi_p(a_{k+1})a_k^{-1/p}\||\nabla f_{k+1}|\|_p\,.$$

Since $a_k = 2^k a_0$, one has $a_k^{-1/p} = 2^{-1/p}(a_{k+2} - a_{k+1})^{(p-1)/p}a_{k+1}^{-1}$. Thus

$$\begin{aligned}\lambda_0 &\leq \sum_{k=0}^{+\infty}\frac{\varphi_p(a_{k+1})}{a_{k+1}}(a_{k+2} - a_{k+1})^{(p-1)/p}\||\nabla f_{k+1}|\|_p\\ &\leq \left(\sum_{k=0}^{+\infty}\left(\frac{\varphi_p(a_{k+1})}{a_{k+1}}\right)^{p/(p-1)}(a_{k+2} - a_{k+1})\right)^{(p-1)/p}\left(\sum_{k=0}^{+\infty}\||\nabla f_{k+1}|\|_p^p\right)^{1/p}.\end{aligned}$$

Then apply the hypothesis on φ_p and the fact that

$$\left(\sum_{k=0}^{+\infty}\||\nabla f_{k+1}|\|_p^p\right)^{1/p} \leq \||\nabla f|\|_p$$

(see [BaCLS] for remarks on this property) to conclude that

$$f_U \leq C\left(\int_{a_0}^{+\infty}\left(\frac{\varphi_p(t)}{t}\right)^{p/(p-1)}dt\right)\||\nabla f|\|_p\,.$$

We have almost reached the conclusion, except that we must get rid of the dependence on a_0 in the right hand side by invoking the assumption on the local geometry on M. Since

$$\int_U |f-f_U|^p dv \le C_U \int_M |\nabla f|^p dv\,,$$

one has

$$f_U^p\big(|U|-a_0\big) \le f_U^p v\big(\{x\in U; f(x)\le \tfrac{f_U}{2}\}\big) \le C'_U \||\nabla f|\|_p^p\,.$$

Finally either $a_0 \ge |U|/2$, and we have proved that

$$f_U \le C\bigg(\int_{|U|/2}^{+\infty}\Big(\frac{\varphi_p(t)}{t}\Big)^{p/(p-1)} dt\bigg)\||\nabla f|\|_p\,,$$

or $a_0 \le |U|/2$, and

$$f_U \le \big(\tfrac{2}{|U|}\big)^p (C'_U)^{1/p}\||\nabla f|\|_p\,.$$

We have till now supposed f non-negative, but since $|\nabla f| = |\nabla |f||$ a.e., we have proved that for any relatively compact open subset U of M,

$$\tfrac{1}{|U|}\int_U |f|dv \le C_U\||\nabla f|\|_p\,,\quad \forall\, f\in \mathcal{C}_0^\infty(M)\,.$$

Therefore M is p-hyperbolic.

REMARKS. (1) If a local Poincaré inequality of the form

$$\int_{B_x(1)} |f-f_{B_x(1)}|^p \le C\int_{B_x(1)} |\nabla f|^p$$

holds uniformly on M, our proof yields the following uniform statement:

$$\frac{1}{V_x(1)}\int_{B_x(1)} |f| \le C\||\nabla f|\|_p\,,\quad \forall\, x\in M\,,\ \forall\, f\in\mathcal{C}_0^\infty(M)\,.$$

(2) If we assume that

$$\int_0^{+\infty}\Big(\frac{\varphi_p(t)}{t}\Big)^{p/(p-1)} dt < +\infty\,,$$

we obtain

$$\|f\|_\infty \le C\||\nabla f|\|_p\,,\quad \forall\, f\in\mathcal{C}_0^\infty(M)\,.$$

The additional hypothesis

$$\int_0^{1}\Big(\frac{\varphi_p(t)}{t}\Big)^{p/(p-1)} dt < +\infty$$

is an assumption on the local isoperimetric profile of M. It is satisfied if M has bounded local geometry and topological dimension greater than p.

Let us compare the necessary condition for p-hyperbolicity given by Corollary 3.2 with the sufficient condition given by Theorem 3.3. Suppose that the volume growth is uniform on M, in the sense that $V_x(r)\simeq V(r)$,

$\forall x \in M$. Suppose further that an L^p pseudo-Poincaré inequality holds on M, i.e.

$$\|f - f_r\|_p \le Cr\||\nabla f|\|_p\,, \quad \forall f \in \mathcal{C}_0^\infty(M)\,,\ r > 0\,, \qquad (PP_p)$$

where $f_r(x) = \frac{1}{V_x(r)} \int_{B_x(r)} f(y)\,dy$. This is the case for instance if M is a Lie group or has non-negative Ricci curvature. Then M satisfies (F^p_φ) with $\varphi = V^{-1}$, up to multiplicative constants (see [CoS1], and [Co2]). In this situation we have the following statement.

Proposition 3.4. *Assume that there exists a C^1 function V on $\mathbb{R}_+$, strictly increasing to $+\infty$, such that*

$$cV(cr) \le V_x(r) \le CV(Cr)\,, \quad \forall\, x \in M\,,\ r \ge 1\,,$$

and that M satisfies the pseudo-Poincaré inequality (PP_p). Then (M,Δ) is p-hyperbolic if and only if

$$\int_1^{+\infty} \left(\frac{r}{V(r)}\right)^{1/(p-1)} dr < +\infty\,.$$

Proof. The necessity follows from Corollary 3.2. Suppose now that

$$\int_1^{+\infty} \left(\frac{r}{V(r)}\right)^{1/(p-1)} dr < +\infty\,.$$

Integration by parts shows that

$$\int_1^{+\infty} \left(\frac{r}{V(r)}\right)^{p/(p-1)} V'(r)dr < +\infty\,,$$

and the change of variable $t = V(r)$ that

$$\int_{V(1)}^{+\infty} \left(\frac{\varphi(t)}{t}\right)^{p/(p-1)} dt < +\infty\,,$$

where $\varphi = V^{-1}$. Now, as we explained above, it follows from the assumptions that $\varphi_p \le \varphi$ up to multiplicative constants. Therefore, according to Theorem 3.3, M is p-hyperbolic.

Remark. If M is a regular cover of a compact manifold, a discrete version of (PP_p) holds on the deck transformation group Γ of M, and the conclusion of Proposition 3.4 follows with V the volume growth function of Γ; see [CoS1] and [Co2].

The following related result has been proved in [Ho4] (in a Riemannian setting): if M satisfies [D] and [P(p)], then it is p-hyperbolic if and only if

$$\int_1^{+\infty} \left(\frac{r}{V_x(r)}\right)^{1/(p-1)} dr < +\infty$$

for some (all) $x \in M$. This result generalizes [Va1, Thm. 2] that treats the case where $p = 2$ and M has non-negative Ricci curvature. It is stronger than ours in the sense that it requires no assumption on the uniformity of the volume growth, but on the other hand (PP_p) may hold in situations where [D] and [P(p)] do not hold, for instance on Lie groups with exponential volume growth.

Another version of the proof of Theorem 3.3 yields the following capacity lower bound, which is of interest in itself. This essentially solves Problem 22 in [Gri2].

Theorem 3.5. *Let $1 < p < +\infty$. Suppose that $G \subset M$ is a relatively compact domain and that $C \subset G$ is compact. Then*

$$\mathrm{Cap}_p(G,C) \geq 2^{-1-2p}\left(\int_{2|C|}^{4|G|}\left(\frac{\varphi_p(t)}{t}\right)^{p/(p-1)} dt\right)^{1-p}. \tag{3.14}$$

In particular, (M,Δ) is p-hyperbolic if

$$\int^{+\infty}\left(\frac{\varphi_p(t)}{t}\right)^{p/(p-1)} dt < +\infty\,,$$

in which case

$$\mathrm{Cap}_p(M,C) \geq 2^{-1-2p}\left(\int_{2|C|}^{+\infty}\left(\frac{\varphi_p(t)}{t}\right)^{p/(p-1)} dt\right)^{1-p}.$$

Proof. Fix $\varepsilon > 0$ and choose a function $u \in C_0^\infty(G)$ such that $u \equiv 1$ in C and

$$\mathrm{Cap}_p(G,C) \geq \int_M |\nabla u|^p\, dv - \varepsilon\,. \tag{3.15}$$

We may assume that $0 \leq u \leq 1$. For each $0 \leq t \leq 1$, write $U(t) = \{x; u(x) > t\}$, and $C(t) = \{x; u(x) \geq t\}$. Let $\lambda_0 = 1/2$, $a_0 = |U(1/2)|$, and, for each integer $i \geq 1$, let

$$\lambda_i = \inf\left\{t > 0\,; |C(t)| \leq 2^i a_0\right\}.$$

We claim that

(i) λ_i is non-increasing in i, $\lambda_i \leq 1/2$, and
(ii) $|U(\lambda_i)| \leq 2^i a_0 \leq |C(\lambda_i)|$.

Claim (i) is obvious. Suppose for a while (ii) holds as well. Fix an integer k such that $2^k a_0 < |G| \leq 2^{k+1} a_0$. For $i = 1, 2, \ldots, k+1$, we set

$$u_i = (u - \lambda_i)_+ \wedge \lambda_{i-1}\,.$$

Observe that $\lambda_{k+1} = 0$. Next we show that, for $i = 1, \ldots, k+1$,

$$\||\nabla u_i|\|_p\, \varphi_p(2^i a_0) \geq (\lambda_{i-1} - \lambda_i)\big(2^{i-1} a_0\big)^{1/p}. \tag{3.16}$$

Since $\{x; u_i(x) > 0\} \subset U(\lambda_i)$, we have

$$\left|\{x; u_i(x) > 0\}\right| \leq 2^i a_0$$

by the left-hand side of (ii). Hence

$$\| |\nabla u_i| \|_p \, \varphi_p(2^i a_0) \geq \|u_i\|_p \, .$$

Since $u_i(x) \geq \lambda_{i-1} - \lambda_i$ if $x \in C(\lambda_{i-1})$, we obtain

$$\|u_i\|_p \geq \big(\lambda_{i-1} - \lambda_i\big) \left|C(\lambda_{i-1})\right|^{1/p} \geq \big(\lambda_{i-1} - \lambda_i\big) \big(2^{i-1} a_0\big)^{1/p} ,$$

where the last inequality follows from the right-hand side of (ii). Thus (3.16) follows. By using (3.16) and Hölder's inequality, we obtain

$$\begin{aligned}
1/2 = \lambda_0 &= \sum_{i=1}^{k+1} (\lambda_{i-1} - \lambda_i) \leq \sum_{i=0}^{k+1} \varphi_p(2^i a_0) \big(2^{i-1} a_0\big)^{-1/p} \| |\nabla u_i| \|_p \\
&\leq 2^{1+1/p} \sum_{i=1}^{k+1} \frac{\varphi_p(2^i a_0)}{2^{i+1} a_0} \\
&\leq 2^{1+1/p} \left(\sum_{i=1}^{k+1} \left(\frac{\varphi_p(2^i a_0)}{2^{i+1} a_0} \right)^{\frac{p}{p-1}} \big(2^{i+1} a_0 - 2^i a_0\big) \right)^{\frac{p-1}{p}} \left(\sum_{i=1}^{k+1} \| |\nabla u_i| \|_p^p \right)^{\frac{1}{p}} \\
&\leq 2^{1+1/p} \left(\sum_{i=1}^{k+1} \int_{2^i a_0}^{2^{i+1} a_0} \left(\frac{\varphi_p(t)}{t} \right)^{p/(p-1)} dt \right)^{(p-1)/p} \| |\nabla u| \|_p \\
&= 2^{1+1/p} \left(\int_{2a_0}^{2^{k+2} a_0} \left(\frac{\varphi_p(t)}{t} \right)^{p/(p-1)} dt \right)^{(p-1)/p} \left(\int_M |\nabla u|^p \, dv \right)^{1/p} .
\end{aligned}$$

Hence

$$\int_M |\nabla u|^p \, dv \geq 2^{-1-2p} \left(\int_{2a_0}^{2^{k+2} a_0} \left(\frac{\varphi_p(t)}{t} \right)^{p/(p-1)} dt \right)^{1-p} . \tag{3.17}$$

The estimate (3.14) then follows from (3.15) and (3.17) by observing that

$$2a_0 = 2\,|U(1/2)| \geq 2\,|C|$$

and

$$2^{k+2} a_0 \leq 4\,|G|$$

and letting $\varepsilon \to 0$.

It remains to prove (ii). Take a sequence $t_{i,j} \nearrow \lambda_i$. Clearly $C(\lambda_i) = \cap_j C(t_{i,j})$. Now, if $t_{i,j} < \lambda_i$, one has by definition

$$|C(t_{i,j})| \geq 2^i a_0 \, .$$

The right-hand side of (ii) follows since

$$|C(\lambda_i)| = \lim_{j \to +\infty} |C(t_{i,j})| \, .$$

The left-hand side of (ii) follows similarly, by taking a sequence $t_{i,j} \searrow \lambda_i$. The theorem is proven.

REMARKS. (1) Condition (3.12) in Theorem 3.3 is not necessary for M to be p-hyperbolic. To see this, we first observe that, for the standard cylinder $\mathbb{R} \times S^{n-1}$, $\varphi_p(t) \geq c\,t$ if t is large enough. A similar estimate remains true if we glue $\mathbb{R} \times S^{n-1}$ with the hyperbolic space H^n to obtain $M = H^n \#(\mathbb{R} \times S^{n-1})$. Now the hyperbolic end, H^n minus a compact set, makes M p-hyperbolic for every $1 < p < +\infty$ while condition (3.12) fails to be true.

(2) The results of section 3.3 also hold in the setting of infinite graphs. Our methods enable one to replace the above p-isoperimetric profile by a *pointed* (or anchored) p-isoperimetric profile where one restricts the supremum in (3.11) to functions whose support is connected and contains a fixed point o. We can therefore improve and generalize the result in [T] (see also [MarMT]), where a transience (2-hyperbolicity) criterion for graphs is formulated in terms of a 1-isoperimetric profile. In a continuous setting, the pointed isoperimetric profile makes no difference since one can always join the connected components of a given set to a fixed point by thin tubes whose measure and surface measure is negligible if the dimension is greater than three (we owe this remark to A. Grigor'yan). But one could still make sense of this by discretization, using, in our general setting, the techniques of [CoS2]. The most general version of the p-hyperbolicity criterion would then be formulated in terms of a modified (in the sense of [ChF]) and pointed p-isoperimetric profile.

4 Harnack Inequalities, Liouville Property

We shall say that (M, Δ) has the (strong) p-Liouville property if every non-negative p-harmonic function on M is constant. This property follows from a uniform elliptic Harnack inequality for the p-Laplace operator. Such inequalities have already been proved in many settings: for a class of elliptic operators generalizing the ordinary p-Laplace operator in the Euclidean space (see the forerunner papers [Mo], for $p = 2$, then [Se] and [Tru] for general p, and the book [HeKM] for an up-to-date exposition), in [CDG] for a class of subelliptic operators in $\mathbb{R}^n$, in [RigSV] for the p-Laplace operator and in [HoR2] for a class of elliptic operators on Riemannian manifolds satisfying [D] and [P(p)] (partial results already appeared in [Ho2]), in [HoSo] on graphs, and more recently in [KiS] for p-quasiminimizers on metric spaces.

4.1 Harnack inequalities under quadratic growth. This subsection contains two results that give Harnack inequalities for p-harmonic functions under the assumption that $\sup_{t>1} t^{-2}V_x(t) < +\infty$, i.e. M has at most quadratic growth at infinity. The first theorem is restricted to dimension 2 and to the Riemannian setting. The second theorem has no dimensional restriction and works in our general subelliptic framework but requires some kind of bounded geometry.

Theorem 4.1. *Assume that (M, g) is a two-dimensional Riemannian manifold with Riemannian measure μ and let o be a fixed point in M. Assume that $dv = m d\mu$ where m is a smooth function bounded below by a positive constant c_0 and that Δ has the form $\Delta = -m^{-1}\mathrm{div}(m\mathcal{B}\,\mathrm{grad})$ for some measurable section $\mathcal{B}$ of the bundle of symmetric endomorphisms such that*

$$C_0^{-2} g \le g(\mathcal{B}\ \cdot\ , \cdot\,) \le C_0^2 g\,.$$

Assume finally that the boundary of any geodesic ball $B_o(t)$ in (M, g) is connected and that

$$\sup_{t>0} t^{-2} V_o(t) = Q < +\infty$$

where $V_o(t) = v(B_o(t))$. Then, for every $R > 0$, any positive p-harmonic function u on $B_o(R)$ satisfies

$$\sup_{B_o(R/2)} u \le C \inf_{B_o(R/2)} u$$

with $C = \exp(16 c_0^{-1} C_0 Q (p-1)^{-1/p})$. In particular, (M, Δ) is p-Liouville.

Remark. For $p = 2$, $\mathcal{B} \equiv \mathrm{Id}$, $m \equiv 1$, this statement is in [ChenY, Prop. 6]. If $p > 2$, the conclusion that M is p-Liouville already follows from Corollary 3.2 since a p-parabolic manifold, admitting no non-trivial positive supersolution, is certainly p-Liouville. For $p \in (1, 2)$, Corollary 3.2 requires a much stronger volume upper bound than the one needed in Theorem 4.1.

Proof. For ease, set $B(s) = B_o(s), V(s) = V_o(s)$. Theorem 3.1 with $\ell = 1$, $r_0 = 3R/4$, $r_1 = R$ implies

$$(p-1)\int_{B(3R/4)} |\nabla \log u|^p dv \le \frac{4^p C_0^p V(R)}{R^p}$$

where the constant C_0 appears because we are using the gradient relative to Δ but the geodesic balls of (M, g). Hence

$$\int_{B(3R/4)} |\nabla \log u|\, dv \le V(R)^{1-\frac{1}{p}} \left(\int_{B(3R/4)} |\nabla \log u|^p dv \right)^{\frac{1}{p}} \le \frac{4 C_0 V(R)}{(p-1)^{1/p} R}\,.$$

It follows that there exists $t \in [R/2, 3R/4]$ such that

$$\int_{\partial B(t)} |\nabla \log u| m \, d\sigma \leq \frac{16 C_0 V(R)}{(p-1)^{1/p} R^2}$$

where $d\sigma$ denote the Riemannian length measure on curves. Thus

$$\int_{\partial B(t)} |\nabla \log u| d\sigma \leq 16 c_0^{-1} C_0 Q (p-1)^{-1/p}. \tag{4.1}$$

Let z_+, z_- be two points such that

$$u(z_+) = \sup_{\partial B(t)} u\,, \quad u(z_-) = \inf_{\partial B(t)} u\,.$$

By the maximum principle (see [HeKM]),

$$\frac{\sup_{B(R/2)} u}{\inf_{B(R/2)} u} \leq \frac{u(z_+)}{u(z_-)}\,.$$

Now, using the fact that $\partial B(t)$ is connected and one dimensional, (4.1) yields

$$\log \frac{u(z_+)}{u(z_-)} \leq \int_{\partial B(t)} |\nabla \log u| d\sigma \leq 16 c_0^{-1} C_0 Q (p-1)^{-1/p}.$$

It follows that

$$\frac{\sup_{B(R/2)} u}{\inf_{B(R/2)} u} \leq \exp\left(16 c_0^{-1} C_0 Q (p-1)^{-1/p}\right).$$

In the next theorem, we drop the assumption that M is 2-dimensional, and that Δ is (essentially) Riemannian. We also relax somewhat the assumption that the boundary of balls are connected. Assumptions 2–4 in the following statement may be seen as bounded geometry hypotheses.

Theorem 4.2. *Let (M, Δ) be defined as in section 2.6. Assume that there exists $s > 0$ such that:*

1. *For any $r > s$ and any two points x, y such that $\rho(o,x) = \rho(o,y) = r$, there is a continuous path $\gamma : [0,1] \to M$ with $\gamma(0) = x$, $\gamma(1) = y$ and*
$$\forall\, t \in [0,1]\,, \quad \gamma(t) \subset B_o(r+s) \setminus B_o(r-s)\,;$$
2. *There exists $c_1, C_1 > 0$ such that $c_1 \leq V_x(s)$ and $V_x(2t) \leq C_1 V_x(t)$ for all $x \in M$ and all $t \in [10^{-2}s, 10^2 s]$;*
3. *There exists $C_2 > 0$ such that, for all $x \in M$, $t \in [10^{-2}s, 10^2 s]$ and $B = B_x(t)$,*
$$\forall\, f \in \mathcal{C}^\infty(2B)\,, \quad \int_B |f - f_B| dv \leq C_2 \int_{2B} |\nabla f| dv\,.$$

4. *There exists C_3 such that any positive p-harmonic function in $B_x(2s)$ satisfies*

$$\sup_{B_x(s)} u \le C_3 \inf_{B_x(s)} u \,.$$

Assume finally that $\sup_{t>1} t^{-2} V_o(t) = Q < +\infty$*. Then there exists a constant C such that any positive p-harmonic function u on $B_o(R)$, $R \ge 128\, s$, satisfies*

$$\sup_{B_o(R/2)} u \le C \inf_{B_o(R/2)} u \,.$$

Proof. Set $B(s) = B_o(s), V(s) = V_o(s)$. Just as in the previous proof, we have

$$\int_{B(3R/4)} |\nabla \log u| \, dv \le \frac{4V(R)}{(p-1)^{1/p} R} \,.$$

Set $A_i = B(t_{i+1}) \setminus B(t_i)$ where $t_i = (R/2) + 16\, i\, s$, $0 \le i < \ell = \lfloor R/(64s) \rfloor$. Then,

$$\sum_{i=0}^{\ell-1} \int_{A_i} |\nabla \log u| \, dv \le \frac{4V(R)}{(p-1)^{1/p} R} \,.$$

It follows that there exists a $j \in \{t_0, \dots, t_{\ell-1}\}$ such that

$$\int_{A_j} |\nabla \log u| dv \le \frac{4V(R)}{(p-1)^{1/p} R\ell} \le \frac{10^3 Q s}{(p-1)^{1/p}} \,. \tag{4.2}$$

Set $t = t_j + 8s$, $S = \{z : \rho(o, z) = t\}$ and $A = \{z : \rho(z, S) \le s\} \subset A_j$. Fix a maximal $s/2$-separated set $X = \{x_0, \dots, x_m\}$ in A (crucial to the proof is the fact that all estimates made below are independent of the number m of points in X). By construction, the balls $B_{x_i}(s)$ cover A and the balls $B_{x_i}(6s)$ are contained in A_j. Moreover, for any given i, all the balls $B_{x_j}(6s)$ that intersect $B_{x_i}(6s)$ are contained in $B_{x_i}(12s)$. As the balls $B_{x_j}(s/2)$ are disjoint, it follows from the volume hypothesis that there are at most C_1^6 balls $B_{x_j}(6s)$ that intersect $B_{x_i}(6s)$, that is,

$$\#\{j \in \{1, \dots, m\} : B_{x_j}(6s) \cap B_{x_i}(6s) \ne \emptyset\} \le C_1^6 \,. \tag{4.3}$$

Say that $x, y \in X$ are neighbors if $\rho(x, y) \le 2s$. We claim that the graph with vertex-set X and an edge from x to y if x, y are neighbors is connected. To prove this claim, let x, y be two elements of X. By construction there exists x', y' such that $x' \in S \cup B_x(s)$, $y' \in S \cup B_y(s)$. By our hypotheses, there is a continuous path joining x' to y' and contained in A. Considering the centers of the balls $B_{x_i}(s)$ that one successively enters when moving along the path from x to y proves the claim.

Now, let z_+, z_- be two points such that

$$u(z_+) = \sup_S u\,, \quad u(z_-) = \inf_S u\,.$$

By the maximum principle,

$$\frac{\sup_{B_o(R/2)} u}{\inf_{B_o(R/2)} u} \le \frac{u(z_+)}{u(z_-)}\,.$$

Further, there exist i_+, i_- such that $z_\pm \in B_{x_{i_\pm}}(s)$. Set $v = \log u$. Lemma 5.3 of [CoS2] gives, for x_i, x_j neighbors,

$$|v_i - v_j| \le 2c_1^{-1}C_1^2C_2 \int_{B_{x_i}(6s)} |\nabla v| dv$$

where v_i is the mean of v over $B_{x_i}(s)$. Here, to apply Lemma 5.3 of [CoS2], we have used hypotheses 2 and 3 of Theorem 4.2. Next, since the graph X is connected and thanks to (4.3),

$$|v_{i_+} - v_{i_-}| \le 2c_1^{-1}C_1^2C_2 \sum_0^m \int_{B_{x_i}(6s)} |\nabla v| dv \le 2c_1^{-1}C_1^8C_2 \int_{A_j} |\nabla v| dv\,.$$

By (4.2), this yields

$$|v_{i_+} - v_{i_-}| \le 10^4 c_1^{-1}C_1^8C_2(p-1)^{-1/p}Qs = C\,.$$

Let ξ be the maximum of u in $B_{x_{i+}}$ and ζ the minimum of u in $B_{x_{i-}}$. Then

$$|\log\xi - \log\zeta| \le |v_{i_+} - v_{i_-}| \le C\,,$$

that is, $\xi \le e^C\zeta$. By assumption 4 of Theorem 4.2 (local Harnack inequality), it follows that

$$u(z_+) \le C_3^2 e^C u(z_-)\,.$$

This gives the desired result.

Remark. The main ingredient in the above proofs is Cheng–Yau's inequality (3.1). As we pointed out at the end of section 3.1 (3.1) may be seen as a consequence of the logarithmic Caccioppoli inequality (3.4). As a matter of fact, (3.4) has earlier been used by Granlund ([Gr]) to obtain Harnack's inequality in $\mathbb{R}^n$ for the n-Laplacian, see also [Ri4, Thm. VI.7.4], and [Ho1, Thm. 5.14].

4.2 A Sobolev inequality. In the next sections, we shall run Moser iteration as in [S1] in order to obtain a Harnack inequality for p-harmonic functions, under the assumption that doubling volume and suitable Poincaré inequalities hold.

The plan of the proof is as follows: doubling and Poincaré imply a family of Sobolev inequalities on balls (see Lemma 4.3 below). From such an inequality, choosing suitable test-functions and using an iterative argument, one estimates the supremum of a supersolution by its L^p norm, and, through an additional iteration inspired from Li-Schoen ([LiS]), by its L^q norm, $0 < q < +\infty$. The companion lower estimate of the infimum of a subsolution by an L^q norm is more involved; it goes through preliminary estimates obtained, first by another choice of test functions, second by another kind of iterative argument, and relies on an abstract lemma of Bombieri–Giusti ([BoG]) that enables one to by-pass the John–Nirenberg lemma. Let us emphasize that we do not assume a priori that our solutions are continuous nor locally bounded, but this is part of the conclusion.

Let us finally point out that instead of Moser's ideas, one could adapt De Giorgi's method as exposed in [Gi] (see [AuC], [KiS] for recent instances of such adaptation).

For $\Omega \subset M$, let $W_0^{1,p}(\Omega)$ be the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $\left(\int_M |\varphi|^p dv + \int_M |\nabla\varphi|^p dv\right)^{1/p}$.

Lemma 4.3. *Fix $R > 0$. Assume that the Poincaré inequalities on balls $[\mathrm{P}(p,R)]$ and the doubling condition $[\mathrm{D}(R)]$ are satisfied. Then there exists $k > 1$ and $S > 0$ such that, for all $x \in M$, all $r \in (0,R)$, all $f \in W_0^{1,p}(B_x(r))$,*

$$\left(\int_{B_x(r)} |f|^{pk} dv\right)^{1/k} \leq \frac{Sr^p}{V_x(r)^{p/\nu}} \int_{B_x(r)} \left[|\nabla f|^p + \frac{1}{r^p}|f|^p\right] dv\,.$$

One can take $k = \nu/(\nu - p)$ with $\nu = \max\{p+1, \log_2 D\}$ where D is the doubling constant; S only depends on D and P_p.

See for instance [BaCLS], [BiM2], [CoG], [HK1], [MS]. The term $\int_B |f|^p\, dv$ can be disposed with by applying Poincaré once again if the space is not compact (if $B_x(2r)$ is not all the space). The condition that f has compact support in $B_x(r)$ can also be disposed with (but we do not need this fact).

4.3 Subsolutions.

Let (M,Δ) be as in section 2.6.

Theorem 4.4. *Fix $B = B_x(r)$ and set $V = V_x(r) = v(B)$. Assume there exists $k > 1$ and $S > 0$ such that, for all $f \in W_0^{1,p}(B)$,*

$$\left(\int_B |f|^{pk} dv\right)^{1/k} \leq \frac{Sr^p}{V^{p/\nu}} \int_B \left[|\nabla f|^p + \frac{1}{r^p}|f|^p\right] dv \tag{4.4}$$

where $\nu = kp/(k-1)$. Then any positive p-subsolution u in B satisfies, for all $q \geq p$ and $0 < \delta < 1$,

$$\sup_{\delta B}\{u^q\} \leq A(p,q,k) S^{\nu/p} (1-\delta)^{-\nu} \frac{1}{V} \int_B u^q dv\,.$$

Proof. For any non-negative test-function $\varphi \in W_0^{1,p}(M,\Delta)$ with compact support in B,

$$\int_B \Gamma(\varphi,u)|\nabla u|^{p-2}dv = \langle \Delta_p u, \varphi\rangle \le 0\,.$$

We shall apply this to a test-function of the form $\varphi = \psi^p G(u)$, with ψ non-negative, and G a differentiable non-negative function to be chosen later. Since

$$\Gamma\big(\psi^p G(u), u\big) = \psi^p G'(u)|\nabla u|^2 + p\psi^{p-1}G(u)\Gamma(\psi,u)\,,$$

we obtain

$$\int_B \psi^p G'(u)|\nabla u|^p\,dv \le p\int_B \psi^{p-1}G(u)|\nabla\psi||\nabla u|^{p-1}dv\,.$$

Assume now that

$$G(t) \le tG'(t)\,,\quad t\ge 0\,. \tag{4.5}$$

It follows that

$$\begin{aligned}\int_B \psi^p G'(u)|\nabla u|^p dv &\le p\int_B \psi^{p-1}uG'(u)|\nabla\psi||\nabla u|^{p-1}dv\\ &\le p\bigg(\int_B \psi^p G'(u)|\nabla u|^p dv\bigg)^{(p-1)/p}\bigg(\int_B |\nabla\psi|^p u^p G'(u)dv\bigg)^{1/p}.\end{aligned}$$

Hence

$$\int_B \psi^p G'(u)|\nabla u|^p dv \le p^p\int_B |\nabla\psi|^p u^p G'(u)dv\,.$$

Now, define $H(t) = \int_0^t G'(s)^{1/p}ds$ so that $G'(u) = H'(u)^p$ and, by Jensen and (4.5), $H(u)\le uH'(u)$. One has

$$\nabla(\psi H(u)) = \psi H'(u)\nabla u + H(u)\nabla\psi$$

therefore

$$\begin{aligned}|\nabla(\psi H(u))|^p &\le 2^{p-1}\big(\psi^p H'(u)^p|\nabla u^p| + H(u)^p|\nabla\psi|^p\big)\\ &\le 2^{p-1}\big(\psi^p G'(u)|\nabla u^p| + u^p G'(u)|\nabla\psi|^p\big)\,.\end{aligned}$$

Thus, if one sets $A_p = 2^{p-1}(1+p^p)$,

$$\int_B \big|\nabla(\psi H(u))\big|^p \le A_p\int_B |\nabla\psi|^p u^p G'(u)dv\,.$$

Setting $\nu = kp/(k-1)$, i.e. $k = \nu/(\nu-p)$, and $\theta = 1 + p/\nu = 2-(1/k)$, (4.4) gives

$$\bigg(\int_B (\psi H(u))^{pk}dv\bigg)^{1/k} \le SV^{-p/\nu}\bigg(A_p r^p\int_B u^p G'(u)|\nabla\psi|^p + (\psi H(u))^p dv\bigg)$$

Fix $0 < \delta < \delta' \le 1$ and pick ψ so that $0 \le \psi \le 1$, $\psi = 1$ on δB, $\psi = 0$ on $(\delta' B)^c$ and $|\nabla\psi| \le [(\delta'-\delta)r]^{-1}$. This gives

$$\left(\int_{\delta B} |H(u)|^{pk} dv\right)^{1/k} \le A'_p S V^{-p/\nu} (\delta'-\delta)^{-p} \int_{\delta' B} u^p G'(u) dv\,.$$

Hölder yields

$$\int_{\delta B} |H(u)|^{p\theta} dv \le \left(\int_{\delta B} |H(u)|^{pk} dv\right)^{1/k} \left(\int_{\delta B} |H(u)|^{p} dv\right)^{(k-1)/k},$$

therefore, using the above inequality and $H(u)^p \le u^p G'(u)$ once again,

$$\int_{\delta B} |H(u)|^{p\theta} dv \le A'_p S V^{-p/\nu} (\delta'-\delta)^{-p} \left(\int_{\delta' B} u^p G'(u) dv\right)^{\theta}. \tag{4.6}$$

Now, for $\alpha \ge p$ and $N > 0$, set

$$H_N(s) = s^{\frac{\alpha}{p}} \text{ if } s \le N\,,\ N^{\frac{\alpha}{p}-1} s \text{ if } s \ge N\,.$$

An easy computation shows that

$$G_N(s) = \int_0^s H'_N(t)^p\, dt$$

equals

$$\left(\tfrac{\alpha}{p}\right)^p \frac{1}{\alpha - p + 1} s^{\alpha - p + 1}$$

if $s \le N$,

$$\left(\tfrac{\alpha}{p}\right)^p \frac{1}{\alpha - p + 1} N^{\alpha - p + 1} + N^{\alpha - p}(s - N)$$

if $s \ge N$. One checks easily that, for all $N > 0$, the functions G_N and H_N satisfy the properties required for G and H above, and that, if $\psi \in \mathcal{C}_0^\infty(B)$, then $\varphi = \psi^p G(u) \in W_0^{1,p}(M,\Delta)$. One can now let N go to $+\infty$ in (4.6) and obtain, for all $\alpha \ge p$,

$$\int_{\delta B} |u|^{\alpha\theta}\, dv \le A'_p S V^{-p/\nu} (\delta'-\delta)^{-p} \left(\left(\frac{\alpha}{p}\right)^p \int_{\delta' B} u^\alpha dv\right)^{\theta}.$$

Let $q \ge p$. Set $\alpha_i = q\theta^i$ and $\sigma_i = (1-\delta)2^{-i}$, $\delta_0 = 1$, $\delta_i = 1 - \sum_1^i \sigma_j$ so that $\lim_{+\infty} \delta_i = \delta$. The last inequality yields

$$\int_{\delta_{i+1} B} |u|^{q\theta^{i+1}} dv \le A'_p S V^{-p/\nu} (\delta_i - \delta_{i+1})^{-p} \left(\left(\frac{\alpha_i}{p}\right)^p \int_{\delta_i B} u^{q\theta^i} dv\right)^{\theta}.$$

This gives

$$\left(\int_{\delta_{i+1} B} |u|^{q\theta^{i+1}} dv\right)^{\theta^{-1-i}}$$

$$\leq (A'_p S V^{-p/\nu})^{\theta^{-1-i}} (1-\delta)^{-p\theta^{-1-i}} 2^{p(i+1)\theta^{-1-i}} \left(\frac{q\theta}{p}\right)^{ip\theta^{-i}} \left(\int_{\delta_i B} u^{p\theta^i} dv\right)^{\theta^{-i}}.$$

Hence

$$\left(\int_{\delta_{i+1}B} |u|^{q\theta^{i+1}} dv\right)^{\theta^{-1-i}}$$
$$\leq (A'_p S V^{-p/\nu})^{\sum \theta^{-j}} (1-\delta)^{-p\sum\theta^{-j}} 2^{p\sum j\theta^{-j}} \left(\frac{q\theta}{p}\right)^{p\sum j\theta^{-j}} \left(\int_B u^q dv\right) \tag{4.7}$$

where all the summations run from 1 to $i+1$. Finally

$$\sup_{\delta B}\{u^q\} \leq A(p,q,k) S^{\nu/p} (1-\delta)^{-\nu} \frac{1}{V} \int_B u^q dv\,.$$

This ends the proof.

Theorem 4.5. *Fix $B = B_x(r)$ and let $q \in (0,p)$. Assume the Sobolev inequality* (4.4). *Then any positive p-subsolution u in B satisfies*

$$\sup_{\delta B}\{u^q\} \leq \frac{C(p,q,k) S^{\nu/p} (1-\delta)^{-\nu}}{V} \int_B u^q dv\,.$$

Proof. Fix $1/2 < \sigma < 1$ and set $\sigma' = \sigma + (1-\sigma)/4$, $\tau = (1-\sigma)/4$. Then, Theorem 4.4 yields

$$\sup_{\sigma B}\{u\} \leq \tau^{-\nu/p} C(p,k) S^{\nu/p^2} V^{-1/p} \|u\|_{p,\sigma' B}\,.$$

where $\|u\|_{\alpha,B} = \left(\int_B |u|^\alpha dv\right)^{1/\alpha}$. Using $\|u\|_p \leq \|u\|_\infty^{1-q/p} \|u\|_q^{q/p}$ and setting

$$J = C(p,k) S^{\nu/p^2} V^{-1/p} \|u\|_{q,B}^{q/p}\,,$$

we obtain

$$\sup_{\sigma B}\{u\} \leq \tau^{-\nu/p} J \Big(\sup_{\sigma' B}\{u\}\Big)^{1-q/p}.$$

We fix $\delta > 1/2$ and set $\sigma_0 = \delta$, $\sigma_{i+1} = \sigma_i + (1-\sigma_i)/4$. This gives $1-\sigma_i = (3/4)^i(1-\delta)$. The above inequality yields

$$\sup_{\sigma_{i-1}B}\{u\} \leq 4^{\nu/p} \left(\tfrac{4}{3}\right)^{\nu i/p} (1-\delta)^{-\nu/p} J \Big(\sup_{\sigma_i B}\{u\}\Big)^{1-q/p}.$$

Hence

$$\sup_{\delta B}\{u\} \leq 4^{\frac{\nu}{p}\sum(j+1)(1-\nu/p)^j} \big((1-\delta)^{-\nu/p} J\big)^{\sum(1-q/p)^j} \Big(\sup_{\sigma_i B}\{u\}\Big)^{(1-q/p)^i}$$

where all the summations run from 0 to $i-1$. When i tends to infinity, we obtain

$$\sup_{\delta B}\{u\} \leq A(p,q,k)\big((1-\delta)^{-\frac{\nu}{p}} J\big)^{\frac{p}{q}} = C(p,q,k) S^{\nu/(p\alpha)} (1-\delta)^{-\frac{\nu}{q}} V^{-\frac{1}{q}} \|u\|_{q,B}\,.$$

4.4 Supersolutions. Let (M,Δ) be as in section 2.6.

Theorem 4.6. *Fix $B = B_x(r)$, and assume the Sobolev inequality* (4.4) *in B with constants S, k and $\nu = kp/(k-1)$. For all $0 < \delta < 1$, $0 < q < +\infty$, any positive p-supersolution u in B satisfies*

$$\sup_{\delta B}\{u^{-q}\} \le C(p,k) S^{\nu/p}(1-\delta)^{-\nu}\frac{1}{V}\int_B u^{-q}dv\,.$$

Remark. One could directly deduce a version of Theorem 4.6 from Theorem 4.4 by noticing that a negative power of a subsolution is a supersolution but this would not yield a constant C independent of q, which we will need below.

Proof. By replacing u with $u+\varepsilon$, one can assume that u is bounded away from 0. For any nonnegative test-function $\varphi \in W_0^{1,p}(M,\Delta)$ with compact support in B,

$$\int \Gamma(\varphi,u)|\nabla u|^{p-2}dv \ge 0\,. \tag{4.8}$$

Let $\beta < 0$, and $\psi \in \mathcal{C}_0^\infty(B)$; one checks easily that $\varphi = -\beta u^{\beta-p+1}\psi^p \in W_0^{1,p}(M,\Delta)$. Setting $w = u^{\beta/p}$, we obtain

$$-\beta(\beta-p+1)\left|\tfrac{p}{\beta}\right|^p\int \psi^p|\nabla w|^p dv - p^2\left|\tfrac{p}{\beta}\right|^{p-2}\int \psi^{p-1}w\Gamma(\psi,w)|\nabla w|^{p-2}dv \ge 0\,.$$

Hence

$$\int \psi^p|\nabla w|^p dv \le \int \psi^{p-1}w|\nabla\psi||\nabla w|^{p-1}dv$$

Here we used the fact that $|\beta|/|\beta-p+1| \le 1$ when $\beta < 0$, $p > 1$. Now, the Hölder inequality gives

$$\int \psi^p|\nabla w|^p dv \le \int |\nabla\psi|^p w^p dv\,.$$

From here, the arguments used for Theorem 4.4 apply, with w instead of u and $G(s) = s$, and give the desired result.

Theorem 4.7. *Fix $B = B_x(r)$, and assume the Sobolev inequality* (4.4) *with constants S, k. Fix $q_0 \in (0, \theta(p-1))$ where $\nu = kp/(k-1)$ and $\theta = 1 + p/\nu$. For all $0 < \delta < 1$ and all $q \in (0, q_0/\theta]$, any positive p-supersolution u in $B_x(r) = B$ satisfies*

$$\left(\int_{\delta B} u^{q_0}dv\right)^{1/q_0} \le \left[C(p,k,q_0)S^{2+\nu/p}(1-\delta)^{-2\nu+p}V^{-1}\right]^{\frac{1}{q}-\frac{1}{q_0}}\left(\int_B u^q dv\right)^{1/q}.$$

Proof. Again, one can assume that u is bounded away from 0. Let $0 < \beta \le q_0\theta^{-1} < (p-1)$ and $0 < \eta = 1 - q_0\theta^{-1}(p-1)^{-1}$. For $\psi \in \mathcal{C}_0^\infty(B)$,

$\varphi = \beta u^{\beta-p+1}\psi^p \in W_0^{1,p}(M,\Delta)$. Set now $w = u^{\beta/p}$; (4.8) yields

$$\eta \int \psi^p |\nabla w|^p dv \le \int \psi^{p-1} w |\nabla \psi| |\nabla w|^{p-1} dv$$

and thus

$$\int \psi^p |\nabla w|^p dv \le \eta^{-p} \int |\nabla \psi|^p w^p dv\,.$$

Using the Sobolev inequality (4.4) as in the proof of Theorem 4.4, we obtain

$$\int_{\sigma B} u^{\beta\theta} dv \le A(p,\eta) S V^{-p/\nu} (\sigma' - \sigma)^{-p} \left(\int_{\sigma' B} u^{\beta} dv \right)^{\theta} \tag{4.9}$$

for all $0 < \sigma < \sigma' \le 1$ and all $\beta \in (0, q_0\theta^{-1}]$. Now, define $q_i = q_0\theta^{-i}$; we shall first prove (a slightly stronger version of) the claimed inequality with $q = q_i$, $i = 1, 2, \ldots$ and then the full result will follow from Jensen's inequality. Thus, fix $i \ge 1$ and apply (4.9) with $\beta = q_i\theta^{j-1}$, $j = 1, \ldots, i$, and $\sigma' = \sigma_{j-1}, \sigma = \sigma_j$ where $\sigma_0 = 1$, $\sigma_{j-1} - \sigma_j = 2^{-j}(1-\delta)$. This yields (observe that $q_i\theta^{j-1} \le q_0/\theta$ for $j = 1, \ldots, i$ as required by (4.9))

$$\int_{\sigma_j B} u^{q_i\theta^j} dv \le A(p,\eta) S V^{-p/\nu} 2^{jp} (1-\delta)^{-p} \left(\int_{\sigma_{j-1} B} u^{q_i\theta^{j-1}} dv \right)^{\theta}$$

for $j = 1, \ldots, i$. Hence

$$\int_{\sigma_i B} u^{q_0} dv \le \left(A(p,\eta) S V^{-p/\nu} \right)^{\sum \theta^j} 2^{\sum (i-j)\theta^j} (1-\delta)^{-p\sum \theta^j} \left(\int_B u^{q_i} dv \right)^{\theta^i}$$

where all the summations run from 0 to $i-1$. Observe now that

$$\sum_0^{i-1} \theta^j = \frac{\theta^i - 1}{\theta - 1} = \frac{\nu}{p}\left(\frac{q_0}{q_i} - 1 \right),$$

$\sum_0^{i-1} (i-j)\theta^j \le C_\theta(\theta^i - 1) = C_\theta((q_0/q_i) - 1)$, and $\sigma_i = 1 - \left(\sum_1^i 2^{-i} \right)(1-\delta) > \delta$. This gives

$$\int_{\delta B} u^{q_0} dv \le \left(A(p,k,\eta) S^{\nu/p} (1-\delta)^{-\nu} V^{-1} \right)^{(q_0/q_i)-1} \left(\int_B u^{q_i} dv \right)^{q_0/q_i}.$$

This ends the proof.

4.5 Harnack inequalities. We start with the following abstract lemma, whose proof is given in [S3] and [S6] up to slight changes in notation; see also [BoG, Thm. 4]. Consider a collection of measurable subsets U_σ, $0 < \sigma \le 1$, of some fixed measure space endowed with a measure v, such that $U_{\sigma'} \subset U_\sigma$ if $\sigma' \le \sigma$. In our application, the space will be M with measure v and U_σ will be σB for some fixed B. For $U \subset M$ and $0 < \alpha \le +\infty$, and f a function defined on U, recall that $\|f\|_{\alpha,U}$ denotes $\left(\int_U |f|^\alpha dv \right)^{1/\alpha}$, with the obvious modification if $\alpha = +\infty$.

LEMMA 4.8. *Fix $0 < \sigma_0 \le 1$. Let γ, K be positive constants and $0 < \alpha_0 \le +\infty$. Let f be a positive integrable function on $U_1 = U$ which satisfies*

$$\|f\|_{\alpha_0, U_{\sigma'}} \le \left[K(\sigma - \sigma')^{-\gamma} v(U)^{-1}\right]^{1/\alpha - 1/\alpha_0} \|f\|_{\alpha, U_\sigma}\,,$$

for all σ, σ', α such that $0 < \sigma_0 \le \sigma' < \sigma \le 1$ and $0 < \alpha \le \min\{1, \alpha_0/2\}$. Assume further that f satisfies

$$\lambda v(\log f > \lambda) \le K\, v(U)$$

for all $\lambda > 0$. Then

$$\|f\|_{\alpha_0, U_{\sigma_0}} \le A v(U)^{1/\alpha_0}$$

where A depends on σ_0, γ, K, and a lower bound on α_0.

Let then (M, Δ) be as in section 2.6. First we derive a weak Harnack inequality for supersolutions.

Theorem 4.9. *Fix $R > 0$. Assume (M, Δ) satisfies the doubling property $[\mathrm{D}(R)]$ and the Poincaré inequality $[\mathrm{P}(p, R)]$ with constants D and P_p respectively. By Lemma 4.3, the Sobolev inequality (4.4) is satisfied for all $x \in M$, $r \in (0, R)$ with $k = \nu/(\nu - p)$, $\nu = \max\{p + 1, \log_2 D\}$, and S depending only on D and P_p. Let $0 < q < (1 + p/\nu)(p - 1)$ and $0 < \delta < 1$. Then, for all $x \in M$, $r \in (0, R)$, any positive p-supersolution u in $B = B_x(r)$ satisfies*

$$\frac{1}{v(\delta B)} \int_{\delta B} u^q\, dv \le C \inf_{\delta B}\{u^q\}\,.$$

Here C depends on p, D, P_p, q, δ.

Proof. We wish to apply Lemma 4.8 to $e^{-c}u$ and $e^c u^{-1}$ where c is a well chosen constant. In fact, we pick

$$c = \frac{1}{V'} \int_{\delta' B} \log u\, dv$$

where $\delta' = (1 + \delta)/2$ and $V' = v(\delta' B)$. We set $U = \delta' B$, $U_\sigma = (\sigma\delta')B$, $0 < \sigma \le 1$, $\sigma_0 = \delta/\delta'$. Theorem 4.7 shows that the first hypothesis of Lemma 4.8 is satisfied by any constant multiple of u with $\alpha_0 = q$, where $0 < q \le q_0 < (1 + p/\nu)(p - 1)$, $\gamma = \nu + p$, $K = C(p, k, q_0) S^{(\nu/p)+1}$. To verify the second hypothesis of Lemma 4.8, we apply $[\mathrm{P}(p, R)]$ to $\log u$ so that

$$\int_{\delta' B} |\log u - c|^p dv \le P' r^p \int_{\delta' B} |\nabla \log u|^p dv\,,$$

then Theorem 3.1 which easily gives

$$\int_{\delta' B} |\nabla \log u|^p dv \le \frac{V}{(p - 1)\left[(1 - \delta')r\right]^p}\,.$$

These two inequalities and doubling yield

$$\int_U |\log u - c|\, dv \le V^{1-1/p}\left(\int_U |\log u - c|^p dv\right)^{1/p} \le C_1 V'$$

where C_1 depends on p, D, P_p, and δ. It follows that

$$\lambda v\left(\{\log(e^{-c}u) \ge \lambda\} \cap U\right) \le C_1 V'$$

for all $\lambda > 0$. Lemma 4.8 yields

$$\|e^{-c}u\|_{q,\delta B} \le C_2 (V')^{1/q}, \tag{4.10}$$

where C_2 depends on p, D, P_p, δ and q. Similarly, Theorem 4.6, Theorem 3.1 and the Poincaré inequality allow us to apply Lemma 4.8 to $e^c u^{-1}$ with $q_0 = +\infty$, $\gamma = \nu$ and $K = C(p,k)S^{\nu/p}$. This gives

$$\|e^c u^{-1}\|_{\infty,\delta B} \le C_3\,, \tag{4.11}$$

where C_3 depends on p, D, P_p and δ. From (4.10) it follows that

$$\left(\frac{1}{v(\delta B)}\int_{\delta B} u^q dv\right)^{1/q} \le C_2 e^c,$$

and from (4.11) that

$$e^c \le C_3 \inf_{\delta B}\{u\}\,.$$

Finally,

$$\left(\frac{1}{v(\delta B)}\int_{\delta B} u^q dv\right)^{1/q} \le C_2 C_3 \inf_{\delta B} u\,.$$

This ends the proof.

As a corollary of Theorems 4.9 and 4.5, we obtain the full Harnack inequality for positive p-harmonic functions under the assumption that the Poincaré inequality and the doubling property are satisfied.

Theorem 4.10. *Assume that, for some $R > 0$, the doubling volume property* [D(R)] *and the Poincaré inequality* [P(p, R)] *are satisfied. Let $0 < \delta < 1$. Then any positive p-harmonic function u in $B = B_x(r)$, $r \le R$ satisfies*

$$\sup_{\delta B} u \le C \inf_{\delta B} u\,.$$

Here C depends on the doubling constant D, the Poincaré constant P_p, p and δ.

Just by letting r go to infinity, one deduces from Theorem 4.10 the following statement.

Theorem 4.11. *Let $1 < p < +\infty$. Suppose that (M, Δ) satisfies* [P(p)] *and* [D]. *Then (M, Δ) is p-Liouville.*

5 Picard Type Theorems

5.1 Quasi-regular mappings. In this section, M and N are complete oriented Riemannian n-manifolds. A continuous mapping $\phi : M \to N$ is said to be quasi-regular if, around each point of M, once seen in some charts, its coordinate functions belong to $W^{1,n}_{\mathrm{loc}}$ and

$$\exists K \geq 1\,, \quad \|d\phi(x)\|^n \leq K J(\phi)(x)\,, \text{ for a.e. } x \in M\,, \tag{5.1}$$

where $\|d\phi(x)\|$ denotes the operator norm of the differential $d\phi(x) : T_xM \to T_{\phi(x)}N$ and $J(\phi)$ is the Jacobian of ϕ. This definition is not quite satisfactory, since ϕ is not assumed to be differentiable. For instance, it is enough to ask first that, seen in charts around every point, ϕ is quasi-regular in the Euclidean sense, which ensures that $d\phi$ exists almost everywhere. This correct, but lengthy, definition can be found in [MatR, p. 277]. For more details and background on this notion, see also [R2] and [Ri4]. Note that quasiregular homeomorphisms are called quasiconformal. There are other equivalent definitions of quasiconformality (at least for certain M and N); (5.1) is usually called the analytic definition. See the discussion in section 5.3.

The completeness assumptions on M and N are superfluous since we can always change the metrics conformally to obtain complete manifolds without affecting either the distortion condition (5.1) or the n-parabolicity (resp. n-hyperbolicity) of M or N. Completeness is assumed only for the conditions [V(n)] and [I(n)] below.

Fix $o \in M$. Denote by [V(n)] the condition

$$\int_1^{+\infty} \frac{dr}{V_o'(r)^{1/(n-1)}} = +\infty$$

where $V_o(r)$ is the volume of the ball of radius r centered at o (see section 3.2 for details).

Denote by [I(n)] the condition

$$\int^{+\infty} \left(\frac{\varphi_n(t)}{t}\right)^{n/(n-1)} dt < +\infty\,,$$

where φ_n is the n-isoperimetric profile of M (see section 3.3 for details).

The connection between the existence of positive n-harmonic (resp. n-supersolutions) and the existence of quasi-regular mappings between manifolds is the following "harmonic morphism" property:

Suppose that on N, Δ_n is given by

$$\Delta_n \varphi = -\mathrm{div}\big(g(\mathcal{B}\,\mathrm{grad}\,\varphi, \mathrm{grad}\,\varphi)^{(n-2)/2}\mathcal{B}\,\mathrm{grad}\,\varphi\big)\,, \tag{5.2}$$

where $\mathcal{B}$ is uniformly comparable to $\mathcal{A} \equiv$ Identity (see section 2.6). Let $u : N \to \mathbb{R}$ be Δ_n-harmonic (resp. a Δ_n-supersolution) and $\phi : M \to N$ a quasi-regular map. Then $u \circ \phi$ is $\tilde{\Delta}_n$-harmonic (resp. a $\tilde{\Delta}_n$-supersolution), where $\tilde{\Delta}_n$, the pull-back of Δ_n by ϕ, is now given on M by

$$\tilde{\Delta}_n \psi = -\mathrm{div}\big(g(\tilde{\mathcal{B}}\,\mathrm{grad}\,\psi, \mathrm{grad}\,\psi)^{(n-2)/2}\tilde{\mathcal{B}}\,\mathrm{grad}\,\psi\big)\,, \tag{5.3}$$

with

$$\tilde{\mathcal{B}}_x = \big[J(\phi)(x)\big]^{2/n} d\phi(x)^{-1}\mathcal{B}_{\phi(x)}\big(d\phi(x)^{-1}\big)^*, \tag{5.4}$$

if $J(\phi)(x)$ exists and is positive, and by $\tilde{\mathcal{B}}_x =$ Identity otherwise. The distortion condition (5.1) implies that $\tilde{\mathcal{B}}$ is also uniformly comparable to the identity with comparison constants only depending on n, K, and the comparison constant of $\mathcal{B}$ on N. The proof of these facts goes back to Reshetnyak ([R1,2]); for a proof in the Euclidean space, that extends readily to our setting, see [HeKM, 14.39 and 14.42].

We shall denote by $\mathcal{D}_M$ the class of all Laplace operators on the n-manifold M associated with $\mathcal{B}$'s that are uniformly comparable to the identity. It is important to notice that the notion of n-parabolicity for (M, Δ) does not depend on the choice of such $\mathcal{B}$, since they all give rise to uniformly comparable gradients.

From now on, we shall say that an n-manifold M is n-parabolic if (M, Δ) is n-parabolic for some (every) $\Delta \in \mathcal{D}_M$.

Slightly abusing the previous terminology, we shall also say that M is n-Liouville if (M, Δ) is (strongly) n-Liouville for every $\Delta \in \mathcal{D}_M$.

We can now state three basic propositions that link the existence of non-trivial qr maps with the n-parabolicity, n-hyperbolicity, and n-Liouville properties of the domain and the target manifolds. Then we shall apply our criteria to each of these propositions.

PROPOSITION 5.1. *If M is n-parabolic and N n-hyperbolic, then every quasi-regular mapping from M to N is constant.*

Proof. Suppose that $\phi : M \to N$ is a qr map. If $\phi(M) = N$, we obtain immediately a contradiction by the "harmonic morphism" property above. If ϕ is non-constant but $\phi(M) \neq N$, choose $y \in \partial(\phi(M))$ and let $g = g(\,\cdot\,, y)$ be a positive Green's function for Δ_n on N (which exists since N is n-hyperbolic, see [Ho1]). Then $g \circ \phi$ is certainly a positive non-constant n-harmonic function on M, which again leads to a contradiction.

Proposition 5.1 is well known. The earliest references are [R1] and [MartRV] for maps $\phi : \mathbb{R}^n \to N \subset \mathbb{R}^n$. In the above more general form, it appeared e.g. in [Ri3, remark on p. 166].

PROPOSITION 5.2. *Let $x_0 \in N$. If M is n-Liouville and N is n-hyperbolic, then every quasi-regular mapping from M to $N \setminus \{x_0\}$ is constant.*

The end of the proof of Proposition 5.1 yields exactly Proposition 5.2. This proposition is well known to specialists; see [HoR1, Thm. 1.12], which is stated for Heisenberg groups, but whose proof is the one given above.

PROPOSITION 5.3. *If M is n-Liouville and N is not, then every quasi-regular mapping from M to N is constant.*

Proof. Suppose that $\phi : M \to N$ is a non-constant qr map. Again, if $\phi(M) = N$, the claim follows directly from the "harmonic morphism" property. Now, since N is not n-Liouville, it cannot be n-parabolic. Therefore, if $\phi(M) \neq N$, Proposition 5.2 applies.

Applications of Proposition 5.1.

Theorem 5.4. *Assume that M satisfies* [V(n)] *and that N satisfies* [I(n)]. *Then every quasi-regular mapping from M to N is constant.*

Proof. According to Corollary 3.2, M is n-parabolic, and according to Theorem 3.3, N is n-hyperbolic. Proposition 5.1 applies.

Theorem 5.4 has been proved in [Gro, Cor. 6.11], see also [P1], under the assumption that N satisfies an isoperimetric inequality at infinity of dimension $D > n$; recall that this is strictly stronger than [I(n)], see section 3.3. Together with the results in [Va2] (see [CoS1] for a simple approach) both Theorem 5.4 and [Gro, Cor. 6.11], imply that if M satisfies [V(n)], N is a Lie group and there exists a non-trivial quasi-regular mapping $\phi : M \to N$, then N has to be of polynomial growth of exponent at most n; the case $M = \mathbb{R}^n$ is already in [VaSC, X.2.2].

One cannot weaken the condition on N in the above statement into a condition on the volume growth only. Indeed, according to an example in [Va4], which is a modification of the example by Greene quoted in section 3.2, $\mathbb{R}^2$ is conformal to a surface N with uniform exponential volume growth. More generally, it is proven in [GrimP] that if M satisfies $\int_1^{+\infty} dr/V_o'(r)^{1/(n-1)} = +\infty$ then any volume growth can be realized by a conformal metric on M.

COROLLARY 5.5. *Assume that M satisfies* [V(n)], *and that N is compact. Suppose there exists a non-constant quasi-regular mapping from M to N. Then $\Gamma = \pi_1(N)$ is virtually nilpotent and its volume growth function satisfies*

$$V_\Gamma(k) = O(k^n)\,.$$

Proof. Let $\phi : M \to N$ be quasi-regular and non-constant. Lift it to $\tilde{\phi} : M \to \tilde{N}$, where $\tilde{N}$ is the universal covering of N. Obviously $\tilde{\phi}$ is quasi-regular and non-constant. Now if the claimed estimate on V_Γ is false, one must have $V_\Gamma(k) \geq ck^{n+1}$, therefore the isoperimetric profile of $\tilde{N}$ satisfies $\varphi_1(t) \leq Ct^{1/(n+1)}$ (see [CoS1]), and in particular [I(n)]. This is a contradiction.

The case $M = \mathbb{R}^n$ of Corollary 5.5 is already in [VaSC, X.5.1].

A version of Proposition 5.1 has been given in [TrV] for mappings satisfying

$$\exists K > 0\,, \quad \|d\phi(x)\|^s \leq KJ(\phi)(x)\,, \text{ for a.e. } x \in M\,,$$

for more general s. Our criteria can be used together with this result to improve non-existence results for such mappings.

Applications of Proposition 5.2.

Theorem 5.6. *Assume that M satisfies* [P(n)] *and* [D], *and that N satisfies* [I(n)]. *Then every quasi-regular mapping from M to $N \setminus \{x_0\}$, $x_0 \in N$, is constant.*

Proof. One applies Theorem 4.11, Theorem 3.3, and Proposition 5.2.

Examples. In Theorems 5.4 and 5.6, one can take N roughly isometric to a Lie group or a finitely generated group, a Riemannian manifold with non-negative Ricci curvature, or a co-compact covering, as soon as their volume growth is at least polynomial of exponent $D > n$. In all these situations N satisfies an isoperimetric inequality at infinity of dimension D (see [CoS1,2]), therefore [I(n)] holds.

Corollary 5.7. *Let G be a simply connected nilpotent non-Abelian Lie group, and let $x_0 \in G$. Then every quasi-regular mapping from G to $G \setminus \{x_0\}$ is constant.*

Proof. Every Lie group with polynomial growth satisfies [D] and [P(p)] for all $p \in [1, +\infty)$. Let n be the topological dimension of G. It follows from the assumption that the volume growth function V of G satisfies $V(r) \geq cr^D$, $r \geq 1$, with $D > n$. Now the isoperimetric dimension at infinity of G is D (this is due to Varopoulos [Va2], see [CoS1] for a simple proof), thus [I(n)] is also satisfied and one can apply Theorem 5.6.

The case where G is the Heisenberg group has been treated in [HoR1, Thm. 1.12]. Note that the above proof works for any Lie group with polynomial growth at infinity of exponent strictly larger than the topological dimension.

Applications of Proposition 5.3. It follows from [P3, pp. 97, 112], that if a simply connected n-dimensional manifold has sufficiently pinched negative sectional curvature, then it admits non-constant Dirichlet finite n-harmonic functions. Now it follows from [Ho1, Thm. 5.9], that it admits non-constant bounded n-harmonic functions as well.

We can therefore state the following.

COROLLARY 5.8. *If M satisfies* [P(n)] *and* [D], *and if N satisfies $-a^2 \leq K \leq -b^2 < 0$, with $a/b < n/(n-1)$, where K is the sectional curvature, then every quasi-regular mapping from M to N is constant.*

Note that we do not need to assume that N is simply connected, because once again one can lift any non-constant quasi-regular map $\phi : M \to N$ to a non-constant quasi-regular $\tilde{\phi} : M \to \tilde{N}$, where $\tilde{N}$ is the universal covering of N.

The case where M is the Heisenberg group and N the hyperbolic space of the same topological dimension has been considered in [Ho2, Cor. 4.13].

Another way to obtain nice applications of Proposition 5.3, in the case of surfaces, is to combine it with the result of [BouE] (see [BBE] for a simple proof) that a regular cover of a compact manifold whose deck transformation group is linear has the strong 2-Liouville property if and only if M is not of exponential growth. One may wonder whether the latter statement holds for the p-Liouville property, which would yield the same applications for n-manifolds.

Proposition 5.2 is an example of results giving upper bounds for the number of omitted values. In [Ri1], Rickman proved that a non-constant qr map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ cannot omit more than a finite number $q(n, K)$ of points, where K is the constant in (5.1). The result is sharp at least in $\mathbb{R}^3$ in the sense that given any positive integer q, there exists a non-constant qr map $\phi : \mathbb{R}^3 \to \mathbb{R}^3$, with K depending on q, omitting at least q points (see [Ri2]). Other proofs of Rickman's Picard theorem are given e.g. in [EL] and [L]; note that the latter is based on a uniform Harnack inequality. See also [HoR2], and [HeH] for further generalizations.

In this section we have only dealt with elliptic operators on M and N since very little is known about quasi-regular maps beyond the Riemannian setting, for instance in a subelliptic context. The difficulty lies in proving the harmonic morphism property starting from a (local) definition like (5.1) without assuming ϕ to be overly smooth. See [Ri4, p. 12], for a discussion on the regularity issue. In [HeH], basic properties of qr-maps on Carnot groups of Heisenberg type were proved under a slightly unsatisfactory regularity assumption on the maps. Recently, Dairbekov was able to overcome this difficulty in [Da]; see also [Vo] for an independent approach.

In the next two sections, we briefly consider two classes of maps for which we can obtain Picard type theorems also in our general setting.

5.2 *p*-harmonic morphisms. One way to overcome the difficulty of proving the harmonic morphism property in a general setting is simply to consider maps that by definition pull-back p-harmonic functions. Say that a continuous map $\phi : M \to N$ is a $(\tilde{\Delta}_p, \Delta_p)$-harmonic morphism if $u \circ \phi$ is $\tilde{\Delta}_p$-harmonic in $\phi^{-1}(\Omega)$ whenever u is Δ_p-harmonic in an open set $\Omega \subset N$; see [HeKM, chapter 13], and also [Lo]. Notice that in this definition, the topological dimensions of M and N need not be the same. Of course, quasiregular maps between Riemannian n-manifolds M and N are $(\tilde{\Delta}_n, \Delta_n)$-harmonic morphisms, where Δ_n and $\tilde{\Delta}_n$ are given by (5.2) and (5.3), respectively.

Assume that (N, Δ) satisfies a local doubling condition and a local Poincaré inequality in the sense that there exists a constant C such that each point $x \in N$ has a neighborhood U so that, for every ball $B_y(2r) \subset U$,

$$v(B_y(2r)) \le C v(B_y(r)) \tag{5.5}$$

and

$$\int_{B_y(r)} |f - f_{B_y(r)}|^p dv \le C r^p \int_{B_y(2r)} |\nabla f|^p dv \tag{5.6}$$

holds for all $f \in \mathcal{C}^\infty(B_y(2r))$.

By Theorem 4.10, we then have a local Harnack inequality for positive Δ_p-harmonic functions on open sets $\Omega \subset N$. Now it is possible to repeat the arguments in [Ho1] (see also [Ho2]) with minor changes to obtain a further characterization of p-hyperbolicity of (N, Δ). Namely, (N, Δ) is p-hyperbolic if and only if there exists a positive Green's function $g = g(\,\cdot\,, x_0)$, for all $x_0 \in N$, for the operator Δ_p on N. In particular, g is then a positive Δ_p-harmonic function on $N \setminus \{x_0\}$ with $\inf_N g = 0$ and

$$\lim_{x \to x_0} g(x, x_0) = \mathrm{Cap}_p\bigl(N, \{x_0\}\bigr)^{1-p},$$

where we make the convention that $0^{1-p} = +\infty$. Furthermore,

$$g(y, x_0) < \lim_{x \to x_0} g(x, x_0)$$

for all $y \ne x_0$. Now the counterpart to Propositions 5.1, 5.2, 5.3 is the following.

Proposition 5.9. *Assume that* (N, Δ) *is* p*-hyperbolic and satisfies* (5.5) *and* (5.6).

(a) *If $(M, \tilde{\Delta})$ is p-parabolic, then every $(\tilde{\Delta}_p, \Delta_p)$-harmonic morphism $\phi : M \to N$ is constant.*
(b) *If $(M, \tilde{\Delta})$ is strongly p-Liouville, then every $(\tilde{\Delta}_p, \Delta_p)$-harmonic morphism $\phi : M \to N \setminus \{x_0\}$, $x_0 \in N$, is constant.*
(c) *If $(M, \tilde{\Delta})$ is strongly p-Liouville but (N, Δ) is not, then every $(\tilde{\Delta}_p, \Delta_p)$-harmonic morphism $\phi : M \to N$ is constant.*

Proof. Let $\phi : M \to N$ be a $(\tilde{\Delta}_p, \Delta_p)$-harmonic morphism. If ϕ is non-constant but $\phi(M) \neq N$ (which is the case in (b)), then all three claims (a)–(c) follow as in the end of Proposition 5.1. Claim (c) is also clear if $\phi(M) = N$. It remains to prove (a) in the case $\phi(M) = N$. Let $g = g(\,\cdot\,, x_0)$ be a Green's function for Δ_p on N. Now $g\circ\phi$ is a positive $\tilde{\Delta}_p$-harmonic function in the open set $\phi^{-1}(N \setminus \{x_0\})$. Let $a = \min\left\{1, \frac{1}{2}\mathrm{Cap}_p(N, \{x_0\})^{1-p}\right\}$, where we make again the convention $0^{1-p} = +\infty$. Then the function

$$u = (g \circ \phi) \wedge a$$

is a non-constant positive $\tilde{\Delta}_p$-supersolution on M. Indeed, u is $\tilde{\Delta}_p$-harmonic, hence a $\tilde{\Delta}_p$-supersolution, in the open set $\phi^{-1}(\{g \neq a\})$. Furthermore, each point $x \in (g\circ\phi)^{-1}(\{a\})$ has a neighborhood where $g\circ\phi$ is $\tilde{\Delta}_p$-harmonic, hence u is a $\tilde{\Delta}_p$-supersolution in this neighborhood, by [HeKM, Thm. 2.3]. This leads to a contradiction.

5.3 Quasi-conformal mappings. If we require $\phi : M \to N$ to be a homeomorphism and consider quasiconformal and quasisymmetric maps, we can also relax the Riemannian assumptions on M and N and still obtain a non-existence result like Proposition 5.1.

Let (X, d) and (Y, d) be metric spaces and let $\eta : [0, +\infty) \to [0, +\infty)$ be a homeomorphism. A homeomorphism $\phi : X \to Y$ is called η-quasisymmetric if

$$\frac{d(\phi(x), \phi(y))}{d(\phi(x), \phi(z))} \leq \eta\left(\frac{d(x,y)}{d(x,z)}\right), \tag{5.7}$$

for every $x, y, z \in X$, $x \neq z$. We refer to [TuV] for the basic theory of quasisymmetric maps. We also say that a homeomorphism $\phi : X \to Y$ is quasiconformal if there exists a constant H such that

$$\limsup_{r\to 0} \frac{\sup\{d(\phi(x), \phi(y)); d(x,y) \leq r\}}{\inf\{d(\phi(x), \phi(y)); d(x,y) \geq r\}} \leq H \tag{5.8}$$

for every $x \in X$. It is well known that for homeomorphisms $\phi : \mathbb{R}^n \to \mathbb{R}^n$, conditions (5.7), (5.8), and, if ϕ is sense-preserving, also the the analytic definition of quasiconformality (5.1) are quantitatively equivalent (see [G] and [V]); note that (5.1) and (5.8) are also equivalent in the Riemannian setting, provided ϕ is sense-preserving. Recently, the equivalence of (5.7) and (5.8) for homeomorphisms $\phi : X \to Y$ has been established in [HeK2] and [HeKST] for a large class of metric spaces X and Y. For earlier related results, see [P2], [KoR], [HeK1], [Mos], [MaM]. For historical remarks and further references, see [HeKST, §9]. Next we shall apply some of the results obtained in [HeK2] and [HeKST] to our setting. Say that (M, Δ) has locally Q-bounded geometry, with $Q > 1$, if there exists a constant $C \geq 1$ such that each point $x \in M$ has a neighborhood U so that, for every ball $B_y(2r) \subset U$,

$$C^{-1} r^Q \leq v(B_y(r)) \leq C r^Q \tag{5.9}$$

and the Poincaré inequality (5.6) holds for $p = Q$.

Proposition 5.10. *Let $(M, \tilde{\Delta})$ and (N, Δ) be of locally Q-bounded geometry, with $Q > 1$. Suppose that $\phi : M \to N$ is quasi-conformal. Then (N, Δ) is Q-parabolic if and only if $(M, \tilde{\Delta})$ is Q-parabolic.*

Proof. Using [HeKST, Thm. 9.8], and [HeK2, Prop. 2.17 and Thm. 5.7], we obtain

$$\begin{aligned} C^{-1}\mathrm{Cap}_Q\big(B_x(R), \bar{B}_x(r)\big) &\leq \mathrm{Cap}_Q\big(\phi(B_x(R)), \phi(\bar{B}_x(r))\big) \\ &\leq C\,\mathrm{Cap}_Q\big(B_x(R), \bar{B}_x(r)\big) \end{aligned}$$

for every $x \in M$ and $R > r > 0$, with constant C independent of x, R and r (see also [Ty, Thm. 1.4]). Letting R go to $+\infty$ yields

$$C^{-1}\mathrm{Cap}_Q\big(M, \bar{B}_x(r)\big) \leq \mathrm{Cap}_Q\big(N, \phi(\bar{B}_x(r))\big) \leq C\,\mathrm{Cap}_Q\big(M, \bar{B}_x(r)\big)$$

which proves the claim.

Final remarks. (1) Assume that $(M, \tilde{\Delta})$ and (N, Δ) have locally Q-bounded geometry and that (N, Δ) is Q-hyperbolic. As explained in section 5.2, N admits a positive Green's function $g = g(\,\cdot\,, x_0)$ for Δ_Q. Then $g \circ \phi$, where ϕ is a quasiconformal map $\phi : M \to N \setminus \{x_0\}$, is a positive quasiminimizer for the Q-Dirichlet integral $\int |\nabla u|^Q \, dv$, see [HoS]. If we suppose in addition that the global doubling condition [D] and Poincaré's inequality [P(p)] hold for some $1 \leq p < Q$, then M admits a uniform Harnack inequality for positive quasiminimizers by [KiS, Corollary 7.5]. Consequently, no quasiconformal map $\phi : M \to N \setminus \{x_0\}$ exists.

(2) It seems plausible that, in our general setting, quasiconformal maps $\phi : M \to N$, with both $(M, \tilde{\Delta})$ and (N, Δ) of locally Q-bounded geometry have the (full) harmonic morphism property as they do in the Riemannian setting. That is, given Δ_p on N, it should be possible to write down explicitly $\tilde{\Delta}_p$, the pull-back of Δ_p by ϕ, and then use the change of variables

formula to obtain the harmonic morphism property; see [HeK2, §7], and [HeKST, §9], for the justification of the change of variables. The Jacobian of ϕ at $x \in M$ in (5.4) should be changed to the volume derivative of ϕ at x but the difficulty is to find the right interpretation of the "differential" $d\phi(x)$ in (5.4). The notion of the Cheeger differential of Lipschitz functions (in a metric space setting) might turn out to be helpful; see [Che] and [HeKST, §10].

5.4 Acknowledgements. We thank Pierre Pansu and Marc Troyanov for useful conversations and references as well as Alexander Grigor'yan for remarks on the manuscript.

References

[A] L. Ahlfors, Sur le type d'une surface de Riemann, C.R.A.S. Paris 201 (1935), 30–32.

[AuC] P. Auscher, T. Coulhon, Gaussian lower bounds for random walks from elliptic regularity, Ann. I.H.P., Proba. Stat. 35 (1999), 605–630.

[BBE] M. Babillot, P. Bougerol, L. Elie, On the strong Liouville property for co-compact Riemannian covers, Rend. Sem. Mat. Fis. Milano LXIV (1994), 77–84.

[BaCLS] D. Bakry, T. Coulhon, M. Ledoux, L. Saloff-Coste, Sobolev inequalities in disguise, Indiana Univ. Math. J. 44:4 (1995), 1033–1074.

[BiM1] M. Biroli, U. Mosco, A Saint-Venant principle for Dirichlet forms on discontinuous media, Ann. Mat. Pura Appl. IV 169 (1992), 125–181.

[BiM2] M. Biroli, U. Mosco, Sobolev inequalities on homogeneous spaces, Potential Analysis 4 (1995), 311–324.

[BoG] E. Bombieri, E. Giusti, Harnack's inequality for elliptic differential equations on minimal surfaces. Invent. Math. 15 (1972), 24–46.

[BouE] P. Bougerol, L. Elie, Existence of non-negative harmonic functions on groups and on covering manifolds, Ann. I.H.P., Proba. Stat. 31 (1995), 59–80.

[Bu] P. Buser, A note on the isoperimetric constant, Ann. Sci. Ecole Norm. Sup. Paris 15 (1982), 213–230.

[CDG] L. Capogna, D. Danielli, N. Garofalo, An embedding theorem and the Harnack inequality for nonlinear subelliptic equations, Comm. P.D.E. 18:9&10 (1993), 1765–1794.

[Ca1] G. Carron, Inégalités isopérimétriques sur les variétés riemanniennes, Thesis, University of Grenoble, 1994.

[Ca2] G. Carron, Inégalités isopérimétriques de Faber–Krahn et conséquences, in "Actes de la table ronde de géométrie différentielle en l'honneur de Marcel Berger", collection SMF séminaires et congrès, 1 (1994), 205–232.

[Ch] I. Chavel, Riemannian geometry: a modern introduction, Cambridge University Press, 1993.

[ChF] I. Chavel, E. Feldman, Modified isoperimetric constants and large time heat diffusion in Riemannian manifolds, Duke Math. J. 64 (1991), 473-499.

[Che] J. Cheeger, Differentiability of Lipschitz functions on metric measure spaces, Geometric And Functional Analysis 9 (1999), 428–517.

[ChenY] S.Y. Cheng, S.-T. Yau, Differential equations on Riemannian manifolds and their geometric applications, Comm. P. Appl. Math. 28 (1975), 333–354.

[Co1] T. Coulhon, Espaces de Lipschitz et inégalités de Poincaré, J. Funct. Anal. 136:1 (1996), 81–113.

[Co2] T. Coulhon, Dimensions at infinity for Riemannian manifolds, Potential Analysis 4:4 (1995), 311–324.

[CoG] T. Coulhon, A. Grigor'yan, Random walks on graphs with regular volume growth, Geometric And Functional Analysis 8 (1998), 656-701.

[CoL] T. Coulhon, M. Ledoux, Isopérimétrie, décroissance du noyau de la chaleur et transformations de Riesz: un contre-exemple, Arkiv Mat. 32 (1994), 163–77.

[CoS1] T. Coulhon, L. Saloff-Coste, Isopérimétrie pour les groupes et les variétés, Rev. Mat. Iberoamer. 9:2 (1993), 293–314.

[CoS2] T. Coulhon, L. Saloff-Coste, Variétés riemanniennes isométriques à l'infini, Rev. Mat. Iberoamer. 11:3 (1995), 687–726.

[Da] N. Dairbekov, Mappings with bounded distortion of two-step Carnot groups, Proceedings on Analysis and Geometry (Novosibirsk Akademgorodok, 1999) (Russian), Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk (1999), 122-155.

[EL] A. Eremenko, J.L. Lewis, Uniform limits of certain $\mathcal{A}$-harmonic functions with applications to quasiregular mappings, Ann. Acad. Sc. Fenn. Ser. A I Math. 16 (1991), 361–375.

[FP] C. Fefferman, D.H. Phong, Subelliptic eigenvalue problems, in "Proceedings of the conference in harmonic analysis in honor of Antoni Zygmund", Wadsworth Math. Ser. (1981), 590–606.

[FS] C. Fefferman, A. Sanchez-Calle, Fundamental solutions for second order subelliptic operators, Ann. Math. 124 (1986), 247–272.

[Fe] J. Fernández, On the existence of Green's function in Riemannian manifolds, Proc. AMS 96:2 (1986), 284–286.

[FeR] J. Fernández, J.M. Rodríguez, Area growth and Green's function on Riemann surfaces, Arkiv Mat. 30:1 (1992), 83–92.

[G] F. Gehring, Rings and quasiconformal mappings in space, Trans. Amer. Math. Soc. 103 (1962), 353–393.

[Gi] M. Giaquinta, Introduction to Regularity Theory for Nonlinear Elliptic Systems, Birkhäuser, (1993).

[GoT1] V. Gol'dshtein, M. Troyanov, The Kelvin–Nevanlinna–Royden criterion for p-hyperbolicity, Math. Z. 232:4 (1999), 607–619.

[GoT2] V. Gol'dshtein, M. Troyanov, Axiomatic theory of Sobolev spaces, Expositiones Math., to appear.

[Gr] S. Granlund, Harnack's inequality in the borderline case, Ann. Acad. Sc. Fenn. Ser. A I Math. 5 (1980), 159–163.

[Gri1] A. Grigor'yan, On the existence of positive fundamental solutions of the Laplace equation on Riemannian manifolds, Math. Sb. 128:170 (1985), 354–363; Engl. transl., Math. USSR Sbornik 56: 2 (1987), 349–358.

[Gri2] A. Grigor'yan, Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds, Bull. Amer. Math. Soc. 36 (1999), 135–249.

[Gri3] A. Grigor'yan, Isoperimetric inequalities and capacities on Riemannian manifolds, in "Operator Theory, Advances and Applications (Special volume dedicated to V.G. Maz'ya), 109, (1999) 139–153.

[GrimP] R. Grimaldi, P. Pansu, Sur la croissance du volume dans une classe conforme, J. Math. Pures Appl. 71 (1992), 1–19.

[Gro] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces, Birkhäuser (1981); Structures métriques pour les variétés riemanniennes, Lecture notes by P. Pansu and J. Lafontaine, Cedic Nathan (1999).

[HK1] P. Hajlasz, P. Koskela, Sobolev meets Poincaré, C.R.A.S. Paris 320 (1995), 1211–1215.

[HK2] P. Hajlasz, P. Koskela, Sobolev met Poincaré, Memoirs of the A.M.S. 145:688 (2000).

[HeH] J. Heinonen, I. Holopainen, Quasiregular mappings on Carnot groups, J. of Geometric Analysis 7:1 (1997), 109–148.

[HeKM] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Oxford University Press, 1993.

[HeK1] J. Heinonen, P. Koskela, Definitions of quasiconformality, Invent. Math. 120 (1995), 61–79.

[HeK2] J. Heinonen, P. Koskela, Quasiconformal maps in metric spaces with controlled geometry, Acta Math. 181 (1998), 1-61.

[HeKST] J. Heinonen, P. Koskela, N. Shanmugalingam, J. Tyson, Sobolev classes of Banach space-valued functions and quasiconformal mappings, J. Anal. Math. 85 (2001), 87–139.

[Ho1] I. Holopainen, Nonlinear potential theory and quasiregular mappings on Riemannian manifolds, Ann. Acad. Sc. Fenn. Ser. A I Math. Diss. 74 (1990), 1–45.

[Ho2] I. Holopainen, Positive solutions of quasilinear elliptic equations on Riemannian manifolds, Proc. London Math. Soc. 3:65 (1992), 651–672.

[Ho3] I. Holopainen, Solutions of elliptic equations on manifolds with roughly Euclidean ends, Math. Z. 217: 3 (1994), 459–477.

[Ho4] I. Holopainen, Volume growth, Green's function, and parabolicity of ends, Duke Math. J. 97:2 (1999), 319–346.

[HoK] I. Holopainen, P. Koskela, Volume growth and parabolicity, Proc. Amer. Math. Soc. 129:11 (2001), 3425–3435.

[HoR1] I. Holopainen, S. Rickman, Quasiregular mappings of the Heisenberg group, Math. Ann. 294 (1992), 625–643.

[HoR2] I. Holopainen, S. Rickman, Ricci curvature, Harnack functions and Picard type theorems for quasiregular mappings, in "Analysis and Topology" (Cazacu, Lehto, Rassias, eds.), World Scientific Publ. (1998), 315–326.

[HoS] I. Holopainen, N. Shanmugalingam, Singular functions on metric measure spaces, preprint.

[HoSo] I. Holopainen, P. Soardi, A strong Liouville theorem for p-harmonic functions on graphs, Ann. Acad. Sc. Fenn. Ser. A I Math. 22 (1997), 205–226.

[Hör] L. Hörmander, Hypoelliptic second order differential equations, Acta Math. 119 (1967), 147–171.

[J] D. Jerison, The Poincaré inequality for vector fields satisfying the Hörmander condition, Duke Math. J. 53 (1986), 503–523.

[JS] D. Jerison, A. Sanchez-Calle, Subelliptic second order differential operators. in "Complexe Analysis III, Proceedings", Springer LNM 1277 (1986), 47–77.

[K] V. Kesel'man, Riemannian manifolds of α-parabolic type, Izv. Vyssh. Uchebn. Zaved. Ser. Mat. 4 (1985), 81–83.

[KiS] J. Kinnunen, N. Shanmugalingam, Regularity of quasi-minimizers on metric spaces, Manuscripta Math. 105:3 (2001), 401–423.

[KoR] A. Korányi, H.M. Reimann, Foundations for the theory of quasiconformal mappings on the Heisenberg group, Adv. Math. 111 (1995), 1–87.

[L] J.L. Lewis, Picard's theorem and Rickman's theorem by way of Harnack's inequality, Proc. Amer. Math. Soc. 122 (1994), 199–206.

[Li] P. Li, Curvature and function theory on Riemannian manifolds, Surveys in Diff. Geom. VII (2000), 71–111.

[LiS] P. Li, R. Schoen, L^p and mean value properties of subharmonic functions on Riemannian manifolds, Acta Math. 153 (1984), 279–301.

[Lo] E. Loubeau, On p-harmonic morphisms, Diff. Geom. and Appl. 12 (2000), 219–229.

[MS] P. Maheux, L. Saloff-Coste, Analyse sur les boules d'un opérateur sous-elliptique, Math. Ann. 303:4 (1995), 713–740.

[MaM] G. Margulis, G. Mostow, The differential of a quasi-conformal mapping of a Carnot-Carathéodory space, Geometric And Functional Analysis 5:2 (1995), 402–433.

[MarMT] S. Markvorsen, S. McGuiness, C. Thomassen, Transient random walks on graphs and metric spaces with applications to hyperbolic surfaces, Proc. London Math. Soc. 3, 64 (1992), 1–20.

[MartRV] O. Martio, S. Rickman, J. Väisälä, Distortion and singularities of quasiregular mappings, Ann. Acad. Sc. Fenn. Ser. A I Math. 465 (1970), 1–13.

[MatR] P. Mattila, S. Rickman, Averages on the counting function of a quasiregular mapping, Acta Math. 143 (1979), 273–305.

[Maz] V. Maz'ya, Sobolev Spaces, Springer Verlag, 1985.

[Mo] J. Moser, On Harnack's theorem for elliptic differential equations, Comm. Pure Appl. Math. 14 (1961), 577–591.

[Mos] G. Mostow, Strong Rigidity of Locally Symmetric Spaces, Annals of Math. Studies, 78, Princeton U.P., 1974.

[NSW] A. Nagel, E. Stein, S. Wainger, Balls and metrics defined by vector fields, Acta Math. 155 (1985), 103–147.

[O] H. Okura, Capacitary inequalities and global properties of symmetric Dirichlet forms, in "Dirichlet forms and stochastic processes (Beijing 1993)" (1995), 291–303.

[OlR] O. Oleinik, E. Radkevic, Second Order Equations with Non-negative Characteristic Form, Plenum Press, 1973.

[Os] Y. Oshima, On conservativeness and recurrence criteria for Markov processes, Potential Analysis 1 (1992), 115–131.

[P1] P. Pansu, An isoperimetric inequality on the Heisenberg group, in "Conference on differential operators on homogeneous groups (Turin, 1983)", Rend. Sem. Mat. Torino, fasc. spec. (1983), 159–174.

[P2] P. Pansu, Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un, Ann. Math. 2, 129 (1989), 1–60.

[P3] P. Pansu, Cohomologie L^p des variétés à courbure négative, cas du degré 1, in "P.D.E and Geometry 1988", Rend. Sem. Mat. Torino, fasc. spec. (1989), 95–120.

[R1] Y. Reshetnyak, On extremal properties of mappings with bounded distortion, (in Russian), Sibirsk. Math. Zh., 10 (1969), 1300–1310.

[R2] Y. Reshetnyak, Space Mappings with Bounded Distortion, Transl. of Math. Monographs 73, AMS, Providence, 1989.

[Ri1] S. Rickman, On the number of omitted values of entire quasiregular mappings, J. d'Analyse Math. 37 (1980), 100–117.

[Ri2] S. Rickman, The analogue of Picard's theorem for quasiregular mappings in dimension three, Acta Math. 154 (1988), 195–242.

[Ri3] S. Rickman, Topics in the theory of quasiregular mappings, in "Conformal Geometry", Aspekte der Math. E 12, Vieweg (1988), 147–189.

[Ri4] S. Rickman, Quasi-regular Mappings, Springer-Verlag, 1993.

[RigSV] M. Rigoli, M. Salvatori, M. Vignati, A note on p-subharmonic functions on complete manifolds, Manuscripta. Math. 92 (1997), 339–359.

[S1] L. Saloff-Coste, Uniformly elliptic operators on Riemannian manifolds, J. Diff. Geom. 36 (1992), 417–450.

[S2] L. Saloff-Coste, A note on Poincaré, Sobolev, and Harnack inequalities, Duke Math. J., I.M.R.N. 2 (1992), 27–38.

[S3] L. Saloff-Coste, Parabolic Harnack inequality for divergence form second order differential operators, Potential Analysis 4:4 (1995), 429–467.

[S4] L. Saloff-Coste, Some inequalities for superharmonic functions on graphs, Potential Analysis 6 (1996), 163–181.

[S5] L. Saloff-Coste, Inequalities for p-superharmonic functions on networks, Rendiconti Sem. Mat. Fis. Milano LXV (1995), 139–158.

[S6] L. Saloff-Coste, Some Aspects of Sobolev Type Inequalities, Cambridge University Press, 2001.

[Se] J. Serrin, A Harnack inequality for nonlinear equations, Bull. AMS 69 (1963), 481–486.

[So] P. Soardi, Parabolic networks and polynomial growth, Colloq. Math. LX-LXI (1990), 65–70.

[St] K.T. Sturm, Sharp estimates for capacities and applications to symmetric diffusions, Prob. Th. and Rel. Fields 103 (1995), 73–89.

[T] C. Thomassen, Isoperimetric inequalities and transient random walks on graphs, Annals of Proba. 20:3 (1992), 1592–1600.

[Tr1] M. Troyanov, Parabolicity of manifolds, Siberian Adv. Math. 9:4 (1999), 125–150.

[Tr2] M. Troyanov, Solving the p-Laplacian on manifolds, Proc. Amer. Math. Soc. 128:2 (1999), 541–545.

[TrV] M. Troyanov, S. Vodop'yanov, Liouville type theorems for mappings with bounded (co-)distortion, preprint (1999).

[Tru] N. Trudinger, On Harnack type inequalities and their application to quasilinear elliptic equations, Comm. in Pure and Appl. Math. XX (1967), 721–747.

[TuV] P. Tukia, J. Väisälä, Quasisymmetric embeddings of metric spaces, Ann. Acad. Sc. Fenn. Ser. A I Math. 5 (1980), 97–114.

[Ty] J. Tyson, Quasiconformality and quasisymmetry in metric measure spaces, Ann. Acad. Sc. Fenn. Ser. A I Math. 23 (1998), 525–548.

[V] J. Väisälä, Lectures on n-dimensional Quasiconformal Mappings, Springer L.N. Math. 229 (1971).

[Va1] N. Varopoulos, Potential theory and diffusion on Riemannian manifolds, Conference in Harmonic Analysis in honor of A. Zygmund, Wadsworth (1983), 821–837.

[Va2] N. Varopoulos, Analysis on Lie groups, J. Funct. Anal. 76:2 (1988), 346–410.

[Va3] N. Varopoulos, Fonctions harmoniques sur les groupes de Lie, C.R. Acad. Sci. Paris 309 (1987), 519–521.

[Va4] N. Varopoulos, Small time Gaussian estimates of heat diffusion kernel, I, The semigroup technique, Bull. Sci. Math. 113:3 (1989), 253–277.

[VaSC] N. Varopoulos, L. Saloff-Coste, T. Coulhon, Analysis and Geometry on Groups, Cambridge University Press, 1992.

[Vo] S. Vodop'yanov, Mappings with bounded distortion and with finite distortion on Carnot groups, Siberian Math. J. 40:4 (1999), 644–678.

[Y] M. Yamasaki, Parabolic and hyperbolic infinite networks, Hiroshima Math. J. 7 (1977), 135–146.

[Z] V. Zorich, Asymptotic geometry and conformal types of Carnot-Carathéodory spaces, Geometric And Functional Analysis 9 (1999), 393–411.

[ZK] V. Zorich, V. Kesel'man, On the conformal type of a Riemannian manifold, Funct. Anal. and Appl. 30:2 (1996), 106–117.

Thierry Coulhon, Départment de Mathématiques, Université de Cergy-Pontoise, 2 rue Adolphe Chauvin, 95302 Pontoise, France
Thierry.Coulhon@math.u-cergy.fr

Ilkka Holopainen, Department of Mathematics, P.O.Box 4, FIN-00014 University of Helsinki, Finland ilkka.holopainen@helsinki.fi

Laurent Saloff-Coste, Department of Mathematics, Malott Hall, Cornell University, Ithaca, NY 14853, USA lsc@math.cornell.edu

Submitted: August 2000
Revision: November 2000