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CENTRAL GAUSSIAN CONVOLUTION SEMIGROUPS ON COMPACT GROUPS: A SURVEY

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This is a survey article on Brownian motions on compact connected groups and the associated Gaussian convolution semigroups. The emphasize is on infinite dimensional groups such as the infinite dimensional torus and infinite products of special orthogonal groups. We discuss the existence of Brownian motions having nice properties such as marginales having a continuous density with respect to Haar measure. We relate the existence of these Brownian motions to the algebraic structure of the group. The results we describe reflect the conflicting effects of, on the one hand, the infinite dimensionality and, on the other hand, the compact nature of the underlying group.

Keywords: Convolution semigroups; Brownian motion; compact groups.

1. Introduction

On $G = \mathbb{R}^n$, up to automorphisms of the group structure, Brownian motion is the unique G-valued process having independent stationary increments, continuous paths, which is nondegenerate and is invariant under the mapping $x \mapsto -x$. For a good choice of Euclidean structure, the one-dimensional marginal at time $t > 0$ is the measure with density

$$\mu_t(x) = \frac{1}{(4\pi t)^{n/2}} \exp(-|x|^2/4t)\,, \tag{1.1}$$

where $|x|$ is the Euclidean length of x (our convention is that Brownian motion is driven by $\sum_1^n \partial_i^2$, not by $1/2\sum_1^n \partial_i^2$).

On a general group G, we call symmetric left-invariant diffusion any G-valued process having independent stationary increments, continuous paths, and which is nondegenerate and invariant under the mapping $x \mapsto x^{-1}$. If the group is compact, we call *Brownian motion* any symmetric left-invariant diffusion which is also invariant under all inner automorphisms, that is, under $x \mapsto axa^{-1}$, $a \in G$.

When the group G is a connected Lie group, the one-dimensional marginals of any Brownian motion have smooth positive densities $x \mapsto \mu_t(x)$, $t > 0$, with respect to Haar measure. For instance, on the n-dimensional torus $\mathbb{T}^n$ identified with $(-\pi, \pi]^n$, the standard Brownian motion has density

$$\mu_t(x) = \left(\frac{\pi}{t}\right)^{n/2} \sum_{k \in 2\pi\mathbb{Z}^n} \exp\left(-\frac{|x+k|^2}{4t}\right)$$

with respect to the normalized Haar measure. In general, there is no simple formula for the density of a Brownian motion on a Lie group but, as t tends to 0, this density resembles (1.1) where n is the topological dimension of G and $|x|$ must be interpreted as a certain bi-invariant Riemannian distance on G. In particlular,

$$\log \mu_t(e) \sim \frac{n}{2} \log \frac{1}{t} \quad \text{as } t \to 0 \tag{1.2}$$

and

$$\lim_{t \to 0} 4t \log \mu_t(x) = -|x|^2 \,. \tag{1.3}$$

The aim of this paper is to survey recent results of the authors on three types of problems concerning Brownian motions on compact groups of infinite dimension: (a) Existence of Brownian motions having certain prescribed properties such as one-dimensional marginals having continuous densities; (b) Relations between various properties, most of them being expressed in terms of the one-dimensional marginals; (c) Relations between certain properties of the one-dimensional marginals and the structure of the underlying group. For instance, are there statements resembling (1.2)–(1.3) that hold for Brownian motions on general compact groups? For various applications to potential theory, sample path regularity and more see, e.g. Refs. 5, 8, 15, 16, 18 and 38. Good examples of infinite dimensional compact groups to keep in mind are the infinite dimensional torus $\mathbb{T}^\infty$, which is the product of countably many circles, and the product $G = \prod_{i=1}^\infty \mathrm{SO}(n_i)$ of special orthogonal groups of various dimensions.

We would like to convey the idea that Brownian motions on general compact connected groups can be studied from a rather concrete viewpoint. See for instance Theorems 2.1, 2.3, 3.1 and the examples given throughout this paper. Some of the key ingredients needed to develop this program are refined results concerning Brownian motions on compact *Lie* groups and how the behavior of the densities of the law of X_t (i.e. the heat kernel) depends of the dimension, see Refs. 7 and 40. As we do not include proofs, these ingredients will not be very visible here but they play a crucial role in Refs. 7–9, 11 and 13. The results described below show that many infinite dimensional compact connected groups carry Brownian motions having very nice properties resembling those of Brownian motions on Lie groups. This is due to the compacity of the underlying group and is in sharp contrast with what happens on infinite dimensional vector spaces.

Three early references relevant to the subject of this paper are Refs. 31, 34 and 44. Heyer's book[36] is a thorough introduction to the fundamental objects that we will discuss in this survey, i.e. Brownian motions and Gaussian convolutions semigroups on compact connected groups. In particular, Secs. 3 and 4 in Chap. 6 of Ref. 36 can be considered as one of the starting points of our investigations. However, most of the issues discussed in this survey are not considered in Ref. 36. One of the more specific roots of our investigations can be found in the independent works of Bendikov and Berg in 1974[1,21] where certain Brownian motions on the infinite dimensional torus are studied in details. The work of Bendikov and Berg, and its potential theoretic relevance, are explained in detail in Ref. 3 which is one of important sources of ideas for the developments presented here.

1.1. *Brownian motions on compact groups*

Let us give a more formal definition of what we call a Brownian motion on a compact group G. Let $X = ((X_t)_{t>0}, \mathbf{P})$, $X_0 = e$, be a G-valued random process.

Definition 1.1. We say that X is a Brownian motion on G if the following properties are satisfied:

(1) X has independent stationary increments, i.e. for any subdivision $0 < t_1 < \cdots < t_n$, $\{X_{t_{i-1}}^{-1} X_{t_i}\}_1^n$ is a family of independent random variables and the law of $X_{t_{i-1}}^{-1} X_{t_i}$ depends only on $s = t_i - t_{i-1}$.
(2) X has continuous paths $t \mapsto X_t(\omega)$, $\mathbf{P}$ a.s.
(3) X is symmetric, i.e. X_t^{-1} has the same law as X_t for all $t > 0$.
(4) X is central, i.e. $a^{-1} X_t a$ has the same law as X_t for all $t > 0$ and $a \in G$.
(5) X is non-degenerate, i.e. for any connected open neighborhood U of the identity, X_t visits any open subset of U with positive probability before it first exits U.

Remarks. (a) We obtain a process having an arbitrary starting point $g \in G$ by considering $(gX_t)_{t>0}$. We call this process X as well.

(b) Property 1 contains the fact that the process X is *left-invariant.* Together with Property 1, Property 4 is equivalent to the requirement that X be also *right-invariant*, hence *bi-invariant.* As we work here on compact groups, Property 4 is a natural requirement. We would like to point out that dropping this condition would considerably change the class of processes one deals with. For instance, on Lie groups, it would allow processes whose infinitesimal generator is subelliptic but not elliptic. In fact, it is a very natural problem to study general symmetric left-invariant diffusions on compact groups and try to generalize to such processes some of the results described in this survey for Brownian motions. This however seems to be a very difficult task.

(c) Property 2 implies that X never leaves the connected component of the neutral element. Hence, there is no loss of generality in assuming that G is connected. The existence of Brownian motions on any compact connected group is proved using the fact that any compact connected group is the projective limit of compact connected Lie groups. See Chap. 6 of Ref. 36 for metrizable groups.

(d) In contrast with the case of $\mathbb{R}^n$, there are often more than one Brownian motion on a given compact Lie group G, even after taking into account all automorphisms of G and linear changes of time scale. For instance, on the two-dimensional torus $\mathbb{T}^2$, Brownian motions are in one-to-one correspondence with flat Riemannian metrics (up to scaling) and classifying Brownian motions up to linear time changes is the same as classifying lattices in $\mathbb{R}^2$ up to isometries and dilations. To understand this phenomenon, consider the torus as $\mathbb{R}^2/\mathbb{Z}^2$. Lift the given Brownian motion to $\mathbb{R}^2$. There is a basis in $\mathbb{R}^2$ in which our Brownian motion has the canonical form. However, the transformation involved in this change of basis will not, in general, leave the lattice $\mathbb{Z}^2$ invariant. Obviously, as far as $\mathbb{T}^2$ is concerned, only those transformations that preserve $\mathbb{Z}^2$ can be allowed, leaving us with an infinite family of different Brownian motions on $\mathbb{T}^2$.

(e) As far as lack of unicity goes, the *Abelian* case is, in some sense, the worst possible case. Indeed, on a compact connected *simple Lie group*, there is (up to time change) a unique Brownian motion. It corresponds to the unique (up to scaling) bi-invariant Riemannian metric. This bi-invariant Riemannian metric can be obtained canonically as (minus) the Killing form. For all of this, see Ref. 35. Note that the requirement that any Brownian motion be invariant under inner automorphisms is crucial to have this unicity on simple compact Lie groups. Any compact connected *semisimple Lie group* G is a quotient of a product of finitely many connected compact simple Lie groups by a finite central subgroup.[35] On such a group, the set of all Brownian motions can be parametrized by a finite sequence of positive numbers (as many as simple factors), each of which is a scaling factor. This fact generalizes, in some sense, to any compact connected semisimple group. See Ref. 37 and Secs. 3 and 4.1 below.

1.2. *Convolution semigroups*

As explained above, the one-dimensional marginals of Brownian motions will be our main object of study in this survey. Obviously, they form convolution semigroup of measures on G. More precisely, one has the following well-known characterization of Brownian motions in terms of convolution semigroups. Recall that a metrizable group is a group whose topology can be defined by a metric. For locally compact groups, this condition is equivalent to having a countable basis for the topology.

Theorem 1.2. *Let G be a compact connected group.*

(1) *Let X be a Brownian motion on G and let μ_t be the law of X_t, $t > 0$. Then*

  1.1 *For any t, $s > 0$, $\mu_t * \mu_s = \mu_{t+s}$.*
  1.2 *$\mu_t \to \delta_e$ weakly as $t \to 0$.*
  1.3 *For any neighborhood V of e,*

$$\lim_{t\to 0} \frac{1}{t}\mu_t(G \setminus V) = 0\,.$$

1.4 *For any* $t > 0$ *and Borel subset* V, $\mu_t(V^{-1}) = \mu_t(V)$, *i.e.* μ_t *is symmetric*;
1.5 *For any* $t > 0$, $a \in G$, *and Borel subset* V, $\mu_t(a^{-1}Va) = \mu_t(V)$, *i.e.* μ_t *is central*;
1.6 *For any* $t > 0$, μ_t *is non-degenerate, i.e. for any* $t > 0$, *the projection of* μ_t *on any Lie quotient of* G *is absolutely continuous with respect to Haar measure.*

(2) *Conversely, if* G *is a metrizable group, for any family* $(\mu_t)_{t>0}$ *of probability measures on* G *satisfying* 1.1–1.6, *there exists a Brownian motion* X *on* G *such that* μ_t *is the law of* X_t, $t > 0$.

Properties 1.1 and 1.2 define convolution semigroups of measures. The additional Property 1.3 defines Gaussian convolution semigroups. Theorem 1.2 asserts that, in the metric case, Brownian motions on G are in one-to-one correspondence with nondegenerate central symmetric Gaussian convolution semigroups. For the Abelian case, see, e.g. Refs. 22 and 25. The non-Abelian case is due to Heyer and Siebert, see Chap. 6 of Ref. 36.

Let G be a compact, connected group. Our main interest is to investigate properties that the convolution semigroup $(\mu_t)_{t>0}$ associated with a Brownian motion X on G might have or not. Let us observe that, in infinite dimension, it is often the case that the measures μ_t, $t > 0$, are singular with respect to Haar measure. For instance, if G is a countable product of identical factors $N_i \cong N$ where N is a compact Lie group, and if μ_t is the product of identical factors μ_t^i on each N_i, then a well-known theorem of Kakutani easily implies that μ_t is singular with respect to Haar measure.

One natural question is that of existence, on any given compact connected group G, of Brownian motions having some of the following properties.

Definition 1.3. We say that a Brownian motion X on G is:

(AC) if, for all $t > 0$, μ_t is absolutely continuous w.r.t. Haar measure.
(CK) if (AC) holds and, for all $t > 0$, μ_t admits a continuous density;
(CKλ*) if (CK) holds and the continuous density $\mu_t(\cdot)$ satisfies

$$\lim_{t\to 0} t^\lambda \log \mu_t(e) = 0$$

(when $\lambda = 1$, we write (CK*) for (CK1*));
(CK#) if (CK) holds and the continuous density $\mu_t(\cdot)$ satisfies

$$\lim_{t\to 0} \sup_{x\in K} \mu_t(x) = 0$$

for each compact K such that $e \notin K$.

In Sec. 4.4, we review what is known about the existence of Brownian motions having such properties. It turns out that there are many Brownian motions on infinite dimensional groups which satisfy some or all of these properties. Properties (AC) and (CK) are very natural properties to consider. The importance of properties such as (CK$*$) and (CK$\#$) first appeared in the early work of Forst,[31] Berg[21,23,24] and Bendikov.[1,2] These properties play an essential role in what follows.

Note that, by comparison with (AC) and (CK), (CK$\lambda*$) is a more quantitative property. One should also stress the difference between properties (CK)–(CK$\lambda*$) and properties (AC)–(CK$\#$). The former are expressed in terms of the on-diagonal behavior of μ_t, i.e. in terms of $\mu_t(e)$ (indeed, (CK) is equivalent to $\mu_t(e) < +\infty$, for all $t > 0$). The latter properties involve the off-diagonal behavior of μ_t.

Anticipating on some of the definitions that will be given below, let us relate the notions introduced above to well-known facts concerning Brownian motions on compact connected Lie groups. Thus, let X be a Brownian motion on a compact connected Lie group G of dimension n. Let $(\mu_t)_{t>0}$ be the associated Gaussian semigroup.

(a) If G is Abelian, then $G = \mathbb{R}^n/\mathbb{Z}^n$ is a finite-dimensional torus. The infinitesimal generator $-L$ of X has the form

$$-L = \sum_{i,j=1}^{n} a_{i,j}\partial_i\partial_j \,,$$

where $a_{i,j} = a_{j,i}$ (by Definition 1.1(3)). It is not very hard to see that Definition 1.1(5) implies that L is elliptic, that is

$$\sum_{i,j} a_{i,j}\xi_i\xi_j > 0 \quad \text{for all } \xi = (\xi_1, \dots, \xi_n) \neq 0\,.$$

(b) If G is simple, Definition 1.1(4) implies that the infinitesimal generator is proportional to the Laplace–Beltrami operator of the bi-invariant Riemannian metric induced by the Killing form on G. By Definition 1.1(5), the proportionality constant is not 0.

(c) By the structure theory of compact connected Lie groups, it follows from the Abelian and simple cases that, in general, the infinitesimal generator $-L$ of X is the Laplace–Beltrami operator of a bi-invariant Riemannian metric on G. In particular, L is an elliptic operator. In the sense of distributions, $(\mu_t)_{t>0}$ solves the heat diffusion equation $(\partial_t + L)u = 0$ with initial data the Dirac mass at the neutral element. Hence classical regularity theory implies that, for each $t > 0$, μ_t admits a smooth positive density w.r.t. Haar measure. Denote this density by $x \mapsto \mu_t(x)$. Then it is well known that

$$\log \mu_t(e) \sim \frac{n}{2} \log \frac{1}{t} \quad \text{as } t \text{ tends to } 0\,.$$

Hence, on a Lie group, property (CK$\lambda*$) is always satisfied for all $\lambda > 0$. In addition, Gaussian heat kernel bounds as developed in Refs. 30 and 45 show that $(\mu_t)_{t>0}$ satisfies property (CK$\#$).

1.3. *Projective limits*

A fundamental structure theorem concerning compact connected groups is that they all arise as projective limits of compact connected Lie groups (see, e.g. Refs. 33, 36, 37 and the reference therein). More precisely, any compact connected group G contains a descending family of compact normal subgroups K_α, $\alpha \in \aleph$, such that $\bigcap_\aleph K_\alpha = \{e\}$ and $G_\alpha = G/K_\alpha$ is a compact connected Lie group. Write $\alpha \prec \beta$ if $K_\alpha \supset K_\beta$. If $\alpha \prec \beta$, let $\pi_{\alpha,\beta}$ be the canonical projection of G_β onto G_α, and π_α be the canonical projection of G onto G_α. It follows from the properties of the family K_α, $\alpha \in \aleph$, that G is the projective limit of $(G_\alpha, \pi_{\alpha,\beta})_{\alpha,\beta\in\aleph;\alpha\prec\beta}$. This means that G is isomorphic to the closed subgroup of the Cartesian product $\prod_\aleph G_\alpha$ whose elements are those $g = (g_\alpha)$ such that, for all pairs $\alpha \prec \beta$ in $\aleph$, $\pi_{\alpha,\beta}(g_\beta) = g_\alpha$. When G is metrizable, one can take $\aleph$ to be at most countable and the sequence K_α to be decreasing.

The projective structure described above allows us to construct the projective Lie algebra $\mathfrak{G}$ of a compact connected group G. The Lie algebra $\mathfrak{G}$ is defined to be the projective limit of the Lie algebras $\mathfrak{G}_\alpha$ equipped with the projection maps $d\pi_{\beta,\alpha}$, $\alpha \prec \beta$. It is independent of the choice of the descending system $(K_\alpha)_\aleph$ and is a linear topological space isomorphic to a product of lines (finite, countable, or uncountable). This follows from Ref. 27 where Born studies carefully the notion of *projective basis* associated with a descending system (K_α) in G (see also Ref. 29). Given a descending system (K_α) as above, a family $Y = (Y_i)_{i\in I}$ of elements of $\mathfrak{G}$ is a projective basis relatively to (K_α) if, for each $\alpha \in \aleph$, the subset I_α of all $i \in I$ such that $d\pi_\alpha(Y_i) \neq 0$ is finite and the family $\{Y_i^\alpha = d\pi_\alpha(Y_i) : i \in I_\alpha\}$ is a vector basis of the Lie algebra $\mathfrak{G}_\alpha$. Born[27] shows that there always exists a descending family (K_α) such that $\mathfrak{G}$ admits projective basis with respect to (K_α). By construction, if $Y = (Y_i)_{i\in I}$ is a projective basis, then $\mathfrak{G} = \{Y = \sum_I y_i Y_i : y_i \in \mathbb{R}\} = \mathbb{R}^I$ with the topology of convergence coordinate by coordinate, i.e. the product topology.

The space $\mathcal{B}(G)$ of Bruhat test functions on G (introduced in Ref. 29) can be defined as the space of all functions f on G for which there exist $\alpha \in \aleph$ and a smooth function ϕ on G_α such that $f = \phi \circ \pi_\alpha$. Given a projective basis $Y = (Y_i)_{i\in I}$, any element $Y = \sum y_i Y_i$ of $\mathfrak{G}$ can be viewed as a left-invariant differential operator on G acting on any Bruhat test function $f = \phi \circ \pi_\alpha$ by

$$Yf = \sum_{i\in I_\alpha} y_i [Y_i^\alpha \phi] \circ \pi_\alpha \,.$$

By definition, a homogeneous left-invariant operator P of order k on G is a formal sum

$$P = \sum_{j\in I^k} a_j Y^j \,,$$

where $Y^j = Y_{i_1} \cdots Y_{i_k}$ if $j = (i_1, \ldots, i_k)$. Its action on the Bruhat test function

$f = \phi \circ \pi_\alpha$ is given by

$$Pf = \sum_{j \in I_\alpha^k} a_j [Y_{i_1}^\alpha \cdots Y_{i_k}^\alpha \phi] \circ \pi_\alpha \,.$$

One can check that the space $\mathcal{B}(G)$ and the notion of homogeneous left-invariant differential operators introduced above are independent of the choice of the descending family (K_α) (see Ref. 29).

Differential operators on G or G/H can obviously be identified if H is a finite normal subgroup of a Lie group G. One generalization of this statement is as follows.

Theorem 1.4. *Let G, $\bar{G}$ be locally compact connected groups such that $G = \bar{G}/N$ where N is a closed normal totally disconnected subgroup of $\bar{G}$. Then, for any left-invariant differential operator P on G there exists a unique left-invariant differential operator $\bar{P}$ on $\bar{G}$ such that*

$$\forall\, f \in \mathcal{B}(G)\,, \quad \bar{P}[f \circ \pi] = (Pf) \circ \pi\,,$$

where π is the canonical projection from $\bar{G}$ onto G.

The proof consists of showing that $G = \bar{G}/N$ and $\bar{G}$ have the same projective Lie algebra.

1.4. *Infinitesimal generators*

Let G be a connected compact group with normalized Haar measure ν. Fix a Brownian motion $X = ((X_t)_{t>0}, \mathbf{P})$ on G. Let μ_t be the law of X_t. Consider the self-adjoint Markov semigroup of operators on $L^2(G, d\nu)$ defined by

$$H_t f(x) = E^{\mathbf{P}}(f(xX_t)) = \int f(xy) d\mu_t(y)\,.$$

The infinitesimal generator $-L$ of H_t is defined by

$$Lf = \lim_{t \to 0} \frac{I - H_t}{t} f$$

with domain $\mathcal{D}$ equals to the set of all functions in $L^2(G, d\nu)$ such that this limit exists in $L^2(G, \nu)$. See Ref. 32. By construction L is a bi-invariant self-adjoint non-negative operator on $L^2(G, \nu)$. In the case of Lie groups, a celebrated (more general) theorem of Hunt describes L as the unique extention of a left-invariant (in our case bi-invariant and symmetric) second-order differential operator. Using the projective structure, this generalizes to all compact connected groups.

Theorem 1.5.[28] *Let G be a compact connected group. Let $Y = (Y_i)_{i \in I}$ be a projective basis. Then, the domain $\mathcal{D}$ of the infinitesimal generator $-L$ of any symmetric Gaussian semigroup contains the set $\mathcal{B}(G)$ of all Bruhat test functions on G as a core. Moreover, restricted to $\mathcal{B}(G)$, L is a left-invariant second-order differential operator of the form*

$$L = -\sum_{i,j} a_{i,j} Y_i Y_j\,,$$

where the matrix $A = (a_{i,j})$ *is real symmetric and satisfies* $\sum_{i,j}, a_{i,j}\xi_i\xi_j \geq 0$ *for all* $\xi = (\xi_i)_{i\in I}$ *with finitely many nonzero coordinates.*

The infinitesimal generator $-L$ can be used to define the so-called intrinsic quasi-distance associated to X which introduces an adapted geometric structure on the group G. Define the field operator $\Gamma(f,f)$ by setting

$$\Gamma(f,f) = -\frac{1}{2}(Lf^2 - 2fLf)$$

for any Bruhat test function f on G.

Definition 1.6. For any x, $y \in G$, set

$$d(x,y) = \sup\{f(x) - f(y) : f \in \mathcal{B}(G)\,, \quad \Gamma(f,f) \leq 1\}\,.$$

Set also $d(x) = d(e,x)$ and $D = \{x \in G : d(x) < \infty\}$.

The quasi-distance d has become a classic object in the context of analysis on local Dirichlet spaces. See, e.g. Ref. 43. Observe that, because L is bi-invariant, d is also bi-invariant, that is $d(x,y) = d(zx,zy) = d(xz,yz)$ or, equivalently, $d(xy) = d(yx)$. It can be shown that D is a dense Borel subgroup of G. It follows that either $D = G$ or $\nu(D) = 0$. Moreover, if $D = G$, then d is bounded on G. For all of this, see Ref. 17. It is an interesting problem to relate properties of d to properties of X. For instance, what can be said about X if d is continuous? Conversely, which property of X implies that d is continuous? These questions will be discussed later on in this survey.

2. The Case of the Infinite Dimensional Torus

To illustrate the above definitions, let us consider the important case where $G = \mathbb{T}^\infty = \mathbb{R}^\infty/\mathbb{Z}^\infty$. In fact, by a theorem of Dixmier (see, e.g. Refs. 36 and 37), $\mathbb{T}^\infty$ and the finite-dimensional torii $\mathbb{T}^n$ are the only compact connected locally connected metrizable Abelian groups. Let X be a Brownian motion on $\mathbb{T}^\infty$ and $(\mu_t)_{t>0}$ be the associated convolution semigroup. These two objects are uniquely determined by an infinite symmetric positive matrix A. Here, positive means that $\sum_{i,j} a_{i,j}\xi_i\xi_j > 0$ for any nonzero vector $\xi \in \mathbb{R}^{(\infty)}$ (i.e. ξ has only finitely many nonzero coordinates). The matrix A can be introduced via Fourier analysis. Namely, the dual of $\mathbb{T}^\infty$ is the group $\mathbb{Z}^{(\infty)}$ and the Fourier transform $\hat{\mu}_t$ is given by

$$\hat{\mu}_t(\theta) = \exp(-t\psi(\theta))\,,$$

where $\psi(\theta)$ is a positive definite quadratic form on $\mathbb{Z}^{(\infty)}$. The matrix A is the matrix of ψ in the standard basis, i.e.

$$\psi(\theta) = \sum_{i,j} a_{i,j}\theta_i\theta_j\,.$$

The matrix A can also be viewed as the matrix of the coefficients of the infinitesimal generator

$$-L = \sum a_{i,j}\partial_i\partial_j$$

acting on functions depending on finitely many coordinates. This second viewpoint illustrates Theorem 1.5 in the easy case of $\mathbb{T}^\infty$.

A natural problem in this setting is to relate properties such as (AC), (CK), (CK$\lambda*$), (CK#) to properties of the matrix A.

2.1. *The diagonal case*

On $\mathbb{T}^\infty$, the existence of Brownian motions having some prescribed properties can be obtained by looking at the diagonal case where $a_{i,i} = a_i$, $a_{i,j} = 0$ for $i \neq j$, for some sequence $\mathbf{a} = (a_i)$ of positive numbers. The study of this case is crucial in discovering interesting relations between different properties and developing reasonable conjectures for the general case. Around 1974, Christian Berg[21] and the first author[1] were the first to study Gaussian semigroups on the infinite-dimensional torus explicitly. They mostly looked at the case when A is diagonal and were motivated by potential theoretic questions raised by the work of Bliedtner.[26] See, e.g. Refs. 1–3, 21, 23 and 24.

Given $\mathbf{a} = (a_i)$, set

$$N(s) = N_{\mathbf{a}}(s) = \sum_{i:a_i\le s} 1 = \#\{i : a_i \le s\}\,.$$

The following theorem gathers some of the noteworthy results (mostly from Refs. 1, 3, 4, 23 and 24) concerning the diagonal case $a_{i,i} = a_i > 0$, $a_{i,j} = 0$ otherwise.

Theorem 2.1.[3,4,7,9] *In the diagonal case on $\mathbb{T}^\infty$, we have the following results*:

(1) *Define $t_* \in [0, +\infty]$ by setting*

$$t_* = \frac{1}{2}\limsup_{s\to\infty} \frac{1}{s}\log N(s)\,.$$

Then μ_t is singular w.r.t. Haar measure for $t < t_*$ and is absolutely continuous w.r.t. Haar measure for all $t > t_*$. For $t > 2t_*$, μ_t admits a continuous density. For $t \in (t_*, 2t_*)$, the density of μ_t is unbounded but belongs to all L^p, $p \in [1, +\infty)$.

(2) (AC) *is equivalent to* (CK) *and holds if and only if*

$$\lim_{s\to\infty} \frac{1}{s}\log N(s) = 0\,.$$

(3) (CK#) *is equivalent to* (CK$*$) *and holds if and only if*

$$\lim_{s\to\infty} \frac{1}{s}N(s) = 0\,.$$

(4) *The quasi-distance* d *is continuous if and only if*

$$\int_0^\infty \frac{N(s)}{s^2} ds < +\infty\,, \quad i.e. \quad \sum_1^\infty \frac{1}{a_i} < \infty\,.$$

Moreover, if d *is continuous, then*

$$\lim_{t \to 0} -4t \log \mu_t(x) = d(x)^2\,.$$

(5) *There exists a positive continuous function* $s \mapsto G(s)$ *satisfying* $G(s) = 1/2 + o(e^{-1/s})$ *as* $s \to 0$ *and* $G(s) = 2se^{-s}(1 + o(1))$ *as* $s \to \infty$ *and such that, for any symmetric diagonal Gaussian semigroup on* $\mathbb{T}^\infty$,

$$\log \mu_s(e) = \int_0^\infty G(st) N(t) \frac{dt}{t}\,.$$

Part 5 allows us to construct Gaussian convolution semigroups having various behaviors. To illustrate some possible behaviors consider the following cases treated explicitly in Ref. 4. Let λ be a positive real parameter.

(1) Set $a_k = k^{1/\lambda}$. Then $N(t) \sim t^\lambda$ which gives

$$\log \mu_t(e) \sim c_\lambda t^{-\lambda} \quad \text{as } t \to 0\,.$$

(2) Property (CK$\lambda *$) is equivalent to $N(t) = o(t^\lambda)$ as t tends to infinity. For instance, taking $a_k = [k \log(1 + k)]^{1/\lambda}$ gives $N(t) \sim \lambda^{-1} t^\lambda [\log(1 + t)]^{-1}$ which, in turn, gives

$$\log \mu_t(e) \sim \frac{c_\lambda t^{-\lambda}}{\log(1 + 1/t)} \quad \text{as } t \to 0\,.$$

(3) Set $a_k = e^{k^{1/\lambda}}$. Then $N(s) \sim [\log(1 + s)]^\lambda$, which yields

$$\log \mu_t(e) \sim c_\lambda [\log(1 + 1/t)]^{1+\lambda} \quad \text{as } t \to 0\,.$$

(4) Set $a_k = e^{e^{k^{1/\lambda}}}$. Then $N(s) \sim [\log_{(2)}(s)]^\lambda$, where $\log_{(2)}(s) = \log(1 + \log(1 + s))$. This yields

$$\log \mu_t(e) \sim c_\lambda \log(1 + 1/t) [\log_{(2)}(1/t)]^\lambda \quad \text{as } t \to 0\,.$$

(5) Set $a_k = [\log(1 + k)]^{1/\lambda}$. Then $N(s) \sim e^{s^\lambda}$. If $\lambda > 1$, then we have $t_* = +\infty$ whereas, for $\lambda = 1$, we have $t_* = 1/2$. If $\lambda \in (0, 1)$, then $t_* = 0$ and

$$\log \log \mu_t(e) \sim c_\lambda t^{-\lambda/(1-\lambda)} \quad \text{as } t \to 0\,.$$

2.2. *The general Abelian case*

Understanding general symmetric Gaussian semigroups on $\mathbb{T}^\infty$ is a delicate question and we can start by stating what appears to be one of the fundamental open problem in this area of research: *Is* (AC) *always equivalent to* (CK)? As we have seen above, in the diagonal case, (AC) is indeed equivalent to (CK).

Introduce the spectral function

$$W(s) = W_A(s) = \#\{\theta \in \mathbb{Z}^{(\infty)} : \langle A\theta, \theta\rangle \le s\}.$$

In words, W counts the number of lattice points in the ellipsoide $\{\xi : \langle A\xi, \xi\rangle \le s\}$ in infinite dimension. It seems that the best one can hope for is to describe the properties of $(\mu_t)_{t>0}$ in terms of W. On the one hand, this is easy for certain properties related to the behavior of $\mu_t(e)$ because

$$\mu_t(e) = \int_0^\infty e^{-st} dW(s).$$

For instance, for any fixed $\lambda \in (0, \infty)$, (CK$\lambda *$) holds true if and only if $\log W(s) = o(s^{\lambda/(1+\lambda)})$. A more subtle argument due to Berg[23] shows that property (CK) is equivalent to the behavior

$$\log W(s) = o(s) \quad \text{as } s \to \infty.$$

On the other hand, characterizing properties such as (AC) or (CK#) is not obvious.

Theorem 2.2.[9] *For any Brownian motion on* $\mathbb{T}^\infty$ *properties* (CK#) *and* (CK$*$) *are equivalent and they hold true if and only if*

$$\log W(s) = o(\sqrt{s}) \quad \text{as } s \to \infty.$$

Yet another equivalent property is that the Green function $q = \int_0^\infty e^{-t}\mu_t dt$ *is absolutely continuous w.r.t. the Haar measure and admits a continuous density on* $G \setminus \{e\}$.

This is a fundamental result and we comment on the proof. That (CK$*$) implies (CK#) holds in greater generality. In particular, it holds for any symmetric Gaussian semigroup on any locally compact metrizable group. See Refs. 9 and 17. The same applies to the fact that (CK#) implies that the Green function q admits a continuous density on $G \setminus \{e\}$. See Ref. 3. To show that (CK$*$) must hold when q admits a continuous density on $G \setminus \{e\}$, we use the inequality

$$\forall x \in \mathbb{T}^n, \quad \forall t > 0, \quad \mu_t(x) \ge \mu_t(e) e^{-d(x)^2/4t}$$

which holds true for any symmetric Gaussian semigroup on any finite dimensional torus $\mathbb{T}^n$ and extends, in some sense, to $\mathbb{T}^\infty$. See Ref. 9.

The same inequality is crucial for the following result.

Theorem 2.3.[9] *Let* X *be a Brownian motion on* $\mathbb{T}^\infty$.

(1) *If the quasi-distance* d *is continuous, then* X *satisfies* (CK$*$).
(2) *If* X *is* (CK$\lambda *$) *for some* $\lambda \in (0, 1)$, *then* d *is continuous and*

$$\lim_{t \to 0} -4t \log \mu_t(x) = d(x)^2.$$

We end this section with some further open questions concerning symmetric Gaussian semigroups on $\mathbb{T}^\infty$. Consider the following *critical times* regarding the appearance of certain nice properties of μ_t as $t > 0$ increases. Let t_{Sing}, (resp. t_{AC}, t_{L^p}, t_{CK}) be the infimum of the times $t > 0$ such that μ_t is not singular with respect to Haar measure (resp. is absolutely continuous, is absolutely continuous with density in L^p, is absolutely continuous with a continuous density). Since $(\mu_t)_{t>0}$ is a convolution semigroup, for any of these properties, μ_t automatically has the property in question for all times t larger than the corresponding critical time. Note also that, by convolution and symmetry, we always have $2t_{L^2} = t_{L^\infty} = t_{\text{CK}}$, see Lemma 4.2 of Ref. 13. Moreover, it is easy to check that

$$t_{L^2} = \limsup_{s \to \infty} \frac{1}{s} \log W(s) \,.$$

In the case of diagonal Gaussian semigroups, Theorem 2.1 shows that

$$t_{\text{Sing}} = t_{\text{AC}} = t_{L^p} = t_{\text{CK}}/2 \,, \tag{2.1}$$

where $p \in [1, \infty)$. Thus it is natural to ask whether (2.1) also holds in the general case. In fact, much weaker questions are open. For instance, does there exists a constant C such that for any symmetric Gaussian semigroup on $\mathbb{T}^\infty$, $t_{\text{CK}} \le C\, t_{\text{Sing}}$? Is it true that $t_{\text{Sing}} < \infty$ implies $t_{\text{CK}} < \infty$? Is it true that $t_{\text{Sing}} < \infty$ implies $t_{\text{AC}} < \infty$? Is it true that $t_{\text{AC}} < \infty$ implies $t_{\text{CK}} < \infty$?

2.3. *Hidden diagonal cases*

The aim of this section is to point out that the diagonal case of Sec. 2.1 covers more cases than one would naively think. The idea indicated below may also prove important in understanding general Brownian motions on $\mathbb{T}^\infty$.

We need to be more explicit about our use of coordinates on $\mathbb{T}^\infty$. By definition, $\mathbb{R}^\infty = \{x = \sum_1^\infty x_i E_i\}$, where E_i is the sequence with a 1 in position i and 0 everywhere else, i.e. "the standard basis" of $\mathbb{R}^\infty$. The topology is the product topology, i.e. the topology of convergence coordinate by coordinate. Set

$$\mathbb{Z}^\infty = \left\{ Z = \sum_i z_i E_i : z_i \in \mathbb{Z} \right\} .$$

Then, by definition, $\mathbb{T}^\infty = \mathbb{R}^\infty / 2\pi \mathbb{Z}^\infty$.

Let us now define a suitable notion of "basis" for $\mathbb{R}^\infty$. Let $F = (F_i)_1^\infty$ be a family of vectors in $\mathbb{R}^\infty$. For any sequence of reals $\mathbf{y} = (y_i)$, set

$$y^n = \sum_1^n y_i F_i$$

and let $x_i^n(\mathbf{y})$, $i = 1, 2, \ldots$, be the coordinates of y^n in the standard basis $E = (E_i)$. We say that F is a basis of $\mathbb{R}^\infty$ if the following two properties are satisfied:

(1) For any $x = \sum_1^\infty x_i E_i \in \mathbb{R}^\infty$, there exists a unique sequence $\mathbf{y} = (y_i)$ of reals such that

$$x = \sum_1^\infty y_i F_i$$

in the sense that $x_i^n(\mathbf{y})$ converges to x_i as n tends to infinity.

(2) For any sequence $\mathbf{y} = (y_i)$ of reals, for each $i = 1, 2, \ldots$, $x_i^n(\mathbf{y})$ converges, as n tends to infinity.

Let us give some examples and counterexamples concerning basis of $\mathbb{R}^\infty$. First, changing the order of the elements of a basis produces a new basis. Second, for any infinite upper triangular matrix C with nonzero diagonal entries, $F_i = \sum_j c_{i,j} E_j$ is a basis of $\mathbb{R}^\infty$. The same conclusion applies, for any infinite matrix C which is block-diagonal with finite dimensional blocks $C_1, C_2, \ldots$ which are all invertible matrices. These three families of examples already yield many nontrivial bases and more are obtained by iterating these constructions. One interesting simple example of basis is $F = (F_i)$ with $F_i = \sum_{j \geq i} E_i$, $i = 1, 2, \ldots$. Let us mention also two examples of families $F = (F_i)$ which *are not basis* in the sense considered above: (1) $F = (F_i)$ with $F_i = \sum_{j=1}^i E_j$, $i = 1, 2, \ldots$; (2) $F = (F_i)$ with $F_1 = E_1$ and $F_i = E_{i-1} + E_i$, $i = 2, 3, \ldots$.

Any basis F as above is as good as the standard basis E to write $\mathbb{R}^\infty$ as a product of lines. However, it is important to realize that, as far as our given infinite dimensional torus $\mathbb{T}^\infty = \mathbb{R}^\infty / 2\pi\mathbb{Z}^\infty$ is concerned, not all changes of basis are admissible. For any basis F, set

$$\mathbf{Z}_F = \left\{ z = \sum z_i F_i : z_i \in \mathbb{Z} \right\} .$$

Of course, given a basis F of $\mathbb{R}^\infty$, $\mathbf{T}_F = \mathbb{R}^\infty / \mathbf{Z}_F$ is an infinite dimensional torus. But, for our study of Gaussian convolution semigroups these different infinite dimensional torii should be viewed as distinct objects. Indeed, for a given coefficient matrices A, let $(\bar{\mu}_t^A)_{t>0}$ denote the associated Gaussian semigroup on $\mathbb{R}^\infty$ and let $(\mu_t^{F,A})_{t>0}$ be the projection on $\mathbf{T}_F$. Then, for a given A, the different semigroups $(\mu_t^{F,A})$ may have very different properties, depending on F. For instance, it is well possible that for one F, $(\mu_t^{F,A})_{t>0}$ is absolutely continuous with respect to Haar measure whereas, for another F, it is singular.

Still, there are many bases F of $\mathbb{R}^\infty$ that are *admissible* with respect to $\mathbb{T}^\infty$ in the sense that $\mathbf{Z}_F = \mathbf{Z}_E$. For such basis, we of course have (with equalities, not isomorphisms)

$$\mathbb{T}^\infty = \mathbf{T}_E = \mathbf{T}_F .$$

Thus, a fundamental problem in the study of Gaussian convolution semigroups on $\mathbb{T}^\infty$ can be formulated informally as follows. *Given a symmetric positive definite matrix A associated in the standard basis E to a Gaussian semigroup $(\mu_t)_{t>0} =$*

$(\mu_t^A)_{t>0}$ *on* $\mathbb{T}^\infty$, *find an admissible basis* F *of* $\mathbb{R}^\infty$ *which is well suited to the study of* $(\mu_t)_{t>0}$.

This seems to be a rather hard problem. Note that, in each basis F, $(\mu_t)_{t>0} = (\mu_t^A)_{t>0}$ will be represented by a different matrix A_F and it is well possible for A_F to be a diagonal matrix for a well chosen basis F. If this is the case, $(\mu_t)_{t>0}$ is a *diagonal* Gaussian convolution semigroup for the product decomposition of $\mathbb{T}^\infty$ associated to F and the results of Sec. 2.1 apply. This leads us to a weaker but more precise form of the problem posed above. *Given a symmetric positive definite matrix* A *associated in the standard basis* E *to a Gaussian semigroup* $(\mu_t)_{t>0} = (\mu_t^A)_{t>0}$ *on* $\mathbb{T}^\infty$, *find whether or not there exists an admissible basis* F *in which the matrix* A_F *representing* $(\mu_t)_{t>0}$ *is diagonal* (*or block diagonal*). We have no good answer to this problem at this writing, but it seems worth giving a simple example.

Consider the following family of matrices. For any non-decreasing sequence $\mathbf{b} = (b_i)$ of positive numbers, set

$$A = A(\mathbf{b}) = \begin{pmatrix} b_1 & b_1 & b_1 & b_1 & b_1 & b_1 & \cdot\cdot \\ b_1 & b_2 & b_2 & b_2 & b_2 & b_2 & \cdot\cdot \\ b_1 & b_2 & b_3 & b_3 & b_3 & b_3 & \cdot\cdot \\ b_1 & b_2 & b_3 & b_4 & b_4 & b_4 & \cdot\cdot \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_5 & \cdot\cdot \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & \cdot\cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\cdot \end{pmatrix}.$$

This is a symmetric positive definite matrix with (i,j)-coefficient $\min\{b_i, b_j\}$ and we can consider the symmetric Gaussian semigroup $(\mu_t)_{t>0} = (\mu_t^A)_{t>0}$ on $\mathbb{T}^\infty$, whose matrix in the standard basis E is A. Set $F_i = \sum_{j\ge i} E_j$, $i = 1, 2, \ldots$. Then $F = (F_i)$ is an admissible basis with respect to $\mathbb{T}^\infty$. Moreover, an easy computation shows that, in F, the matrix A_F representing $(\mu_t)_{t>0}$ is diagonal with entries $a_i = b_i - b_{i-1}$, $i = 1, 2, \ldots$, with the convention that $b_0 = 0$. Thus we can apply the various results of Theorem 2.1. To illustrate this, assume that $b_i = i^\alpha$, with $\alpha \in (0, \infty)$. In this case, Theorem 2.1 gives the following results.

(1) If $\alpha \in (0, 1]$, then μ_t is singular for all $t > 0$.
(2) If $\alpha \in (1, \infty)$, then μ_t is absolutely continuous and admits a continuous density for all $t > 0$. Moreover, $\log \mu_t(e) \sim c_\alpha t^{1/(\alpha-1)}$ as t tends to 0.
(3) If $\alpha \in (2, \infty)$, then the associated intrinsic distance d is continuous and we have

$$\forall\, x \in \mathbb{T}^\infty, \quad \lim_{t\to 0} -4t \log \mu_t(x) = d(x)^2 .$$

3. The Semisimple Case

3.1. *Simple Lie groups*

Recall that a Lie algebra $\mathcal{G}$ over a field of characteristic 0 is simple if it is not Abelian and has no ideals except $\{0\}$ and $\mathcal{G}$. See e.g. Refs. 35 and 37. A simple

Lie group is a Lie group whose Lie algebra is simple. Simple Lie algebras and simple connected Lie groups are classified. Here, we are interested in simple compact connected Lie groups. The simple compact connected Lie groups come in four infinite series and three exceptional series, depending on the type of the corresponding complexe simple Lie algebra (obtained by complexification of the Lie algebra of the group). The infinite series $A_\ell, B_\ell, C_\ell, D_\ell$ correspond to the classical groups. Type A_ℓ corresponds to $\mathrm{SU}(\ell+1)$, type B_ℓ to $\mathrm{SO}(2\ell+1)$ and its two-cover $\mathrm{Spin}(2\ell+1)$, type C_ℓ to $Sp(\ell)$ and type D_ℓ to $\mathrm{SO}(2\ell)$ and its two-cover $\mathrm{Spin}(2\ell)$. The exceptional groups correspond to simple complex Lie algebras of type G_2, F_4 and E_i, $i = 6, 7, 8$. For each simple complex Lie algebra, there is a unique simply connected compact group associated to it and all other compact groups in the class are finite central quotients of this group.

Each simple compact Lie group comes equipped with a canonical metric given by (minus) the Killing form which is, up to multiplication by a positive scalar, the unique bi-invariant Riemannian metric on G. Let us spell this out in the case of $G = \mathrm{SO}(m)$, $m \geq 2$. The dimension of $\mathrm{SO}(m)$ is $n = \binom{m}{2}$. The Lie algebra $\mathfrak{G}$ of $\mathrm{SO}(m)$ is the space of real skew symmetric matrices with the exponential map given by

$$\exp : \mathfrak{G} \to G\,, \quad M \mapsto \exp(M) = \sum_0^\infty \frac{M^k}{k!}\,.$$

It is simple for $m \neq 2, 4$. Let $\{E_{i,j} : 1 \leq i < j \leq m\}$ be the natural basis of $\mathfrak{G}$ where $E_{i,j}$ has all entries zero except the (i,j) and (j,i) entries which respectively equal 1 and -1. The metric induced by the Killing form is

$$B(M,N) = 2(m-2)\sum_{i<j} M_{i,j}N_{i,j} = -(m-2)\mathrm{Tr}(MN)\,.$$

This metric has Ricci curvature equals to $B/4$. Let $g = (g_{i,j})$ be an element of $\mathrm{SO}(m)$ and denote by g_i the column vector $g_i = (g_{k,i})_{1\leq k\leq m}$. The normalized Haar measure is given by

$$dg = \frac{\Gamma(m/2)\Gamma((m-1)/2)\cdots\Gamma(3/2)\Gamma(1)}{2^{m-1}\pi^{n/2}}\left|\bigwedge_{i<j} g_j^t dg_i\right|\,.$$

Write $m = 2p$ or $2p+1$ and let $e^{\pm i\theta_i}$, $\theta_i \in [0, 2\pi)$, $1 \leq i \leq p$, denote the eigenvalues of g, (there is one additional eigenvalue 1 in the odd case). The by-invariant distance d on G induced by the metric B is given by

$$d(e,g)^2 = 2(m-2)\sum |\theta_i|_1^2\,,$$

where

$$|\theta|_1 = \min\{\theta, 2\pi - \theta\}\,.$$

Thus, the diameter is

$$\operatorname{diam}(\mathrm{SO}(m)) = \begin{cases} \pi\sqrt{m(m-2)} & \text{if } m \text{ is even} \\ \pi\sqrt{(m-1)(m-2)} & \text{if } m \text{ is odd} \end{cases}$$

and $\operatorname{diam}(\mathrm{SO}(m)) \sim \pi m \sim \pi\sqrt{n/2}$ as m tends to infinity ($n = \binom{m}{2} = \dim(\mathrm{SO}(m))$).

Although we will not go into any detail here, let us emphasize that the results that we will present below concerning compact connected semisimple groups depend in a crucial way on a uniform analysis of the densities of the Gaussian semigroups associated with Brownian motions on compact connected simple Lie groups. Here uniform refers to the time parameter $t \in (0, +\infty)$ and to the underlying group. In particular, the role played by the dimension of the underlying group must be controlled.

3.2. *Semisimple groups*

For any compact connected group, any element of the algebraic commutator group (i.e. the group G' generated by commutators) is a commutator. It follows that G' is closed and $G'' = G'$. By definition, a compact connected group is semisimple if $G = G'$. This coincides with the usual definition in the case of Lie groups. See Ref. 37.

Semisimple compact connected groups have a very nice structure: up to quotient by a central subgroup they are direct products of compact connected simply connected *simple Lie* groups. For our purpose it will be sufficient to describe this structure for metrizable groups, i.e. groups which admit a countable basis for their topology. If G is a compact connected metrizable semisimple group, there exists a finite or infinite sequence (Σ_k) of compact connected simply connected simple Lie groups such that

$$G = \Sigma/\Xi\,, \quad \Sigma = \prod \Sigma_k\,,$$

where Ξ is a central closed subgroup of Σ. As the center Z of Σ is the product $Z = \prod Z_k$ of the centers of the Σ_k's and each Z_k is finite, it follows that $\Xi \subset Z$ is totally diconnected. (When G is not metrizable, the product above can be over an uncountable number of factors.)

To write G as a projective limit of Lie groups, consider the decreasing sequence (K_j) of compact subgroups where K_j is the image in G of $\prod_{k\geq j} \Sigma_k$. Let $\mathfrak{G}$ be Lie algebra of G. It can be identified with the Cartesian product of the Lie algebras of the Σ_k. Let Y be a projective basis of $\mathfrak{G}$ obtained by listing consecutively linear bases of the Lie algebras of the groups Σ_k, $k = 1, 2, \ldots$. Now, if X is a Brownian motion on $G = \prod \Sigma_k/\Xi$, its infinitesimal generator $-L$ is a second-order differential operator as described in Theorem 1.5. Because this operator is bi-invariant, it is not hard to see that it must split as a sum of bi-invariant second-order differential operators defined on the simple factors Σ_k. But, on each simple factor Σ_k, any bi-invariant second-order differential operator equals a scalar multiple of the Killing

Laplace–Beltrami operator L_k. Thus there exists a sequence $\alpha = (\alpha_k)$ of positive reals such that

$$L_\alpha = \sum_k \alpha_k L_k$$

and the central Gaussian semigroup $(\mu_t)_{t>0}$ associated with the Brownian motion X is given on the set $\mathcal{B}(G)$ of Bruhat test functions by

$$f * \mu_t = e^{-tL_\alpha} f\,.$$

Let n_k be the dimension of Σ_k. It turns out that the two sequences (n_k) and (α_k) suffice to characterize many properties of $(\mu_t)_{t>0}$ through the behavior of the function

$$N_\alpha(s) = \sum_{k:\alpha_k \le s} n_k\,.$$

Theorem 3.1.[7,8,11,13] *Let G be a connected compact semisimple metrizable group. Let X be a Brownian motion on G with one-dimensional marginals $(\mu_t)_{t>0}$.*

(1) *Define $t_0 \in [0, +\infty]$ by setting*

$$t_0 = \limsup_{s\to\infty} \frac{1}{s} \log N_\alpha(s)\,.$$

Then, if $t < t_0/4$, μ_t is singular w.r.t. Haar measure whereas, for $t > 2t_0$, μ_t is absolutely continuous with respect to Haar measure and its density belongs to all L^p, $p \in [1, +\infty)$. For $t > 4t_0$, μ_t admits a continuous density.

(2) *The properties* (AC) *and* (CK) *are equivalent. They hold true if and only if*

$$\lim_{t\to\infty} \frac{1}{t} \log N_\alpha(t) = 0\,.$$

Another equivalent property is that the Green function $q = \int_0^\infty e^{-t}\mu_t dt$ is absolutely continuous w.r.t. Haar measure.

(3) *The properties* (CK#) *and* (CK$*$) *are equivalent. They hold true if and only if*

$$\lim_{t\to\infty} \frac{1}{t} N_\alpha(t) = 0\,.$$

Another equivalent property is that the Green function q defined above is absolutely continuous w.r.t. Haar measure and admits a continuous density on $G \setminus \{e\}$.

(4) *The associated quasi-distance d is continuous if and only if*

$$\int_0^\infty N_\alpha(s) \frac{ds}{s^2} < \infty\,, \quad \text{i.e.} \quad \sum_{k\ge 1} \frac{n_k}{\alpha_k} < \infty\,.$$

This is also equivalent to $\int_0^1 \log \mu_t(e) dt < \infty$.

(5) *If* X *satisfies* (CK), *then there exist constants* c_i, $1 \le i \le 4$ *such that, for all* $t \in (0,1)$,

$$c_1 N_\alpha^\#(c_2/t) \le \log \mu_t(e) \le c_3 \widehat{N_\alpha^\#}(c_4/t)$$

where

$$N_\alpha^\#(s) = \int_0^s N_\alpha(t)\frac{dt}{t}\,, \quad \widehat{N_\alpha^\#}(s) = \int_0^\infty N_\alpha^\#(st)e^{-t}dt\,.$$

The fifth statement of this theorem can be used to obtain existence and non-existence results. For instance, it is used in the proof of Theorem 4.4 to exhibit nice central Gaussian semigroups on any metrizable connected locally connected compact group. It is also used to show that, even in infinite dimension, the structure of the underlying group imposes some restrictions on the type of behaviors that can occur for $\mu_t(e)$. See Sec. 4.5.

Next, we illustrate Theorem 3.1 by looking at the simple case where $G = \prod_1^\infty \mathrm{SO}(n_i)$ with $n_i = 5 + [i^\sigma]$ for some $\sigma > 0$. Fix also $\gamma > 0$.

(1) Let $\alpha = (\alpha_i)$ with $\alpha_i = (1+i)^\gamma$. Then $N_\alpha(s) \sim \frac{1}{4\sigma}s^{(1+2\sigma)/\gamma}$ and we have:

  (a) For all σ, $\gamma > 0$, for all $t > 0$, μ_t is absolutely continuous w.r.t. Haar measure and admits a continuous density $x \mapsto \mu_t(x)$. The Green function q is absolutely continuous w.r.t. Haar measure.

  (b) If $1 + 2\sigma \ge \gamma$, then the intrinsic distance d is infinite almost everywhere. The (1-excessive) density of the Green functions q has poles in $G \setminus \{e\}$. In fact, it has a dense set of poles if $1 + 2\sigma > \gamma$. See Ref. 7.

  (c) If $1 + 2\sigma < \gamma$, then the associated intrinsic distance d is continuous and defines the topology of G. Moreover, there are constants c, $C \in (0,\infty)$ depending only on σ, γ such that

$$\forall\, t \in (0,1)\,, \quad ct^{-\lambda} \le \log \mu_t(e) \le Ct^{-\lambda}\,,$$

  where $\lambda = (1+2\sigma)/\gamma$. The (1-excessive) density of the Green function q is continuous on $G \setminus \{e\}$.

(2) Set $\alpha_i = e^{i^\gamma}$. Then $N_\alpha(s) \sim \frac{1}{4\sigma}(\log s)^{(1+2\sigma)/\gamma}$. In this case, μ_t has a continuous density for all $t > 0$ and satisfies

$$c\log(1+1/t)^{1+\lambda} \le \log \mu_t(e) \le C\log(1+1/t)^{1+\lambda}$$

with $\lambda = (1+2\sigma)/\gamma$. The intrinsic distance d is continuous and defines the topology of G. The Green function is absolutely continuous and its (1-excessive) density is continuous on $G \setminus \{e\}$.

(3) Set $\alpha_i = [\log(3+i)]^\gamma$. Then $N_\alpha(s) \sim \frac{1}{4\sigma}\exp((1+2\sigma)s^{1/\gamma})$. It follows that the intrinsic distance d is infinite almost everywhere.

  (a) If $\gamma \in (0,1)$, then μ_t is singular for all $t > 0$.

(b) If $\gamma = 1$, then $t_0 = 1+2\sigma$. For $t < (1+2\sigma)/4$, μ_t is singular. For $t > 4(1+2\sigma)$, μ_t has a continuous density. For all $\gamma \in (0,1]$, the Green function q is not absolutely continuous w.r.t. Haar measure.

(c) If $\gamma > 1$, then μ_t is absolutely continuous with a continuous density for all $t > 0$. The Green function $q = \int_0^\infty e^{-t}\mu_t dt$ is absolutely continuous with respect to Haar measure but has a dense set of poles in G.[7]

4. General Compact Connected Groups

4.1. *The structure of compact connected groups*

In order to study in detail Brownian motions on compact connected groups one needs some understanding of the structure of such groups. For a comprehensive treatment, see Ref. 37.

Let G be a compact connected group. In such a group, the commutator subgroup G' of G is closed in G. Let Z^0 be the connected component of the center of G and $H = Z^0 \cap G'$. Set also $\mathbf{H} = \{(h,-h) : h \in H\} \subset Z^0 \times G'$. Then the Levy–Malcev decomposition asserts that

$$G \cong \frac{Z^0 \times G'}{\mathbf{H}} . \tag{4.1}$$

Moreover, G' is semisimple (i.e. $(G')' = G'$) and thus has a decomposition of the form

$$G' = \Sigma/\Xi \quad \text{with} \quad \Sigma = \prod_k \Sigma_k , \tag{4.2}$$

where each Σ_k is a simple, connected, simply connected Lie group with (finite) center Z_k and Ξ is a closed subgroup of the center $\prod Z_k$ of Σ. Of course, the number of simple factors can be finite or infinite, countable or uncountable. However, if G is metrizable then the number of factors in (4.2) is finite or countable. In any case, setting

$$\bar{G} = Z^0 \times \Sigma , \tag{4.3}$$

we see that $\bar{G}$ is a "kind of cover" of G having the same structure that in the case of Lie groups. The important caveat here is that $\bar{G}$ is not a cover in the proper sense, let alone a finite cover. Indeed, the group H (resp. Ξ) is, in general, not a finite group but a closed central totally disconnected (not discrete) subgroup of $Z^0 \times G'$ (resp. Σ). It follows that the kernel N of the natural projection $\bar{G} \to G$ is, in general, a central closed totally disconnected subgroup.

Besides the Levy–Malcev decomposition (4.1) there is an equally important description of G known as the Borel–Hofmann–Scheerer decomposition. See Theorem 9.39 of Ref. 37. It asserts that G is the semi-direct product

$$G \cong T \ltimes G' , \tag{4.4}$$

where T is a compact connected Abelian subgroup of G, isomorphic to the quotient $A = Z^0/H$, $H = Z^0 \cap G'$. It is worth noting that A is also naturally isomorphic to G/G'. We call G' the semisimple part of G and A the Abelian part of G.

The two descriptions of G given by (4.1) and (4.4) complement each other. For instance, even if G is locally connected, it might happen that Z^0 is not. However, by (4.4), G is locally connected if and only if $A = Z^0/H$ is locally connected (Corollary 9.66 of Ref. 37). The reader who wants more detail on this should consult Chap. 9 of Ref. 37.

4.2. *The dimensional spectrum*

Assuming that G is a metrizable group, the Levy–Malcev decomposition (4.1)–(4.2) allows us to define the notion of *dimensional spectrum* of G. The Abelian group Z_0 is the projective limit of torii $(T_i)_{i\in I}$ (torii are the only Abelian connected compact Lie groups). We define the dimension of Z_0 to be

$$n_0(G) = \sup_{i\in I} \dim(T_i)\,.$$

Each Σ_k in (4.2) is a compact simple Lie group. Let n_k be its dimension. We say that (n_k) is a *semisimple dimension sequence* for G. A group G has many dimension sequences that differ by the ordering of their terms. To obtain a uniquely defined invariant of G, we proceed as follows. It is well known that the set of compact connected simply connected simple Lie groups, up to isomorphisms, is a countable set $\mathcal{SL}$. Order this set so that the topological dimension is a non-decreasing function on $\mathcal{SL}$. We denote by $\mathcal{N}$ the image of the topological dimension as a map from $\mathcal{SL}$ to the integers. $\mathcal{N}$ contains all the numbers of the form $n(n+2)$ with $n \geq 1$, $n(2n+1)$ with $n \geq 2$, $n(2n-1)$ with $n \geq 4$, and 14, 52, 78, 133, 248, corresponding to the exceptional groups (it is not really important for our purpose to know $\mathcal{N}$ precisely). Now, for G as above, we can define the semisimple dimensional spectrum $\sigma(G)$ of G to be the sequence $(\sigma(\ell))_{\ell\in\mathcal{N}}$ where $\sigma(\ell)$ equals the multipicity of $\ell \in \mathcal{N}$ in any semisimple dimension sequence (n_k) for G. The full dimensional spectrum is the function $\sigma : \{1\} \cup \mathcal{N} \to \{0, 1, 2, \dots\} \cup \{\infty\}$ given by $1 \mapsto n_0(G)$ and $\ell \mapsto \sigma(\ell)$ otherwise. We will say that the dimensional spectrum of G contains infinite multiplicity if at least one of the $\sigma(\ell)$, $\ell \in \{1\} \cup \mathcal{N}$ is infinite. Otherwise we will say that G has a finite multiplicity dimensional spectrum. Note that each dimension in $\mathcal{N}$ occurs for at most a (small) finite number of not isomorphic Lie groups. Thus, infinite multplicity occurs only if some simple Lie group occurs with infinite multiplicity in the decomposition of G. If G has a finite multiplicity dimensional spectrum, we denote by $n^{\uparrow} = (n_1^{\uparrow}, n_2^{\uparrow}, \dots)$ any *nondecreasing* semisimple dimensional sequence of G.

To give an example, consider

$$G = \mathrm{SO}(2) \times \mathrm{SO}(3) \times \mathrm{SO}(4) \times \cdots \times \mathrm{SO}(n) \times \cdots,$$

the product of all the special orthogonal groups. Recall that $\mathrm{SO}(2) = \mathbb{T}$ is a circle, that $\mathrm{SO}(4)$ is not simple but is covered by $\mathrm{SU}(2) \times \mathrm{SU}(2)$, and that $\mathrm{SO}(n)$ has dimension $\binom{n}{2}$. Thus the dimensional spectrum of G has $\sigma(1) = 1$, $\sigma(3) = 3$ (because of one $\mathrm{SO}(3)$ and two $\mathrm{SU}(2)$ factors), and $\sigma(\ell) = 1$ if $\ell = \binom{n}{2}$ for some $n \geq 5$ and 0 otherwise. The dimensional spectrum of this group has finite multiplicity.

4.3. *Examples of metrizable compact connected groups*

An Abelian compact connected Lie group is a torus $\mathbb{T}^n$ where n, the topological dimension, is an integer. Compact connected Lie groups are finite quotients of direct products of the form $\mathbf{T} \times \Sigma_1 \times \cdots \times \Sigma_k$ where $\mathbf{T}$ is a finite dimensional torus and each Σ_i is a simple, simply connected, compact connected Lie group. A good example of simple, simply connected, compact connected Lie group is $\mathrm{SU}(n)$.

By a theorem of Dixmier, an Abelian compact connected locally connected metrizable group is a torus $\mathbb{T}^n$ where n is either an integer or ∞ (countable ∞). Moreover, any Abelian compact connected metrizable group can be realized as a closed subgroup of $\mathbb{T}^\infty$.

The simplest compact connected Abelian groups that are not locally connected are solenoids. To describe these, let $a = (a_1, a_2, \ldots)$ be a sequence of natural integers and define $a_\ell^n = \prod_{\ell+1}^n a_i$. Consider the projective sequence $(T_k, \pi_{\ell,k})$, $\ell \leq k$ where, for each k, T_k is the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $\pi_{\ell,k} : T_k \to T_\ell$ is the map $z \mapsto z^{a_\ell^k}$.

By definition, the a-adic solenoid S_a is the projective limit of the projective sequence $(T_k, \pi_{\ell,k})$, $\ell \leq k$. In more concrete terms, it means that S_a is the subgroup of $\prod_1^\infty T_k$ of those points $x = (x_1, x_2, \ldots)$ such that $\pi_{\ell,k}(x_k) = x_\ell$ for all $\ell \leq k$. It follows that S_a is a compact connected group having a countable basis for its topology. The following well-known alternative description of S_a shows that it is not locally connected. Let Δ_a be the set of all a-adic numbers of the form $z = \sum_0^\infty z_i a_0^i$, $0 \leq z_i < a_{i+1}$. The topology of Δ_a is defined by a metric and, in this metric, the distance from z to the neutral element 0 is $|z| = (a_0^\ell)^{-1}$ where ℓ is the smallest integer i such that $z_i \neq 0$. The integers $\mathbb{Z}$ form a dense subset of Δ_a. With this notation,

$$S_a \cong (\mathbb{R} \times \Delta_a)/H\,,$$

where H is the discrete subgroup $\{(h, -h) : h \in \mathbb{Z}\}$. The natural homomorphism that leads to the isomorphism above is $\psi : \mathbb{R} \times \Delta_a \to S_a$ given by

$$\psi : (\phi, k) \mapsto x = (x_1, x_2, \ldots) \quad \text{with } x_j = \exp\left(\frac{2\pi i}{a_0^j}\left(\phi + \sum_0^{j-1} k_\ell a_0^\ell\right)\right).$$

Set $\bar{\Delta}_a = \{0\} \times \Delta_a$ and $\underline{\Delta}_a = \psi(\bar{\Delta}_a)$. Then $\underline{\Delta}_a$ is a closed subgroup of S_a, isomorphic to Δ_a. Moreover, $\underline{\Delta}_a$ is the projective limit of finite cyclic groups $\mathbb{Z}/a_0^n\mathbb{Z}$ with projection maps $\pi_{\ell,n} : \mathbb{Z}/a_0^n\mathbb{Z} \to \mathbb{Z}/a_0^\ell\mathbb{Z}$. Finally, $S_a/\underline{\Delta}_a \cong \mathbb{T}$. See, e.g. Ref. 15 where there is also a short discussion of Gaussian measures on solenoids.

Next we present a class of examples of non-Abelian compact connected locally connected groups that was brought to our attention by K. Hofmann. See pp. 488–489 of Ref. 37. Fix a sequence of natural integers $a = (a_k)$ and set $a_\ell^k = \prod_{\ell+1}^k a_i$ as above. Let Σ_a be the product group

$$\Sigma_a = \prod_1^\infty \mathrm{SU}(a_0^k)\,.$$

Obviously, this group is semisimple, i.e. $\Sigma_a = \Sigma_a'$. The center $Z(\mathrm{SU}(n))$ of $\mathrm{SU}(n)$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ through the isomorphism

$$j \mapsto e^{2\pi i j/n} I_n\,,$$

where I_n is the n by n identity matrix. The center $Z(\Sigma_a)$ of Σ_a is the direct product of the finite groups $Z(\mathrm{SU}(a_0^k))$. For $\ell \le k$, consider the map $p_{\ell,k} : Z(\mathrm{SU}(a_0^k)) \to Z(\mathrm{SU}(a_0^\ell))$ defined by

$$p_{\ell,k} : e^{2\pi i j/a_0^k} I_{a_0^k} \mapsto e^{2\pi i j/a_0^\ell} I_{a_0^\ell}\,.$$

The sequence $(Z(\mathrm{SU}(a_0^k)), p_{\ell,k})$ is a projective sequence and we denote by D_a its projective limit. It is clear that D_a is a compact central totally disconnected subgroup of Σ_a. It is also easy to see that D_a is isomorphic to the subgroup $\underline{\Delta}_a$ of the solenoide S_a. The isomorphism is given

$$r : \underline{\Delta}_a \to D_a\,, \quad (x_k) \mapsto (x_k I_{a_0^k})\,.$$

We are now ready to define the class of examples that we call Hofmann groups. Set

$$G_a = \frac{S_a \times \Sigma_a}{\mathbf{H}_a}$$

with

$$\mathbf{H}_a = \{(z^{-1}, r(z)) : z \in \underline{\Delta}_a\} \subset \underline{\Delta}_a \times D_a\,.$$

Thus, G_a is a compact connected group having a countable basis for its topology. Its semisimple part is

$$G_a' = (\{0\} \times \Sigma_a)\mathbf{H}_a/\mathbf{H}_a \cong \Sigma_a$$

whereas the connected component of the neutral element in the center is

$$Z_a^0 = (S_a \times \{e\})\mathbf{H}_a/\mathbf{H}_a \cong S_a\,.$$

Thus, Z_a^0 is not locally connected. The intersection of G_a' and Z_a^0 is

$$H_a = Z_a^0 \cap G_a' = (\underline{\Delta}_a \times D_a)/\mathbf{H}_a \cong \underline{\Delta}_a \cong D_a\,.$$

The Abelian part of G_a is

$$A_a = G_a/G'_a = \left(\frac{Z_a^0 \times \Sigma_a}{\mathbf{H}_a}\right) \Big/ \left(\frac{\underline{\Delta}_a \times \Sigma_a}{\mathbf{H}_a}\right)$$

$$\cong \left(Z_a^0 \times \Sigma_a\right) \Big/ (\underline{\Delta}_a \times \Sigma_a) \cong S_a/\underline{\Delta}_a \cong \mathbb{T}\,.$$

That is, A_a is isomorphic to a circle. By the Borel–Hofmann–Scheerer theorem,[37] the group G_a is isomorphic to the semidirect product of $G'_a \rtimes_\phi A_a$. It follows that G_a is locally connected.

Using the decompositions $G_a = (S_a \times \Sigma_a)/\mathbf{H}_a$, $\Sigma_a = \prod_1^\infty \mathrm{SU}(a_0^k)$, we can write any generator of a symmetric central Gaussian semigroup in the form

$$-L = \beta\partial^2 - \sum_1^\infty \beta_k L_k\,,$$

where ∂ refers to differentiation in the one real variable corresponding to S_a and L_k is the canonical Killing Laplace–Beltrami operator on $\mathrm{SU}(a_0^k)$.

4.4. *General existence results*

Let us start this section with a result from the 70's due to Heyer and Siebert.

Theorem 4.1. (Chap. 6 of Refs. 36 and 41) *If there exists a Brownian motion on G such that the law μ_1 of X_1 is absolutely continuous w.r.t. Haar measure, then G is locally connected and metrizable.*

Conversely, if G satisfies these two topological properties, then there are Brownian motions on G such that μ_t is absolutely continuous and has a continuous density for all $t \geq 1$.

For details and further discussion, see Chap. 6 of Refs. 36 and 10. Because of this result we will concentrate in what follows on the case of locally connected connected compact metrizable groups. Note that this theorem fails to prove the existence of (AC) Brownian motions, let alone the existence of (CK) or (CK$\lambda*$) Brownian motions.

The next definition introduces further interesting types of on-diagonal bounds that complement property (CK$\lambda*$). See Definition 1.3.

Definition 4.2. Let ψ be a nondecreasing, non-negative function on $(0, +\infty)$. We say that a (CK) Brownian motion X on G with continuous density $x \mapsto \mu_t(x)$ is:

(CK$_\psi$) if it satisfies $\sup_{t\in(0,1)} \psi(1/t)^{-1} \log \mu_t(e) < \infty$;
(CK$_\psi^*$) if it satisfies $\lim_{t\to 0} \psi(1/t)^{-1} \log \mu_t(e) = 0$;
(CK$_\psi^\approx$) if it satisfies $c \leq \psi(1/t)^{-1} \log \mu_t(e) \leq C$ for two constants $0 < c \leq C < +\infty$.

Typical functions ψ of interest are as follows: $\psi(t) = c\log(t)$; $\psi(1+t) = ct^\lambda$, $\lambda > 0$, more generally, ψ a regularly varying function of index $\lambda > 0$; ψ a slowly varying function such as $\exp(c[\log(1+t)]^\alpha)$, $\alpha \in (0,1)$, or $c(\log(1+t))^\alpha(\log(1+\log(1+t)))^\beta$, $\alpha \geq 1$, $\beta \in \mathbb{R}$.

To illustrate these definitions let us state the following simple result.

Proposition 4.3. *Let G be a compact connected group. Then a necessary and sufficient condition for the existence of a Brownian motion on G satisfying* ($\mathrm{CK}_{\log}$) *is that G is a Lie group. Moreover, if G is a Lie group, all Brownian motions on G satisfy* ($\mathrm{CK}^{\approx}_{\log}$).

The sufficiency and the second part of the proposition follow from (1.2). For the necessity we refer to Ref. 17 where it is shown that, if G is not a Lie group but carries a (CK) Brownian motion with density $x \mapsto \mu_t(x)$, then

$$\lim_{t\to 0} \frac{\log \mu_t(e)}{\log(1+1/t)} = +\infty\,. \tag{4.5}$$

We now state our main existence result.

Theorem 4.4.[11] *Let G be a compact, connected, locally connected metrizable group. Then:*

(1) *There exist many* (CK) *or even* (CK$*$) *Brownian motions on G.*
(2) *In fact, for any increasing positive function v tending to infinity as t tends to infinity there exists a* (CK_ψ) *Brownian motion on G where*
$$\psi(t) = \log(1+t)v(t)\,.$$
(3a) *If G is not a Lie group and if ψ is a regularly varying function of index $\lambda > 0$, then there exists a Brownian motion X on G satisfying* (CK_ψ) *and*
$$\limsup_{t\to 0} \psi(1/t)^{-1}\log \mu_t(e) > 0\,.$$
(3b) *Finally, if G is not a Lie group, for any increasing positive function v tending to infinity as t tends to infinity there exists a* ($\mathrm{CK}^{\approx}_\phi$) *Brownian motion on G where ϕ is a certain slowly varying function such that $\phi(t) \leq \log(1+t)\ v(t)$.*

Statements 1, 2 and 3b above are increasing in strength and difficulty. The first statement already improve upon Siebert's result asserting the existence of a Brownian motion on G with continuous density for $t > 1$.

Statements 2 and 3 of Theorem 4.4 are sharp in various ways. First, for G not a Lie group, Statement 2 should be compared with (4.5). Statement 3a cannot be improved much. See Theorem 4.5 below.

Statement 3b is rather subtle. It says that, for any positive v increasing to infinity, one can find a (CK) Brownian motion $(X_t)_{t>0}$ and a slowly varying function ϕ bounded above by $\psi(t) = \log(1+t)v(t)$ such that

$$c\phi(1/t) \leq \log \mu_t(e) \leq C\phi(t)\,,$$

where μ_t is the law of X_t, $t > 0$, and $x \mapsto \mu_t(x)$ is the continuous density of μ_t with respect to Haar measure. However, Theorem 4.4 does not give any indication on how small ϕ might be compared to ψ. In fact, in some cases, ϕ has to be chosen much smaller than ψ. See Theorem 4.7.

4.5. *Existence results involving the dimensional spectrum*

Given a reasonable function ψ such as $\psi(t) = t^\lambda$, $\lambda > 0$, and an infinite dimensional compact connected locally connected metrizable group G, it is natural, in view of Theorem 4.4, to ask whether there exists any Brownian motion satisfying $(\mathrm{CK}^{\approx}_{\psi})$ on G. The answer is no.

Theorem 4.5.[11] *Let G be an infinite dimensional compact connected locally connected metrizable group. Let ψ be any fixed regularly varying function of index $\lambda > 0$, e.g. $\psi(t) = t^\lambda$. Then there exists a Brownian motion on G satisfying* $(\mathrm{CK}^{\approx}_{\psi})$ *if and only if either the dimensional spectrum of G* (*see Sec.* 4.2) *contains infinite multiplicity or it has finite multiplicity and*

$$\sup_k \frac{S^{\uparrow}_{k+1}}{S^{\uparrow}_k} < +\infty\,,$$

where $S^{\uparrow}_k = \sum_1^k n^{\uparrow}_j$ and $(n^{\uparrow}_j)$ is the nondecreasing semisimple dimension sequence of G (*see Sec.* 4.2).

It should be noted that the necessary and sufficient condition in this theorem is independent of λ. To give concrete examples, for each $\lambda \in (0,\infty)$, the group $G = \prod_1^\infty \mathrm{SO}(n_i)$ carries Brownian motions satisfying $(\mathrm{CK}\lambda)$ if $n_i = i^k$ or $n_i = k^i$ for some fixed positive integer k, but does not if $n_i = 2^{i^2}$, $i = 1, 2, \ldots$.

The next two results are in the same vein and further results of this type can be found in Ref. 11.

Theorem 4.6.[11] *Let G be an infinite dimensional compact connected locally connected metrizable group. A necessary and sufficient condition for G to carry a* (CK) *Brownian motion such that, for some $0 < a < b < \infty$,*

$$\forall\, t \in (0,1)\,, \quad t^{-a} \le \log \mu_t(e) \le t^{-b}\,,$$

is that either the dimensional spectrum of G contains infinite multiplicity or it has finite multiplicity and

$$\sup_k \frac{\log S^{\uparrow}_{k+1}}{\log S^{\uparrow}_k} < +\infty\,.$$

For instance, if $G = \prod_1^\infty \mathrm{SO}(n_i)$ with $n_i = 2^{2^{i^2}}$ then, for any $0 < a < b < \infty$, there is no Brownian motion on G satisfying $t^{-a} \le \log \mu_t(e) \le t^{-b}$ for all $t \in (0,1)$.

Theorem 4.7.[11] *Let v be any fixed increasing slowly varying function and set*

$$\psi(t) = \log(1+t)v(t)\,.$$

Then there are infinite dimensional compact connected locally connected metrizable groups on which there is no $(\mathrm{CK}_{\psi}^{\approx})$ *Brownian motion.*

The main ingredients in the proof of these results are the structure of Brownian motion described in Ref. 11 and the two-sided inequality stated in Theorem 3.1(5).

4.6. *On-diagonal behavior and hypoellipticity*

One main reason to try to understand the on-diagonal behavior of $(\mu_t)_{t>0}$, i.e. the behavior of the funtion $t \mapsto \mu_t(e)$ as t tends to zero, is that it relates to other properties of the associated Brownian motion and infinitesimal generator, leading to a better understanding of these properties. We close this survey by considering one such question: the *hypoellipticity* of the infinitesimal generators of Brownian motions on compact groups. A more complete discussion of hypoellipticity in this context can be found in Refs. 14 and 20. To define what we mean here by hypoellipticity we need two ingredients: a space of "distributions" and a space of "regular" functions. We will not give a general definition of what we mean by "distributions" or "regular" functions. Instead, we give a list of examples.

Spaces of distributions In Ref. 29, Bruhat introduces the space $\mathcal{B}'(G)$ of Bruhat distributions on G, defined as the space of all continuous linear functionals on the space of Bruhat test functions $\mathcal{B}(G)$. The linear space $\mathcal{B}'(G)$ is equipped with the strong dual topology. For reasons that will become clear later, we need to consider smaller spaces of distributions. Thus, we will consider the space $\mathcal{M}(G)$ of all (signed) Borel measures and the function spaces $L^p(G)$, $1 \le p \le \infty$, as spaces of distributions.

Spaces of regular functions There are many different ways to define smoothness in infinite dimensions. Here, we only consider the following scale of function spaces. Fix a projective basis $Y = (Y_i)_I$ in the Lie algebra of G (see Sec. 1.3). For $k = 0, 1, 2, \ldots$, let $\mathcal{C}_Y^k(G)$ be the closure of $\mathcal{B}(G)$ for the system of seminorms $\|Y^j f\|_\infty$, $j \in I^n$, $n \le k$, where $Y^j = Y_{i_1} \cdots Y_{i_n}$ if $j = (i_1, \ldots, i_n) \in I^n$. Define $\mathcal{C}_Y^\infty(G)$ to be the inductive limit of the spaces $\mathcal{C}_Y^k(G)$. Thus $\mathcal{C}_Y^0(G) = \mathcal{C}(G)$ is the space of all continuous functions, $\mathcal{C}_Y^1(G)$ is the space of all continuous functions having a continuous derivative in each of the directions Y_i, $i \in I$, etc. Except for $k = 0$, these spaces depend very much on the choice of the projective basis Y. For the present purpose, we will consider each of the spaces $\mathcal{C}_Y^k(G)$, $k = 0, 1, 2, \ldots, \infty$, as a space of regular functions.

In the following definition, for any open set Ω, $\mathcal{B}_0(\Omega)$ denotes the space of all Bruhat test functions with compact support in Ω.

Definition 4.8. Let $-L$ be the infinitesimal generator of a Brownian motion on a compact connected group G. Let $\mathcal{A} \subset \mathcal{B}'(G)$ be a space of distributions and $\mathcal{S}$ be a space of regular functions. We say that L is $\mathcal{A}$-$\mathcal{S}$-hypoelliptic if, for any distributions $U \in \mathcal{A}$, $F \in \mathcal{B}'(G)$, such that $LU = F$ in the sense of Bruhat distributions on G, and for any open set Ω, the condition

$$\forall\, \phi \in \mathcal{B}_0(\Omega)\,, \quad \phi F \in \mathcal{S}$$

implies

$$\forall\, \phi \in \mathcal{B}_0(\Omega)\,, \quad \phi U \in \mathcal{S}\,.$$

In the following theorem, we drop the reference to the group G in the notation of functions and distribution spaces. Thus, $L^p = L^p(G)$, $\mathcal{C} = \mathcal{C}(G)$, $\mathcal{B}' = \mathcal{B}'(G)$ and $\mathcal{M} = \mathcal{M}(G)$.

Theorem 4.9.[14,19,20] *Let G be a compact connected group. Let $-L$ be the infinitesimal generator of a Brownian motion on G with associated Gaussian semigroup $(\mu_t)_{t>0}$.*

(1) *L is L^∞-$\mathcal{C}$-hypoelliptic if and only if $(\mu_t)_{t>0}$ satisfies property* (AC). *In particular, a necessary condition for L to be L^∞-$\mathcal{C}$-hypoelliptic is that G be locally connected and metrizable.*
(2) *For any fixed $p \in [1,\infty)$, L is L^p-$\mathcal{C}$-hypoelliptic if and only if $(\mu_t)_{t>0}$ satisfies* (CK$*$).
(3) *L is $\mathcal{B}'$-$\mathcal{C}$-hypoelliptic if and only if G is a Lie group.*
(4) *Assume that $(\mu_t)_{t>0}$ satisfies* (CK$*$). *Let $X = (X_i)_I$ is a projective basis in which* (*such a projective basis always exists*) *$L = -\sum_I X_i^2$. Then, for any $k = 0, 1, \ldots, \infty$, L is $\mathcal{M}$-$\mathcal{C}^k_X$-hypoelliptic.*

Statement 3 shows that the hypoellipticity theory involving general Bruhat distributions is not relevant in infinite dimension. Statements 1 and 2 show that dealing with bounded measurable functions as distributions is very different from dealing with, say, square integrable functions. Statements 2 and 4 show that hypoellipticity for square integrable functions and for measures are the same notion in our context. The papers Refs. 19 and 20 introduce much larger spaces $\mathcal{A}$ of distributions for which $\mathcal{A} - \mathcal{C}^k$-hypoellipticity holds if and only if (CK$*$) holds but these spaces $\mathcal{A}$ depend on the operator L.

From a more technical viewpoint, one of the crucial ingredients for the proof of Statement 4 of Theorem 4.9 is a set of Gaussian type upper bounds for the density of the semigroup $(\mu_t)_{t>0}$ and its space and time derivatives. See Ref. 12.

5. Conclusion

Brownian motions on compact connected groups form one of the most natural class of stochastic processes related to an algebraic structure. This class of processes contains Brownian motion on the circle, on finite dimensional torii, on SO(n) and other simple compact Lie groups, as well as processes on some infinite dimensional groups such as the infinite dimensional torus $\mathbb{T}^\infty$.

It is natural to look at these processes as a whole and try to understand them as such. One of the fundamental first step is then to study the convolution semigroups of measures associated to these processes. When G is a compact Lie group, this is a well-developed area of study, closely connected to the analysis of the heat diffusion equation on G and its fundamental solution, the heat kernel. This, in turn, relates in various ways to the algebraic structure of G, to its geometry, to its representation theory, and a great wealth of results have been developed in this direction. However, until recently, very little was known — besides existence — concerning Brownian motions on compact connected groups that are not Lie groups.

After the seminal work of Greenander,[34] some attention was given to the study of invariant Markov processes on groups, see, e.g. Ref. 39. The basic definitions of invariant Markov processes and Brownian motions on groups are developed in Heyer's book.[36] Chapter 6, Secs. 3–4 of Ref. 36 describes the earliest important results — see Theorem 4.1 above — concerning Brownian motions on compact connected groups, taken as a whole.

After Ref. 36, no improvement upon Theorem 4.1 appeared until Refs. 8 and 11 where Theorem 4.4 is proved. Nevertheless, the first evidence that there should exist a rich theory of Brownian motions on some compact connected groups predate the publication of Ref. 36 and is found in the Ph.D. works of Bendikov and Berg who, independently, around 1974, studied certain properties of some specific Brownian motions on the infinite dimensional torus. A detailed account of this can be found in Ref. 3.

The spirit of the work presented in this survey is to pursue and develop in the infinite dimensional case the connections that are well established in the finite dimensional case between, on the one hand, Brownian motion and, on the other hand, analysis, algebraic structure and geometry.

In order to do this, one is led to revisit the classical finite dimensional case and develop new results concerning Brownian motions on Lie groups where the emphasize is not on a given Brownian motion on a given compact Lie group but on Brownian motions on compact Lie groups. To explain, consider the following question. Let $\mu_t(x)$ denote the heat kernel associated with the Killing Riemannian metric on a compact simple Lie group, taken with respect to the normalized Haar measure. Of course, $\lim_{t\to 0}\mu_t(e) = \infty$. What is, approximately, the size of $\mu_1(e)$? The answer is that there are some constants c, $C \in (0,\infty)$ such that

$$cn \le \log \mu_1(e) \le Cn\,,$$

where n is the dimension of the group and c, C are independent of the group (see Ref. 7). Such uniform information is crucial for most of the results in this survey.

Several tentalizing open questions concerning Brownian motions on compact connected groups are still open. Moreover, a perhaps even more natural class of processes is the class of general invariant diffusions obtained by dropping the forth condition in Definition 1.1. As already pointed out after Definition 1.1, generalizing the results presented in this survey to this wider class of processes seems to be a very challenging question.

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