Loop Spaces and Operads

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This document serves as an introduction to operads and their algebras, along with basic examples. We review the theory necessary to show May's recognition principle for loop spaces; namely, that every loop space $\Omega^k X$ is an algebra for the little k-cubes operad, and every connected little k-cubes algebra has the weak homotopy type of a k-fold loop space [May72]. We then follow Berger and Moerdijk [BM02] and present a model structure on the category of operads which allows us to show that every loop space can be rectified to a topological monoid.

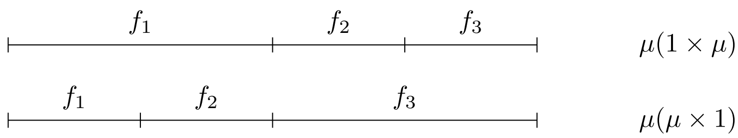
Recognizing loop spaces

In algebraic topology, one of the main ways to probe a space X is by looking at maps from spheres $S^n \to X$. If we fix a point $* \in X$, we can define its nth loop space $\Omega^n X$ as the space of all maps $S^n \to X$ taking the north pole to $*$, with the compact open topology. Loop spaces are of great importance, and so it makes sense to wonder when a given space X is of the homotopy type of a k-fold loop space $\Omega^k Y$ for some other space Y.

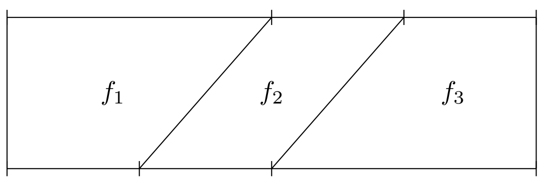
Let's start by looking at ΩY. Some properties are easy to pin down; first on the list is the fact that we have a binary operation

$$\mu : \Omega Y \times \Omega Y \to \Omega Y$$

given by concatenation of loops. As we well know, this operation is not associative: $\mu(1 \times \mu)$ and $\mu(\mu \times 1)$ are not equal, but instead differ by a change of parametrization.



However, there is a homotopy h between these maps, which we can easily represent as



and which makes μ a *homotopy associative* operation, that is, associative up to homotopy.

What happens when we try to multiply four maps? Well, if we remember our choice of homotopy h for multiplying three maps, we get homotopies

$$f_1(f_2(f_3f_4)) \xrightarrow{\mu(1\times h)} f_1((f_2f_3)f_4) \xrightarrow{h(1\times\mu\times 1)} (f_1(f_2f_3))f_4 \xrightarrow{\mu(h\times 1)} ((f_1f_2)f_3)f_4$$

$$f_1(f_2(f_3f_4)) \xrightarrow{h(1\times 1\times \mu)} (f_1f_2)(f_3f_4) \xrightarrow{h(\mu\times 1\times 1)} ((f_1f_2)f_3)f_4$$

and once again, we can find a homotopy between these homotopies! Iterating this process, we see that by remembering our choices of homotopies along the way, we get that loop composition is an operation that's associative up to homotopy, with homotopies between these homotopies, and homotopies between those, and so on.

How can we keep track of all of this structure in a reasonable way? Note that we can encode each possible way of composing n loops as an embedding of n copies of the unit interval I into itself, as we did in a previous picture for two cases of $n = 3$. Asking for these higher homotopies that we find in the loop space scenario essentially boils down to making sure each of these spaces of n embeddings is contractible.

What about iterated loop spaces? Following the same train of thought, we can encode a composition of n maps $S^k \to X$ as an embedding of n copies of the cube I^k into itself. Aside from having all higher homotopies for associativity, we know that a space that can be delooped twice will have a product that's commutative up to homotopy. Moreover, one can see that the number of times the space can be delooped corresponds to how many levels of higher homotopies one has for commutativity, and this in turn is given by the connectivity of these spaces of embeddings.

By this point, it looks like there has to be a connection between loop spaces and some adequate spaces encoding embeddings, since these seem to capture many essential properties. But even if this connection is apparent, it is still far from obvious to what extent they can actually be used to characterize (iterated) loop spaces. Our goal will be to show a sketch of the following result:

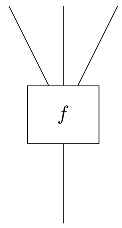
Theorem 1 (May). *There exists an operad $\mathcal{C}_k$ suitably encoding embeddings of copies of I^k into itself, with the property that every loop space $\Omega^k X$ is a $\mathcal{C}_k$-algebra. "Conversely", every connected $\mathcal{C}_k$-algebra has the weak homotopy type of a k-fold loop space.*

Operads and their algebras

A key component in our main theorem is the notion of *operad* and that of *algebra* for an operad, so we will start by introducing them, assuming basic notions from category theory.

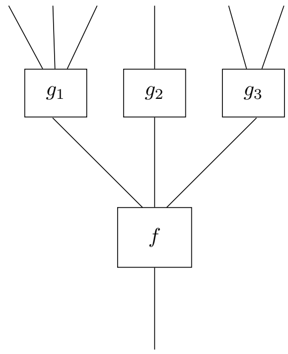
Conceptually, an operad $\mathcal{O}$ consists of a family of sets $\{\mathcal{O}(n)\}_{n\geq 0}$, where each $\mathcal{O}(n)$ is interpreted as a set of abstract "n-ary operations", together with structure maps indicating how composition behaves.

The following insightful representation, found in a review written by John Baez[Bae02], is good to keep in mind before moving on to a more formal definition: if we think about these abstract n-ary operations as black boxes with one wire coming in for each input and one coming out for the output,

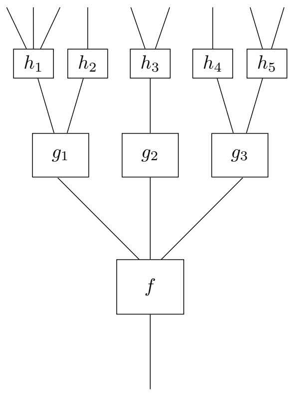


we require the following:

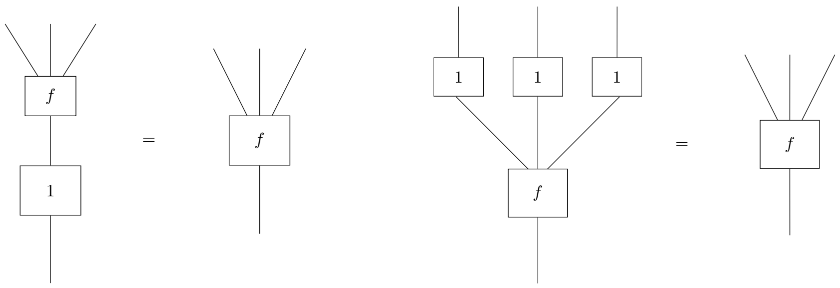
- There should be a way to compose an n-ary operation with n other operations, which we can interpret as the grafting of the boxes



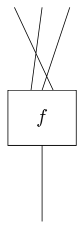
- Composition should be associative, allowing us to make sense of a picture like



- There must exist a unary operation that acts as an identity for composition in the two obvious ways.

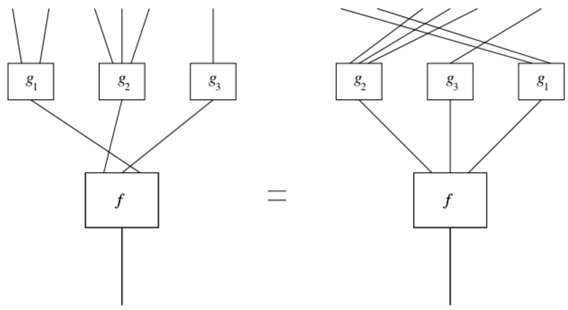


- Permuting the inputs of an n-ary operation gives us a new operation



  in such a way that this constitutes a right action of the permutation group S_n on the set $\mathcal{O}(n)$.

- Compositions must be compatible with the actions, as represented by



Let's formally define all of this!

Definition 2. An operad in a (strict) symmetric monoidal category $\mathcal{C}$ is a family of objects $\{\mathcal{O}(n)\}_{n\geq 0}$, together with morphisms

$$\gamma_{n;m_1,\ldots,m_n} : \mathcal{O}(n) \otimes \mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n) \longrightarrow \mathcal{O}(m_1 + \cdots + m_n)$$

satisfying the following axioms:

- *Associativity.* The following diagram commutes

$$\begin{array}{c}
\mathcal{O}(n)\otimes\bigotimes_{i=1}^{n}\mathcal{O}(m_i)\otimes\bigotimes_{i=1}^{n}(\mathcal{O}(k_{i,1})\otimes\cdots\otimes\mathcal{O}(k_{i,m_i})) \\
\swarrow^{\gamma_{n;m_1,\dots,m_n}\otimes 1} \qquad\qquad \searrow^{\rho} \\
\mathcal{O}(m_1+\cdots+m_n)\otimes\bigotimes_{i=1}^{n}(\mathcal{O}(k_{i,1})\otimes\cdots\otimes\mathcal{O}(k_{i,m_i})) \qquad \mathcal{O}(n)\otimes\bigotimes_{i=1}^{n}(\mathcal{O}(m_i)\otimes\mathcal{O}(k_{i,1})\otimes\cdots\otimes\mathcal{O}(k_{i,m_i})) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow 1\otimes\bigotimes_{i=1}^{n}\gamma_{m_i;k_{i,1},\dots,k_{i,m_i}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathcal{O}(n)\otimes\bigotimes_{i=1}^{n}\mathcal{O}(k_{i,1}+\cdots+k_{i,m_i}) \\
\searrow_{\gamma_{m_1+\cdots+m_n;k_{1,1},\dots,k_{n,m_n}}} \qquad\qquad \swarrow_{\gamma_{n;k_{1,1}+\cdots+k_{1,m_1},\dots,k_{n,1}+\cdots+k_{n,m_n}}} \\
\mathcal{O}(k_{1,1}+\cdots+k_{n,m_n})
\end{array}$$

where ρ uses the symmetry in $\mathcal{C}$ to permute the factors accordingly.

- *Unit.* If 1 is the unit object of $\mathcal{C}$, then there exists a morphism $\eta : 1 \to \mathcal{O}(1)$ such that the compositions

$$\mathcal{O}(n)\otimes 1^{\otimes n} \xrightarrow{1\otimes\eta^{\otimes n}} \mathcal{O}(n)\otimes\mathcal{O}(1)\otimes\cdots\otimes\mathcal{O}(1) \xrightarrow{\gamma_{n;1,\dots,1}} \mathcal{O}(n)$$

$$1\otimes\mathcal{O}(m) \xrightarrow{\eta\otimes 1} \mathcal{O}(1)\otimes\mathcal{O}(m) \xrightarrow{\gamma_{1;m}} \mathcal{O}(m)$$

  are the right and left unit morphisms given by the monoidal structure in $\mathcal{C}$.

- *Action.* Each $\mathcal{O}(n)$ admits a right action by S_n.

- *Equivariance.* Given $\sigma \in S_n$, the following diagram commutes

$$\begin{array}{ccc}
\mathcal{O}(n)\otimes\mathcal{O}(m_1)\otimes\cdots\otimes\mathcal{O}(m_n) & \xrightarrow{1\otimes\rho} & \mathcal{O}(n)\otimes\mathcal{O}(m_{\sigma^{-1}(1)})\otimes\cdots\otimes\mathcal{O}(m_{\sigma^{-1}(n)}) \\
 & & \downarrow{\scriptstyle \gamma_{n;m_{\sigma^{-1}(1)},\dots,m_{\sigma^{-1}(n)}}} \\
\downarrow{\scriptstyle \sigma\otimes 1} & & \mathcal{O}(m_1+\cdots+m_n) \\
 & & \downarrow{\scriptstyle \sigma_{m_1,\dots,m_n}} \\
\mathcal{O}(n)\otimes\mathcal{O}(m_1)\otimes\cdots\otimes\mathcal{O}(m_n) & \xrightarrow{\gamma_{n;m_1,\dots,m_n}} & \mathcal{O}(m_1+\cdots+m_n)
\end{array}$$

  where $\sigma_{m_1,\dots,m_n}$ takes a tuple of $m_1+\cdots+m_n$ elements, splits it into n blocks of lengths $m_1,\dots,m_n$ (in that order) and permutes the blocks by applying the action of σ.

Operads form a category $\mathbf{Op}(\mathcal{C})$, where a morphism of operads is a collection of morphisms $\{\mathcal{O}(n)\to\mathcal{P}(n)\}_{n\geq 0}$ preserving compositions, units, and the action of the S_n's.

Let's take a look at some examples.

Example 3. When it comes to examples, perhaps the most notable case is the one of the *endomorphism operad*. For an object X in a symmetric monoidal closed category $\mathcal{C}$, the operad $\mathcal{E}nd_X$ is defined as

$$\mathcal{E}nd_X(n) = \mathrm{Hom}(X^{\otimes n}, X)$$

where composition

$$\gamma_{n,m_1,\ldots,m_n} : \mathcal{E}nd_X(n) \otimes \mathcal{E}nd_X(m_1) \otimes \cdots \otimes \mathcal{E}nd_X(m_n) \longrightarrow \mathcal{E}nd_X(m_1 + \cdots + m_n)$$

is given as follows:

If $\mathcal{C} = \mathbf{Set}$, then

$$\gamma_{n,m_1,\ldots,m_n}(f, g_1, \ldots, g_n) = f \circ (g_1 \times \cdots \times g_n)$$

More generally, the element

$$\gamma_{n,m_1,\ldots,m_n} \in \mathcal{C}(\mathcal{E}nd_X(n) \otimes \mathcal{E}nd_X(m_1) \otimes \cdots \otimes \mathcal{E}nd_X(m_n), \mathcal{E}nd_X(m_1 + \cdots + m_n))$$

is defined as the transpose of the map

$$\hat{\gamma} \in \mathcal{C}\big(\big(\mathcal{E}nd_X(n) \otimes \mathcal{E}nd_X(m_1) \otimes \cdots \otimes \mathcal{E}nd_X(m_n)\big) \otimes X^{\otimes m_1 + \cdots + m_n}, X\big)$$

given by the composition

$$\begin{array}{c} \big(\mathcal{E}nd_X(n) \otimes \mathcal{E}nd_X(m_1) \otimes \cdots \otimes \mathcal{E}nd_X(m_n)\big) \otimes X^{\otimes m_1 + \cdots + m_n} \\ \big\downarrow \\ \mathcal{E}nd_X(n) \otimes (\mathcal{E}nd_X(m_1) \otimes X^{\otimes m_1}) \otimes \cdots \otimes (\mathcal{E}nd_X(m_n) \otimes X^{\otimes m_n}) \\ \big\downarrow {\scriptstyle 1 \otimes ev^{\otimes n}} \\ \mathcal{E}nd_X(n) \otimes X \otimes \cdots \otimes X \\ \big\downarrow {\scriptstyle ev} \\ X \end{array}$$

The unit $\eta : 1 \to \mathcal{E}nd_X(1) = \mathrm{Hom}(X, X)$ is the transpose of the unit isomorphism $1 \otimes X \simeq X$, which in the case of **Set** turns out to be the identity map, and the right action of S_n on $\mathcal{E}nd_X$ is given by acting on $X^{\otimes n}$ using the monoidal structure of $\mathcal{C}$.

We will soon be able to appreciate the importance of this operad in the bigger picture.

Example 4. Denote by $\mathbf{S}$ the cartesian category whose objects are the natural numbers, identified with the sets $[n] = \{1, \ldots, n\}$, and whose only maps are the permutations S_n.

Permutation groups form an operad over the category $\mathbf{S}$, with

$$\mathcal{O}(n) = S_n$$

and composition given by

$$\gamma_{n,m_1,\ldots,m_n} : S_n \times S_{m_1} \times \cdots \times S_{m_n} \to S_{m_1 + \cdots + m_n}$$

$$(\sigma, \rho_1, \ldots, \rho_n) \mapsto \sigma_{m_1,\ldots,m_n} \circ (\rho_1 \times \cdots \times \rho_n)$$

so, split the tuple of $m_1 + \cdots + m_n$ numbers into n blocks of lengths $m_1, \ldots, m_n$, act within the i-th block as prescribed by ρ_i, and then permute the blocks according to σ.

Example 5. The *associative operad* is defined by

$$\mathcal{A}ss(n) = \coprod_{S_n} 1$$

where compositions and S_n actions are induced by the operad structure of the permutation groups as shown in example 4.

Example 6. The *commutative operad* is

$$\mathcal{C}om(n) = 1$$

the monoidal unit, where all compositions consist of the unit isomorphisms

$$1 \otimes (1 \otimes \cdots \otimes 1) \xrightarrow{\simeq} 1$$

The following is a historically relevant example, being the main reason operads were defined and what convinced many mathematicians that their study was worthwhile –not to mention, why we're doing all this in the first place.

Example 7. Let I^k denote the standard k-dimensional unit cube in R^k. A little k-cube is a linear embedding $c : I^k \to I^k$ with parallel axes, that is, a map

$$c(t_1, \ldots, t_n) = (c^1(t_1), \ldots, c^n(t_n)), \quad t_i \in I$$

where $c^i : I \to I$ is a linear function of the form

$$c^i(t) = (1-t)x_i + ty_i \text{ with } 0 \le x_i < y_i \le 1$$

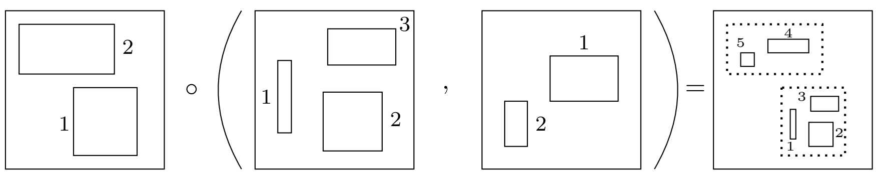
This means each function c^i takes the interval $[0,1]$ to an interval $[x_i, y_i]$ inside of $[0,1]$.

The *little k-cubes* operad $\mathcal{C}_k$ consists of the spaces

$$\mathcal{C}_k(n) = \mathbf{lEmb}(\coprod_n I^k, I^k)$$

of linear embeddings , so an element is an n-tuple $(c_1, \ldots, c_n)$ of little k-cubes, such that the interior of the images $c_i(I^k)$ are mutually disjoint. As a topological space, we identify $\mathcal{C}_k(n)$ with a subspace of $\mathbb{R}^{2kn}$ using the $\{x_i, y_i\}_{i=1}^k$ as coordinates.

An element of $\mathcal{C}_k(n)$ looks like a k-cube with n little disjoint k-cubes inside it. The symmetric groups S_n act on $\mathcal{C}_k(n)$ by permuting the labels of the n little cubes, and composition is defined by iterating embeddings as shown in the following picture:



Note that for every k there exists an injective operad morphism $\mathcal{C}_k \hookrightarrow \mathcal{C}_{k+1}$ given by "embedding into the equator", and it is with respect to these inclusions that we define $\mathcal{C}_\infty = \operatorname{colim} \mathcal{C}_k$.

Operads are not all that interesting by themselves; usually the real purpose is to study their algebras. The following is the definition of algebra over an operad, which mirrors that of a module over a ring. Recall that one can define a module M over a ring R to be an abelian group along with a morphism

$$R \to \mathrm{Hom}(M, M)$$

where we consider the Hom functor internal to abelian groups. Similarly;

Definition 8. Let $\mathcal{O}$ be an operad in a symmetric monoidal closed category $\mathcal{C}$. An algebra for $\mathcal{O}$ is an object X together with an operad morphism

$$\alpha : \mathcal{O} \to \mathcal{E}nd_X$$

that is, a collection of morphisms $\alpha_n : \mathcal{O}(n) \to \mathrm{Hom}(X^{\otimes n}, X)$ that is compatible with compositions, units, and the actions of S_n.

Equivalently, using the adjunction we can describe an algebra structure on X as a family of morphisms

$$\alpha_n : \mathcal{O}(n) \otimes X^{\otimes n} \to X$$

which are compatible with compositions, the unit, and are S_n-equivariant (here the action of S_n on the target X is the trivial one).

Given an operad $\mathcal{O}$, its algebras form a category which we denote $\mathcal{O}$-**Alg**. A morphism of $\mathcal{O}$-algebras is a map $\varphi : X \to Y$ in $\mathcal{C}$ that makes the following diagram commute

$$\begin{array}{ccc} \mathcal{O}(n) \otimes X^{\otimes n} & \xrightarrow{\alpha_n^X} & X \\ {\scriptstyle 1\otimes\varphi^{\otimes n}}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathcal{O}(n) \otimes Y^{\otimes n} & \xrightarrow{\alpha_n^Y} & Y \end{array}$$

Note that a map of operads $\mathcal{O} \to \mathcal{P}$ induces a "change of rings" type of map $\mathcal{P}$-**Alg** $\to$ $\mathcal{O}$-**Alg** given by $\mathcal{O} \to \mathcal{P} \to \mathcal{E}nd_X$.

We can now understand why $\mathcal{E}nd_X$ is so important: every object X has a canonical $\mathcal{E}nd_X$-algebra structure via $\mathrm{id} : \mathcal{E}nd_X \to \mathcal{E}nd_X$, or equivalently, via the evaluation morphisms $\mathrm{Hom}(X^{\otimes n}, X) \otimes X^{\otimes n} \to X$, and thus $\mathcal{E}nd_X$ is in some sense the universal operad acting on X, since definition 8 tells us that every other operad action must factor through that of $\mathcal{E}nd_X$.

Also, we can see that the definition of operad algebras is what motivates the idea of operads as abstract n-ary operations, since having an algebra structure on X means that each element of $\mathcal{O}(n)$ will be given an interpretation as an element of $\mathrm{Hom}(X^{\otimes n}, X)$. With this in mind, it is only natural that $\mathcal{E}nd_X$, the object of all n-ary operations on X, plays a major role in the theory.

Example 9. Algebras over $\mathcal{A}ss$ are associative unital monoids, explaining this operad's name.

Proof. [Fre17, Prop. 1.1.17] Let M be an associative unital monoid, with multiplication $\mu : M \otimes M \to M$ and unit $\eta : 1 \to M$. We want to define an algebra structure on M, that is, a collection of maps

$$\mathcal{A}ss(n) \to \mathcal{E}nd_M(n)$$

Given a permutation $\sigma \in S_n$, we associate to its corresponding element in $\mathcal{A}ss(n)$ the morphism $f_\sigma : M^{\otimes n} \to M$ that takes an n-tuple, permutes it via σ, and multiplies the elements by iterating μ. If our category had elements, the explicit expression for f would be

$$f(m_1, \dots, m_n) = m_{\sigma(1)} \dots m_{\sigma(n)}$$

When $n = 0$, we associate to the unique (trivial) permutation $\mathrm{id}_0 \in S_0$ the unit map $\eta : 1 = M^{\otimes 0} \to M$.

In the other direction, given an algebra structure on M we can consider the map $\eta : 1 \to M$ associated to the trivial permutation $\mathrm{id}_0 \in S_0$, and the map $\mu : M \otimes M \to M$ associated to the trivial permutation $\mathrm{id}_2 \in S_2$. The map associated to the trivial permutation $\mathrm{id}_1 \in S_1$, which denotes the unit of the operad $\mathcal{A}ss$, acts as the identity map on M.

From the identities

$$\mathrm{id}_2(\mathrm{id}_0, \mathrm{id}_1) = \mathrm{id}_1 = \mathrm{id}_2(\mathrm{id}_1, \mathrm{id}_0)$$

and

$$\mathrm{id}_2(\mathrm{id}_2, \mathrm{id}_1) = \mathrm{id}_3 = \mathrm{id}_2(\mathrm{id}_1, \mathrm{id}_2)$$

in $\mathcal{A}ss$ we (respectively) deduce the unit condition

$$\mu \circ (\eta \otimes 1_M) \simeq 1_M \simeq \mu \circ (1_M \otimes \eta)$$

and the associativity relation

$$\mu \circ (\mu \otimes 1_M) = \mu \circ (1_M \otimes \mu)$$

in M. □

Note that the associative operad has exactly $n!$ elements in its set on n-ary operations, representing the fact that in an associative monoid we have precisely $n!$ ways of multiplying n elements, one for each possible ordering, since parenthesising is rendered useless because of the associativity.

Example 10. Similarly to the previous example, it's not hard to show that algebras over $\mathcal{C}om$ are commutative (associative, unital) monoids.

All the same arguments apply, with the following addition: the instance

$$\begin{array}{ccc} \mathcal{C}om(2) \otimes M^{\otimes 2} & \xrightarrow{1 \otimes \sigma} & \mathcal{C}om(2) \otimes M^{\otimes 2} \\ \downarrow{\scriptstyle \sigma \otimes 1} & & \downarrow{\scriptstyle \alpha} \\ \mathcal{C}om(2) \otimes M^{\otimes 2} & \xrightarrow{\quad \alpha \quad} & M \end{array}$$

of the equivariance axiom for $\sigma = (1, 2)$ implies that

$$\alpha \circ (\sigma \otimes 1)(\{*\} \otimes m_1 \otimes m_2) = m_1 m_2$$

is equal to

$$\alpha \circ (1 \otimes \sigma)(\{*\} \otimes m_1 \otimes m_2) = \alpha(\{*\} \otimes m_2 \otimes m_1) = m_2 m_1$$

and then the binary operation must be commutative.

One of the following sections will be devoted to characterizing algebras for the little k-cubes operad. But before delving into that, we must establish a relation between operads and another familiar type of structure.

Operads and monads

Let's start by recalling some basic definitions.

Definition 11. Let $\mathcal{C}$ be any category. A monad in $\mathcal{C}$ is an endofunctor $T : \mathcal{C} \to \mathcal{C}$ together with a multiplication transformation $\mu : T^2 \Rightarrow T$ and a unit transformation $\eta : 1 \Rightarrow T$ making the diagrams below commute

$$\begin{array}{ccccc} T & \xRightarrow{\eta_T} & T^2 & \xLeftarrow{T\eta} & T \\ & \overset{\text{id}}{\searrow} & \Downarrow \mu & \overset{\text{id}}{\swarrow} & \\ & & T & & \end{array} \qquad\qquad \begin{array}{ccc} T^3 & \xRightarrow{T\mu} & T^2 \\ \Downarrow \mu_T & & \Downarrow \mu \\ T^2 & \xRightarrow{\mu} & T \end{array}$$

Definition 12. If $T : \mathcal{C} \to \mathcal{C}$ is a monad, then a T-algebra is an object X in $\mathcal{C}$ together with an algebra structure map $\theta : TX \to X$ such that the following diagrams are commutative

$$\begin{array}{ccc} X & \xrightarrow{\eta} & TX \\ & \overset{\text{id}}{\searrow} & \downarrow \theta \\ & & X \end{array} \qquad\qquad \begin{array}{ccc} T^2X & \xrightarrow{T\theta} & TX \\ \downarrow \mu & & \downarrow \theta \\ TX & \xrightarrow{\theta} & X \end{array}$$

For any object X in $\mathcal{C}$, it is always the case that $(TX, \mu : T^2X \to TX)$ is a T-algebra, which we call the free T-algebra generated by X.

We are interested in two ways through which we can define a monad.

Proposition 13 (Monad associated to an adjunction)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ and $U : \mathcal{D} \to \mathcal{C}$ be adjoint functors, that is, there exists an isomorphism*

$$\mathcal{D}(FC, D) \cong \mathcal{C}(C, UD)$$

or equivalently, unit and counit natural transformations $\eta : 1_{\mathcal{C}} \Rightarrow UF$ and $\epsilon : FU \Rightarrow 1_{\mathcal{D}}$ such that the compositions

$$F \xrightarrow{F\eta} FUF \xrightarrow{\epsilon_F} F$$

$$U \xrightarrow{\eta_U} UFU \xrightarrow{U\epsilon} U$$

are the identity transformations on F and G.

Then $UF : \mathcal{C} \to \mathcal{C}$ is a monad, with multiplication $\mu = U\epsilon F : UFUF \Rightarrow UF$ and unit $\eta : 1 \Rightarrow UF$.

Example 14. Let Σ denote the reduced suspension functor, and recall the adjunction

$$\text{Hom}_*(\Sigma X, Y) \cong \text{Hom}_*(X, \Omega Y)$$

of pointed maps and pointed spaces. Note that

$$\text{Hom}_*(\Sigma^2 X, Y) \cong \text{Hom}_*(\Sigma X, \Omega Y) \cong \text{Hom}_*(X, \Omega^2 Y)$$

and so by iterating this adjunction we get that

$$\text{Hom}_*(\Sigma^k X, Y) \cong \text{Hom}_*(X, \Omega^k Y)$$

Therefore, proposition 13 implies $\Omega^k\Sigma^k$ has a canonical monad structure. Consider the maps

$$\sigma_k : \Omega^k\Sigma^k X \longrightarrow \Omega^{k+1}\Sigma^{k+1} X$$

taking a map $f \in \Omega^k\Sigma^k X = \text{Hom}_*(S^k, \Sigma^k X)$ to the map $\sigma_k(f) \in \Omega^{k+1}\Sigma^{k+1}X = \text{Hom}_*(S^{k+1}, \Sigma^{k+1}X)$ given by

$$S^{k+1} = S^k \wedge S^1 \xrightarrow{f\wedge 1} \Sigma^k X \wedge S^1 = \Sigma^{k+1} X$$

Note that each σ_k is an inclusion, and it's with respect to these that we define

$$\Omega^\infty\Sigma^\infty = \text{colim}\, \Omega^k\Sigma^k$$

This will also be a monad, with product $\mu_\infty = \text{colim}\, \mu_k$ and unit $\eta_\infty = \text{colim}\, \eta_k$.

Now, given an operad $\mathcal{O}$, we will show a way to define a monad whose algebras coincide with those of $\mathcal{O}$. Define maps $\sigma_i : \mathcal{O}(j) \to \mathcal{O}(j-1)$ for $0 \leq i < j$ by $\sigma_i(f) = \gamma(f, s_i)$ where

$$s_i = 1^i \times * \times 1^{j-1-i} \in \mathcal{O}(1)^i \times \mathcal{O}(0) \times \mathcal{O}(1)^{j-1-i}$$

Also, let $s_i : X^{j-1} \to X^j$ be the map

$$s_i(x_1, \ldots, x_{j-1}) = (x_1, \ldots, x_i, *, x_{i+1}, \ldots, x_{j-1})$$

Proposition 15 (Monad associated to an operad)**.** *[May72, Constr. 2.4] Let $\mathcal{O}$ be an operad, and X an object in $\mathcal{C}$. We define $O : \mathcal{C} \to \mathcal{C}$, the monad associated to the operad $\mathcal{O}$, as*

$$OX = \coprod_j \mathcal{O}(j) \times X^j / \sim$$

where

$$(\sigma_i f, x) \sim (f, s_i x) \quad \text{and} \quad (f\sigma, x) \sim (f, \sigma x)$$

If we denote the image of (f, x) in OX by $[f, x]$, then, given a map $g : X \to X'$, define $Og : OX \to OX'$ by $Og[f, x] = [f, g^j x]$. For the monad structure, we define the multiplication and unit transformations by

$$\mu([f, [f^1, x^1], \ldots, [f^j, x^j]]) = [\gamma(f, f^1, \ldots, f^j), (x^1, \ldots, x^j)]$$

$$\eta(x) = [\eta(1), x]$$

(note that $O^2X = \coprod_j \mathcal{O}(j) \times (OX)^j/\sim = \coprod_j \mathcal{O}(j) \times (\coprod_i \mathcal{O}(i) \times X^i/\sim)^j/\sim$)

The following result shows the relation between the operad $\mathcal{O}$ and its associated monad O.

Theorem 16. *[May72, Prop. 2.8] Let $\mathcal{O}$ be an operad and O its associated monad. Then there is a one-to-one correspondence between $\mathcal{O}$-algebras and O-algebras.*

Proof. Let X be an $\mathcal{O}$-algebra, with structure maps $\alpha_n : \mathcal{O}(n) \times X^n \to X$. We define a map $\alpha : OX = \coprod_j \mathcal{O}(j) \times X^j / \sim \longrightarrow X$ by $\alpha[f, x] = \alpha_n(f, x)$.

Conversely, given an O-algebra X with algebra map $\alpha : OX = \coprod_j \mathcal{O}(j) \times X^j / \sim \longrightarrow X$, we can make it into an $\mathcal{O}$-algebra by defining structure maps $\alpha_n : \mathcal{O}(n) \times X^n \to X$ as $\alpha_n(f, x) = \alpha[f, x]$.

It's a routine problem to check that α is well defined, and that all the corresponding diagrams commute. □

The bar construction

Let's go back to the simplest instance of our problem: given a space X, we want to know if X is a loop space; that is, if there exists a space Y such that $X \simeq \Omega Y$. When our space is a topological group G, the answer is affirmative: we can define its classifying space BG, and we know that $G \simeq \Omega BG$ [Hat02, Prop. 4.66]. With this in mind, our approach will be to show that all $\mathcal{C}_k$-algebras admit a generalization of a classifying space satisfying this same property.

Let's start by recalling the construction of the space BG. If we look at G as a one-object category, then BG is defined as the geometric realization of the nerve of G. Explicitly,

$$BG = |B_* G|$$

where $B_* G$ is the simplicial space with components

$$B_n G = G^n,$$

face maps $d^i : B_{n+1} G \to B_n G$ given by

- $(g_0, \dots, g_n) \mapsto (g_1, \dots, g_n)$, for $i = 0$
- $(g_0, \dots, g_n) \mapsto (g_0, \dots, g_i g_{i+1}, \dots, g_n)$, for $0 < i < n$
- $(g_0, \dots, g_n) \mapsto (g_0, \dots, g_{n-1})$, for $i = n$

and degeneracy maps $s^i : B_n G \to B_{n+1} G$ given by

$$(g_0, \dots, g_{n-1}) \mapsto (g_0, \dots, g_{i-1}, 1, g_i, \dots, g_{n-1})$$

In order to generalize this construction, we introduce the following notion.

Definition 17. Given a monad T on $\mathcal{C}$, a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be a T-functor if T acts on F on the right via a natural transformation $\lambda : FT \Rightarrow F$, which is required to satisfy analogue conditions for those of a T-algebra structure map. Namely, the following diagrams must commute:

$$\begin{array}{ccc} F & \xRightarrow{F\eta} & FT \\ & {\scriptstyle 1}\searrow & \Downarrow{\scriptstyle \lambda} \\ & & F \end{array} \qquad\qquad \begin{array}{ccc} FT^2 & \xRightarrow{F\mu} & FT \\ \Downarrow{\scriptstyle \lambda T} & & \Downarrow{\scriptstyle \lambda} \\ FT & \xRightarrow{\lambda} & T \end{array}$$

Remark 18. It follows from this definition, and that of a monad, that T is always a T-functor.

Now, given a triple (F, T, X) where T is a monad, F a T-functor and X a T-algebra, we can construct a simplicial object $B_*(F, T, X)$ where

$$B_q(F, T, X) = FT^q X,$$

the face maps are given by

- the T-functor structure map $FT \to F$, for $i = 0$
- the multiplication $T^2 \to T$, for $0 < i < q$
- the algebra structure map $TX \to X$, for $i = q$

and the degeneracy maps are given by $\eta : 1 \to T$.

Example 19. If G is a topological group, we can use the group product and unit to define a monad $G : \mathbf{Top} \to \mathbf{Top}$ by $G(X) = G \times X$.

This allows us to recover the usual classifying space of G:

$$BG = |B_*(*, G, *)|$$

The main result

We finally have all the tools we need to give a sketch of the proof of the following result:

Theorem 20. *[May72, Thm. 13.1] Let $\mathcal{C}_k$ denote the little k cubes operad, for $1 \leq k \leq \infty$. Then every loop space $\Omega^k X$ is a $\mathcal{C}_k$-algebra, and every connected $\mathcal{C}_k$-algebra has the weak homotopy type of a k-fold loop space.*

Proof. The easy part: defining an action $\alpha_k : \mathcal{C}_k \to \mathcal{E}nd_{\Omega^k X}$.

First consider $k < \infty$; for a fixed n, we define

$$\alpha_{k,n} : \mathcal{C}_k(n) \otimes (\Omega^k X)^n \longrightarrow \Omega^k X$$

as follows: denote an element $\lambda \in (\Omega^k X)^n$ as a tuple $(\lambda_1, \ldots, \lambda_n)$ of maps $\lambda_i : (I^k, \partial I^k) \to (X, *)$, and a little k-cube in $C_k(n)$ as $(c_1 \ldots, c_n)$, where c_i are the numbered smaller cubes inside I^k. Then, $\alpha_{k,n}((c_1 \ldots, c_n), \lambda)$ is defined as the map $(I^k, \partial I^k) \to (X, *)$ which is λ_i (rescaled) on c_i, and which maps everything in the complement of the n cubes to the basepoint $*$. Staring at the action for a bit is probably enough to convince yourself that this actually works.

Note that in the case $X = \Omega Y$, the actions are coherent

$$\begin{array}{ccc} \mathcal{C}_k(n) \otimes (\Omega^k \Omega Y)^n & \xrightarrow{\alpha_{k,n}} & \Omega^k \Omega Y \\ {\scriptstyle \sigma \times 1} \downarrow & \nearrow {\scriptstyle \alpha_{k+1,n}} & \\ \mathcal{C}_{k+1}(n) \otimes (\Omega^k \Omega Y)^n & & \end{array}$$

where $\sigma : \mathcal{C}_k \hookrightarrow \mathcal{C}_{k+1}$ denotes the embedding, and so they induce an action $\mathcal{C}_\infty(n) \otimes (\Omega^\infty X)^n \longrightarrow \Omega^\infty X$.

A sketch of the sophisticated part:

Due to construction 15 and theorem 16, we know there exists a monad C_k whose algebras are the algebras of the operad $\mathcal{C}_k$. Recalling the monad $\Omega^k \Sigma^k$

from example 14, we can define a morphism of monads $\rho_k : C_k \to \Omega^k \Sigma^k$ given by the composition

$$C_k X \xrightarrow{C_k \eta} C_k \Omega^k \Sigma^k X \xrightarrow{\theta_k} \Omega^k \Sigma^k X$$

(recall that an action map of the operad $\mathcal{C}_k$ over Y corresponds to an algebra-structure map $C_k Y \to Y$).

With considerable effort, it's possible to show ([May72, Thm. 6.1]) that if X is connected, then $\rho_k : C_k X \to \Omega^k \Sigma^k X$ is a weak homotopy equivalence.

We now use the bar construction to find a "simplicial resolution" of our space X. From remark 18, we know that every monad T is a T-functor; in particular, C_k is a C_k-functor and it makes sense to consider the simplicial space

$$B_*(C_k, C_k, X)$$

Let X_* denote the simplicial object with X at every level and $1 : X \to X$ as face and degeneracy maps. The map

$$B_*(C_k, C_k, X) \to X_*$$

$$C_k^{q+1} X \xrightarrow{\mu^q} C_k X \xrightarrow{\theta_k} X$$

is a homotopy equivalence, with inverse

$$X_* \to B_*(C_k, C_k, X)$$

$$X \xrightarrow{\eta^{q+1}} C_k^{q+1} X$$

Recalling that, by definition of algebra over a monad, the following diagrams are commutative

$$\begin{array}{ccc} X & \xrightarrow{\eta} & C_k X \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \theta_k} \\ & & X \end{array} \qquad\qquad \begin{array}{ccc} C_k^2 X & \xrightarrow{\mu} & C_k X \\ \downarrow{\scriptstyle C_k \theta_k} & & \downarrow{\scriptstyle \theta_k} \\ C_k X & \xrightarrow{\theta_k} & X \end{array}$$

we can see that

$$X \xrightarrow{\eta^{q+1}} C_k^{q+1} X \xrightarrow{\mu^q} C_k X \xrightarrow{\theta_k} X$$

is the identity on X. For the other composition, we can define a simplicial homotopy

$$h_i : C_k^{q+1} X \xrightarrow{\mu^i} C_k^{q+1-i} X \xrightarrow{\eta^{i+1}} C_k^{q+2} X$$

and see with some work that everything checks out.

For completeness' sake –and because I find it impossible to remember– we include the definition of simplicial homotopy. As expected, it can be defined by mimicking the definition of homotopy in **Top**, or by using the combinatorial structure of simplicial objects.

Definition 21. A homotopy between two morphisms of simplicial objects $f, g : X \to Y$ is a morphism $H : X \times \Delta[1] \to Y$ that makes the following diagram commute

$$\begin{array}{ccccc} X \simeq X \times \Delta[0] & \xrightarrow{\mathrm{id} \times \delta^1} & X \times \Delta[1] & \xleftarrow{\mathrm{id} \times \delta^0} & X \times \Delta[0] \simeq X \\ & {\scriptstyle f} \searrow & \downarrow {\scriptstyle H} & \swarrow {\scriptstyle g} & \\ & & Y & & \end{array}$$

Equivalently, a simplicial homotopy is a family of morphisms $h_i : X_n \to Y_{n+1}$ for $i = 0, \ldots, n$ and for every n, such that

$$d_0 h_0 = f_n, \quad d_{n+1} h_n = g_n$$

and

$$d_i h_j = \begin{cases} h_{j-1} d_i, & i < j \\ d_i h_{i-1}, & i = j \neq 0 \\ h_j d_{i-1}, & i > j + 1 \end{cases}$$

$$s_i h_j = \begin{cases} h_{j+1} s_i, & i \leq j \\ h_j s_{i-1}, & i > j \end{cases}$$

Going back to our proof, so far we have homotopies

$$X \cong |X_*| \longleftarrow |B_*(C_k, C_k, X)|$$

Next, we can see that the monad map $\rho_k : C_k \to \Omega^k \Sigma^k$ constructed previously, together with its transpose $\rho_k^{\#} : \Sigma^k C_k X \to \Sigma^k X$ via the adjunction $\Sigma^k \dashv \Omega^k$ make $\Omega^k \Sigma^k$ and Σ^k into C_k-functors:

$$\Omega^k \Sigma^k C_k \xrightarrow{\Omega^k \Sigma^k \rho_k} \Omega^k \Sigma^k \Omega^k \Sigma^k \xrightarrow{\mu} \Omega^k \Sigma^k$$

$$\Sigma^k C_k \xrightarrow{\rho_k^{\#}} \Sigma^k$$

Therefore, we can consider the simplicial spaces

$$B_*(\Omega^k \Sigma^k, C_k, X) \quad , \quad B_*(\Sigma^k, C_k, X)$$

Since $\rho_k : C_k X \to \Omega^k \Sigma^k X$ is a weak homotopy equivalence, it induces a weak homotopy equivalence

$$B_*(C_k, C_k, X) \longrightarrow B_*(\Omega^k \Sigma^k, C_k, X)$$

Finally, using the fact that there exists a weak homotopy equivalence $|\Omega\{Y_n\}| \cong \Omega|\{Y_n\}|$ whenever $\{Y_n\}$ is a nice enough simplicial object, we get a chain of weak homotopy equivalences

$$X \cong |X_*| \longleftarrow |B_*(C_k, C_k, X)| \longrightarrow |B_*(\Omega^k \Sigma^k, C_k, X)| \longrightarrow \Omega^k |B_*(\Sigma^k, C_k, X)|$$

which gives us a precise description of X as a k-fold loop space!

□

Remark 22. The use of the monad $\Omega^k\Sigma^k$ is far from coincidental. Note that a map $X \to X'$ determines and is determined by a loop map $\Omega^k X \to \Omega^k X'$, and so

$$\mathrm{Hom}_*(\Sigma^k X, Y) \simeq \mathrm{Hom}_{\mathrm{Loops}}(\Omega^k\Sigma^k X, \Omega^k Y)$$

But we also know that

$$\mathrm{Hom}_*(\Sigma^k X, Y) \simeq \mathrm{Hom}_*(X, \Omega^k Y)$$

so we conclude that

$$\mathrm{Hom}_*(X, \Omega^k Y) \simeq \mathrm{Hom}_{\mathrm{Loops}}(\Omega^k\Sigma^k X, \Omega^k Y)$$

which makes $\Omega^k\Sigma^k X$ the free k-fold loop space generated by X.

Rectifying loop spaces

From now on we will focus on the little intervals operad $\mathcal{C}_1$. From theorem 20, we know that every $\mathcal{C}_1$-algebra X is weakly equivalent to a loop space, $X \simeq \Omega Y$. We also know from our previous discussion that loop spaces are associative and unital up to coherent homotopy, but never strictly. What is the relation between C_1-algebras and strict topological monoids?

If we start with a topological monoid M with unit $1 \in M$ and product $m : M \times M \to M$, then we can give M a $\mathcal{C}_1$-algebra structure as follows:

$$\begin{aligned}
\mathcal{C}_1(0) &\to \mathrm{Hom}(*, X), & * &\mapsto 1\\
\mathcal{C}_1(1) &\to \mathrm{Hom}(X, X), & c &\mapsto \mathrm{id}_X\\
\mathcal{C}_1(2) &\to \mathrm{Hom}(X^2, X), & c &\mapsto m\sigma_c\\
\mathcal{C}_1(3) &\to \mathrm{Hom}(X^3, X), & c &\mapsto m(1 \times m)\sigma_c\\
&\vdots
\end{aligned}$$

Where σ_c denotes the permutation that gives c its numbering, read left to right. We are not making any arbitrary choices here, since the associativity of m ensures that all possible ways to multiply n elements will coincide. Our goal will be to show the closest thing to a converse that we are able to get; namely, that every *cofibrant* $\mathcal{C}_1$-algebra can be rectified to a topological monoid.

Fortunately, we know of an operad whose precise purpose is to model associative unital monoids in any category: the associative operad (recall examples 5 and 9). Thus, our strategy will be to relate the operads $\mathcal{C}_1$ and $\mathcal{A}ss$ in a way that gives us the relation that we need between their categories of algebras.

Here is a glimpse of how this will work: we will define a model structure on the category of operads over a given symmetric monoidal closed model category, and show that both our operads satisfy a weaker version of cofibrancy. Next, we will construct a weak equivalence

$$\mathcal{C}_1 \xrightarrow{\sim} \mathcal{A}ss$$

and explain how this implies we have a Quillen equivalence

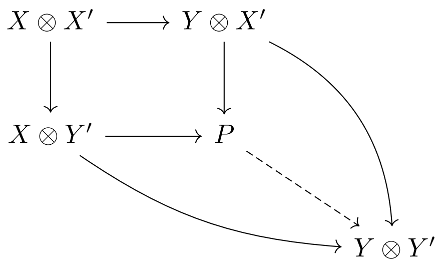
$$\mathcal{C}_1\text{-}\mathbf{Alg} \rightleftarrows \mathcal{A}ss\text{-}\mathbf{Alg}$$

for a suitable model structure on the categories of algebras, which will yield our rectification property. This section follows [BM02].

Some preliminaries on model categories

Definition 23. A monoidal model category is a closed symmetric monoidal category together with a model structure satisfying the *pushout-product* axiom:

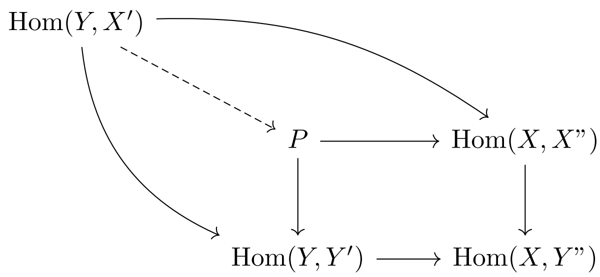
Given cofibrations $f : X \hookrightarrow Y$ and $f' : X' \hookrightarrow Y'$, the induced map from the pushout



is a cofibration, and it is trivial if either f or f' are. Formally we should also require another axiom, which becomes irrelevant if we assume the unit object is cofibrant.

Remark 24. It's tedious (but not hard) to show that a category will satisfy the pushout-product axiom if and only if it satisfies its dual, the *pull back-Hom* axiom:

For a cofibration $f : X \hookrightarrow Y$ and a fibration $f' : X' \twoheadrightarrow Y'$, the induced map to the pull back



is a fibration, and it is trivial if either f or f' are.

We now show a handy result about model categories, from which we deduce a lemma we will use on multiple occasions.

Lemma 25. *[Ken Brown's lemma][Hov99, 1.1.12] Let $\mathcal{C}$ be a model category and $\mathcal{D}$ a category with weak equivalences which satisfy the two out of three axiom. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a functor which takes trivial cofibrations between cofibrant objects to weak equivalences. Then F takes all weak equivalences between cofibrant objects to weak equivalences (and the dual version is, of course, also true).*

Proof. Suppose $f : X \to Y$ is a weak equivalence of cofibrant objects. Factor the map $(f, 1_B) : X \coprod Y \to Y$ as

$$X \coprod Y \xrightarrow[f]{} Z \xrightarrow[g]{\sim} Y$$

The pushout diagram

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \coprod Y \end{array}$$

shows that the inclusion maps $i_1 : X \to X \coprod Y$ and $i_2 : Y \to X \coprod Y$ are cofibrations. By the two out of three axiom, both qi_1 and qi_2 are weak equivalences, hence trivial cofibrations (of cofibrant objects). By hypothesis, we then have that both $F(qi_1)$ and $F(qi_2)$ are weak equivalences. Since $F(pqi_2) = F(1_B)$ is also a weak equivalence, we conclude from the two out of three axiom that $F(p)$ is a weak equivalence, and hence that $F(f) = F(pqi_1)$ is a weak equivalence, as required. □

Lemma 26. *[BM02, 2.3] Let $f : X \to Y$ be a map between cofibrant objects of a monoidal model category. If f is a weak equivalence, then for every fibrant object Z, the induced map $f^* : Z^Y \to Z^X$ is a weak equivalence.*

Proof. If f is a trivial cofibration, we can apply the pull back-Hom axiom to the maps $X \overset{\sim}{\hookrightarrow} Y$ and $Z \twoheadrightarrow 1$ and conclude that $f^* : Z^Y \to Z^X$ must be a weak equivalence. Ken Brown's lemma (25) allows us to conclude the proof □

The following transfer principle will be crucial in constructing the desired model structures on the categories of operads and operad algebras:

Theorem 27. *Let $\mathcal{D}$ be a cofibrantly generated model category and let*

$$F : \mathcal{D} \leftrightarrows \mathcal{E} : G$$

be an adjunction with left adjoint F and right adjoint G. Assume that $\mathcal{E}$ has small colimits and finite limits. Define a map f in $\mathcal{E}$ to be a weak equivalence (resp. fibration) iff $G(f)$ is a weak equivalence (resp. fibration). Then this defines a cofibrantly generated model structure on $\mathcal{E}$ provided

1. *the functor F preserves small objects;*
2. *$\mathcal{E}$ has a fibrant replacement functor;*
3. *$\mathcal{E}$ has functorial path-objects for fibrant objects, that is, for every fibrant X there exists a functorial factorization of its diagonal as*

$$X \xrightarrow{\sim} Path(X) \twoheadrightarrow X \times X$$

The proof of this theorem can be easily adapted from [SS00, Lemma 2.3], where the authors assume every object to be fibrant. Clearly, the first hypothesis is there so that we can use the small object argument for $F(I)$ and $F(J)$, where I (resp. J) denote the generating sets of cofibrations (resp. trivial cofibrations) in $\mathcal{D}$. The other two, which are perhaps a bit obscure at a first glance, are used to obtain some notion of right homotopy, and relate the maps in $F(J)$ to trivial cofibrations in $\mathcal{E}$.

Model structure on operads and their algebras

For a discrete group G, denote by $\mathcal{E}^G$ the category of objects of $\mathcal{E}$ with a (right) G-action. If $\mathcal{E}$ is a symmetric monoidal closed category, then so is $\mathcal{E}^G$, and we have a free-forgetful adjunction

$$F : \mathcal{E} \leftrightarrows \mathcal{E}^G : U$$

where $F(X) = \coprod_G X$. If we consider the permutation groups, we can define the category of collections by

$$\mathbf{Coll}(\mathcal{E}) = \prod_{n\geq 0} \mathcal{E}^{S_n}$$

If our free-forgetful adjunction satisfies the hypotheses for the transfer principle, then the model structure on $\mathcal{E}^G$ will induce a product model structure on $\mathbf{Coll}(\mathcal{E})$, and we can apply our transfer principle once again to the free-forgetful adjunction

$$F : \mathbf{Coll}(\mathcal{E}) \leftrightarrows \mathbf{Op}(\mathcal{E}) : U$$

to obtain a model structure on the category of operads over $\mathcal{E}$. Here U is the functor that forgets the unit and compositions, and F is the free operad functor.

Since the model structure on $\mathbf{Op}(\mathcal{E})$ relies on it, we take a moment to describe the construction of the free operad functor

$$F : \mathbf{Coll}(\mathcal{E}) \to \mathbf{Op}(\mathcal{E})$$

Let $\mathbb{T}$ be the category of finite rooted planar trees, and non-planar isomorphisms. Each edge has an orientation (going downwards), and any tree will have three kinds of edges: internal edges with a vertex at the beginning as well as at the end, input edges with a vertex only at the end, and one outgoing edge, called the output of the tree, with the root vertex as its beginning and no vertex at its end.

Any tree T with a root having n incoming edges decomposes canonically into n trees $T_1, \ldots, T_n$ whose outputs are grafted upon the inputs of the tree t_n with one vertex and n inputs. We denote this grafting operation by $T = t_n(T_1, \ldots, T_n)$.

Note that the number of vertices of each T_i is strictly less than the number of vertices of T, which allows for inductive definitions. Any tree isomorphism $\varphi : T \to T'$ decomposes as $\varphi = \sigma(\varphi_1, \ldots, \varphi_n)$ with isomorphisms $\sigma : t_n \to t_n$ and $\varphi_i : T_i \to T'_{\sigma(i)}$.

For any collection C we define a functor $\underline{C} : \mathbb{T}^{op} \to \mathcal{E}$ by setting inductively $\underline{C}(|) = 1$ and

$$\underline{C}(T) = \underline{C}(t_n(T_1, \ldots, T_n)) = C(n) \otimes \underline{C}(T_1) \otimes \cdots \otimes \underline{C}(T_n)$$

On morphisms, for a map $\varphi : T \to T'$, we get again by induction

$$\underline{C}\varphi = \sigma\underline{C}(\varphi_1, \ldots, \varphi_n) = \sigma \otimes \underline{C}\varphi_{\sigma^{-1}(1)} \otimes \cdots \otimes \underline{C}\varphi_{\sigma^{-1}(n)}$$

Finally, define $\mathbb{T}(n) = \{T \in \mathbb{T} \ : \ T \text{ has } n \text{ inputs}\}$. The inputs of a tree in $\mathbb{T}(n)$ admit $n!$ different numberings, so we associate to it the object $\coprod_{S_n} 1$. Then, an automorphism $\varphi : T \to T$ induces a permutation on the factors of $\coprod_{S_n} 1$.

We use all this information to define the free operad functor,

$$F : \mathbf{Coll}(\mathcal{E}) \to \mathbf{Op}(\mathcal{E})$$

$$FC(n) = \coprod_{[T]\in \mathbb{T}(n)/\sim} \underline{C}(T) \otimes_{\mathrm{Aut}(T)} \amalg_{S_n} 1$$

where $\mathbb{T}(n)/\sim$ denotes the isomorphism classes of trees.

Now that we have a good understanding of the key adjunction involved, let's start by proving we can define a model structure on $\mathcal{E}^G$ by transfer.

Proposition 28. *For any cofibrantly generated, symmetric monoidal closed model category $\mathcal{E}$ with cofibrant unit, the adjunction $F : \mathcal{E} \leftrightarrows \mathcal{E}^G : U$ satisfies the hypotheses of the transfer principle.*

Proof. Since limits and colimits in $\mathcal{E}^G$ are computed pointwise, we trivially have $U(\operatorname{colim} X_i) = \operatorname{colim} U(X_i)$ and so U preserves (filtered) colimits.

Now, $\mathcal{E}$ is cofibrantly generated and therefore has a fibrant replacement functor (the one given by the small object argument, if nothing else). Given, $X \in \mathcal{E}^G$, consider its fibrant replacement in $\mathcal{E}$,

$$X \xrightarrow{\sim} X^{\sim} \twoheadrightarrow 1$$

Then, $X^{\sim} \in \mathcal{E}^G$ by composing the functors

$$\begin{array}{ccccc} G & \longrightarrow & \mathcal{E} & \xrightarrow{(\ )^{\sim}} & \mathcal{E} \\ * & \longmapsto & X & \longmapsto & X^{\sim} \end{array}$$

and $X \xrightarrow{\sim} X^{\sim}$ is a map in $\mathcal{E}^G$ due to the fact that $(\ )^{\sim}$ is functorial.

Let X be a fibrant object in $\mathcal{E}^G$. Consider the folding map $1 \coprod 1 \to 1$ and factor it using the model structure in $\mathcal{E}$ as

$$1 \coprod 1 \hookrightarrow J \xrightarrow{\sim} 1$$

(note that 1 is also in $\mathcal{E}^G$, with the trivial action). Then, we have

$$X \simeq X^1 \longrightarrow X^J \longrightarrow X^{1 \amalg 1} \simeq X \times X$$

where, like before, one can easily see these to be G-maps between G-objects. It remains to show that this composition consists of a weak equivalence followed by a fibration (since it was already constructed in a functorial manner).

For the first map, note that the fact that 1 is cofibrant implies $1 \coprod 1$ is too, and so the cofibration $1 \coprod 1 \hookrightarrow J$ makes J cofibrant as well. Then, since X is fibrant we can use lemma 26 for the weak equivalence $J \xrightarrow{\sim} 1$ to get that $X^1 \longrightarrow X^J$ is a weak equivalence.

Finally, we apply the pull back-Hom axiom to the maps $1 \coprod 1 \hookrightarrow J$ and $X \twoheadrightarrow 1$ and conclude that $X^J \longrightarrow P = X^{1 \amalg 1}$ is a fibration. $\square$

Now we move on to the second part of our proof: we have shown that $\mathbf{Coll}(\mathcal{E})$ is a (cofibrantly generated) model category, and so we apply the transfer principle to the adjunction

$$F : \mathbf{Coll}(\mathcal{E}) \leftrightarrows \mathbf{Op}(\mathcal{E}) : U$$

to define a model structure on the category of operads.

Theorem 29. *[BM02, Thm. 3.2] Let $\mathcal{E}$ be a cofibrantly generated, cartesian closed model category with cofibrant unit, possessing a symmetric monoidal fibrant replacement functor. Then, there exists a cofibrantly generated model structure on $\mathbf{Op}(\mathcal{E})$ where a map $\mathcal{P} \to \mathcal{Q}$ is a weak equivalence (resp. fibration) if and only if the map $\mathcal{P}(n) \to \mathcal{Q}(n)$ is a weak equivalence (resp. fibration) in $\mathcal{E}$, for every n.*

Proof. Once again, we will check the hypotheses for the transfer principle. First, note that the forgetful functor trivially preserves (filtered) colimits.

Let's define a fibrant replacement functor in $\mathbf{Op}(\mathcal{E})$. For an operad $\mathcal{P}$, we define its fibrant replacement $\mathcal{P}^{\sim}$ as the operad $\mathcal{P}^{\sim}(n) = (\mathcal{P}(n))^{\sim}$. Here the S_n action is given by

$$\mathcal{P}^{\sim}(n) \xrightarrow{\sigma^{\sim}} \mathcal{P}^{\sim}(n)$$

using the fact that $(\ )^{\sim}$ is a functor, and the operad structure maps are defined as

$$\begin{array}{c}
\mathcal{P}^{\sim}(n) \times \mathcal{P}^{\sim}(m_1) \times \cdots \times \mathcal{P}^{\sim}(m_n) \\
\Big\downarrow \simeq \\
(\mathcal{P}(n) \times \mathcal{P}(m_1) \times \cdots \times \mathcal{P}(m_n))^{\sim} \\
\Big\downarrow \gamma^{\sim} \\
(\mathcal{P}(m_1 + \cdots + m_n))^{\sim} \\
\| \\
\mathcal{P}^{\sim}(m_1 + \cdots + m_n)
\end{array}$$

where the first isomorphism is given by $(\ )^{\sim}$ being symmetric monoidal. Note that this is functorial because $(\ )^{\sim}$ is, and that it was defined to be an operad map.

Now, given a fibrant operad $\mathcal{P}$ (that is, an operad such that each $\mathcal{P}(n)$ is fibrant in $\mathcal{E}$), we will define a functorial path object

$$\mathcal{P} \xrightarrow{\sim} Path(\mathcal{P}) \twoheadrightarrow \mathcal{P} \times \mathcal{P}$$

This will be done just like in Prop. 28: we factor the folding map $1 \coprod 1 \to 1$ using the model structure in $\mathcal{E}$ as

$$1 \coprod 1 \hookrightarrow J \overset{\sim}{\twoheadrightarrow} 1$$

and then claim that our path object sequence will be given by

$$\mathcal{P} \simeq \mathcal{P}^1 \longrightarrow \mathcal{P}^J \longrightarrow \mathcal{P}^{1 \coprod 1} \simeq \mathcal{P} \times \mathcal{P}$$

Here we define $\mathcal{P}^X$ as the operad $\mathcal{P}^X(n) = \mathrm{Hom}(X, \mathcal{P}(n))$, with S_n action given by

$$\mathrm{Hom}(X, \mathcal{P}(n)) \xrightarrow{\mathrm{Hom}(X,\sigma)} \mathrm{Hom}(X, \mathcal{P}(n))$$

and operad structure given by

$$\begin{array}{c} \mathrm{Hom}(X, \mathcal{P}(n)) \times \mathrm{Hom}(X, \mathcal{P}(m_1)) \times \cdots \times \mathrm{Hom}(X, \mathcal{P}(m_n)) \\ \downarrow \simeq \\ \mathrm{Hom}(X, \mathcal{P}(n) \times \mathcal{P}(m_1) \times \cdots \times \mathcal{P}(m_n)) \\ \downarrow \mathrm{Hom}(X,\gamma) \\ \mathrm{Hom}(X, \mathcal{P}(m_1 + \cdots + m_n)) \end{array}$$

where the isomorphism is due to the fact that, in a cartesian closed category, exponentiation is symmetric monoidal.

It's easy to see from the definition that all maps in our sequence are operad maps, so it suffices to show that the composition consists of a weak equivalence followed by a fibration (since it was already constructed in a functorial manner).

For the first map, the fact that 1 is cofibrant implies $1 \coprod 1$ is too, and so the cofibration $1 \coprod 1 \hookrightarrow J$ makes J cofibrant as well. Then, since each $\mathcal{P}(n)$ is fibrant, we can use lemma 26 for the weak equivalence $J \xrightarrow{\sim} 1$ to get that each $\mathcal{P}(n)^1 \longrightarrow \mathcal{P}(n)^J$ is a weak equivalence, and therefore, $\mathcal{P}^1 \longrightarrow \mathcal{P}^J$ is a weak equivalence.

Finally, we apply the pull back-Hom axiom to the maps $1 \coprod 1 \hookrightarrow J$ and $\mathcal{P}(n) \twoheadrightarrow 1$ for each n and conclude that each $\mathcal{P}(n)^J \longrightarrow \mathcal{P}(n)^{1 \coprod 1}$ is a fibration, and then so is $\mathcal{P}^J \longrightarrow \mathcal{P}^{1 \coprod 1}$. □

Example 30. Let **Top** denote the category of compactly generated Hausdorff topological spaces. This is a cofibrantly generated, cartesian closed model category, where every object is fibrant. Therefore, topological operads admit a model structure.

Now that we have defined a model structure on the category of operads (over certain model categories), we proceed to define a model structure on the category of algebras for a (certain type of) operad. Once again, we will do this by using the transfer principle for the free-forgetful adjunction

$$F_{\mathcal{O}} : \mathcal{E} \rightleftarrows \mathcal{O}\text{-}\mathbf{Alg} : U$$

where $F_{\mathcal{O}}(X) = \coprod_j \mathcal{O}(j) \times X^j / \sim$, with relations as in Prop. 15.

Definition 31. An operad $\mathcal{O}$ in $\mathbf{Op}(\mathcal{E})$ is called *admissible* if the category $\mathcal{O}$-**Alg** carries a model structure which is transferred from $\mathcal{E}$ along the free-forgetful adjunction.

The following result shows that, in the context with which we are dealing, all operads are admissible.

Theorem 32. *Let $\mathcal{E}$ be a category under the hypotheses of Theorem 29; that is, a cofibrantly generated, cartesian closed model category with cofibrant unit, possessing a symmetric monoidal fibrant replacement functor. Then all operads in $\mathbf{Op}(\mathcal{E})$ are admissible.*

Proof. We will show that the conditions for the transfer principle are satisfied for any operad $\mathcal{O}$. Note that the forgetful functor $U : \mathcal{O}\text{-}\mathbf{Alg} \to \mathcal{E}$ preserves filtered colimits.

To define a fibrant replacement functor on $\mathcal{O}$-**Alg**, let X be an $\mathcal{O}$-algebra and consider its fibrant replacement in $\mathcal{E}$, $X \xrightarrow{\sim} X^{\sim}$. If $\mathcal{O} \xrightarrow{\sim} \mathcal{O}^{\sim}$ denotes the fibrant replacement that we get from Thm. 29, then $X^{\sim}$ has an $\mathcal{O}^{\sim}$-algebra structure given by

$$\mathcal{O}^{\sim}(n) \times (X^{\sim})^n \xrightarrow{\simeq} \mathcal{O}(n)^{\sim} \times (X^n)^{\sim} \xrightarrow{\simeq} (\mathcal{O}(n) \times X^n)^{\sim} \longrightarrow X^{\sim}$$

where the existence of the two isomorphisms is ensured by the fact that the fibrant replacement functor in $\mathcal{E}$ is monoidal.

Then, the operad map $\mathcal{O} \xrightarrow{\sim} \mathcal{O}^{\sim}$ induces an $\mathcal{O}$-algebra structure on $X^{\sim}$, by

$$\mathcal{O}(n) \times (X^{\sim})^n \to \mathcal{O}^{\sim}(n) \times (X^{\sim})^n \to X^{\sim}$$

which makes the map $X \xrightarrow{\sim} X^{\sim}$ a weak equivalence of $\mathcal{O}$-algebras.

Now suppose X is fibrant. In the same way as before, we factor the folding map $1 \coprod 1 \to 1$ as

$$1 \coprod 1 \hookrightarrow J \xrightarrow{\sim} 1$$

using the model structure in $\mathcal{E}$, and consider

$$X \simeq X^1 \longrightarrow X^J \longrightarrow X^{1 \coprod 1} \simeq X \times X$$

The first map is a weak equivalence by lemma 26, and the second map is a fibration, as we can see from an application of the pull back-Hom axiom to the maps $1 \coprod 1 \hookrightarrow J$ and $X \twoheadrightarrow 1$.

This construction is clearly functorial, so in order for it to work, we just need to show this factorization lives in the category $\mathcal{O}$-**Alg**. But X^J can be made into an $\mathcal{O}$-algebra by

$$\begin{array}{c}
\mathcal{O}(n) \times \mathrm{Hom}(J, X)^n \\
\Big\downarrow{\scriptstyle \alpha \times \varphi} \\
\mathrm{Hom}(X^n, X) \times \mathrm{Hom}(J^n, X^n) \\
\Big\downarrow{\scriptstyle \simeq} \\
\mathrm{Hom}(X^n, X) \times \mathrm{Hom}(J^n, X^n) \times 1 \\
\Big\downarrow{\scriptstyle 1 \times 1 \times d} \\
\mathrm{Hom}(X^n, X) \times \mathrm{Hom}(J^n, X^n) \times \mathrm{Hom}(J, J^n) \\
\Big\downarrow{\scriptstyle \circ} \\
\mathrm{Hom}(J, X)
\end{array}$$

where $\alpha : \mathcal{O}(n) \to \mathrm{Hom}(X^n, X)$ is the algebra map, $\varphi : \mathrm{Hom}(J, X)^n \to \mathrm{Hom}(J^n, X^n)$ is the map that defines a morphism coordinate-wise, $d : 1 \to \mathrm{Hom}(J, J^n)$ is the map that chooses the diagonal, and $\circ$ is composition. It's easy to check that this is actually an algebra structure map, and that it makes the maps $X \to X^J$ and $X^J \to X \times X$ into algebra maps. □

Example 33. Since the category of compactly generated Hausdorff spaces satisfies the hypotheses of Theorem 29, we conclude that all topological operads must be admissible.

The category of algebras for a cofibrant operad will behave nicely (homotopically speaking) but being cofibrant is, in practice, too strong a condition sometimes. Fortunately, full cofibrancy is not always necessary, and we can weaken it to the following.

Definition 34. An operad $\mathcal{O}$ in $\mathbf{Op}(\mathcal{E})$ is said to be Σ-*cofibrant* if its underlying collection is cofibrant for the model structure in $\mathbf{Coll}(\mathcal{E})$.

Recall that a model category is called *left proper* if weak equivalences are preserved by pushouts along cofibrations.

Theorem 35. *[BM02, Thm. 4.4] Let $\mathcal{E}$ be a left proper monoidal closed model category with cofibrant unit, and*

$$\mathcal{P} \xrightarrow{\sim} \mathcal{Q}$$

be a weak equivalence between Σ-cofibrant operads. Then the base-change adjunction

$$\varphi_! : \mathcal{P}\text{-}\mathbf{Alg} \rightleftarrows \mathcal{Q}\text{-}\mathbf{Alg} : \varphi^*$$

is a Quillen equivalence.

Proof. First, recall that by the definition of the model structures, an algebra map $f : X \to Y$ is a weak equivalence (resp. fibration) if and only if it is so as a map in $\mathcal{E}$, forgetting the algebra structure. Then, since $\varphi^* f = f$, we see that φ^* preserves weak equivalences and fibrations, and so the base-change adjunction is a Quillen pair.

Since φ^* also reflects weak equivalences, the derived adjunction will be an equivalence if (and only if) the unit induces a weak equivalence

$$X \to \varphi^* R\varphi_! X$$

for each cofibrant $\mathcal{P}$-algebra X.[Hov99, 1.3.16]

This is a very technical condition to verify, that uses the fact that the model category is left proper, and the operads are Σ-cofibrant. □

With these results in hand, we are finally ready to start proving our rectification property; we will do so by using Theorem 35. First of all, note that the operads involved, $\mathcal{C}_1$ and $\mathcal{A}ss$, are admissible since by Example 33, all topological operads are. We will start by showing that they are also Σ-cofibrant.

Let's convince ourselves that every connected component of $\mathcal{C}_1(n)$ is a CW-complex. For each n, the space of embeddings $\mathcal{C}_1(n)$ will have one connected component for each possible ordering of $\{1, 2, \ldots, n\}$. Let's look at one of those components; say, the one indexed by $\mathrm{id} \in S_n$. If the starting point of the ith embedded little interval is denoted by x_i, and the endpoint by y_i, then this connected component can be identified with the subspace of $[0, 1]^{2n}$ of vectors $(x_1, y_1, \ldots, x_n, y_n)$ such that

$$x_1 < y_1 \leq x_2 < y_2 \leq \cdots \leq x_n < y_n$$

That is, we consider the hyperplanes $x_1 = y_1$, $y_1 = x_2, \ldots, x_n = y_n$ that divide $[0, 1]^{2n}$ into sections, and our connected component will be the chunk of space given by the correct intersection of sections. This is clearly seen to be a CW-complex (and moreover, contractible, which we will need in the future).

Now, consider a commuting diagram of collections and collection maps

$$\begin{array}{ccc} \emptyset & \longrightarrow & C \\ \big\downarrow & & \big\downarrow f \\ \mathcal{C}_1 & \xrightarrow{g} & D \end{array}$$

where f is a trivial fibraton. We want to define a lift $h : \mathcal{C}_1 \to C$ in $\mathbf{Coll}(\mathbf{Top})$, that is, a family of maps $h_n : \mathcal{C}_1(n) \to C(n)$ that preserve the S_n actions and such that

$$\begin{array}{ccc} & & C(n) \\ & \overset{h_n}{\nearrow} & \big\downarrow f_n \\ \mathcal{C}_1(n) & \xrightarrow{g_n} & D(n) \end{array}$$

commutes. Let $\mathcal{C}_1(n)^{\mathrm{id}}$ denote the connected component of $\mathcal{C}_1(n)$ whose little intervals are numbered by $\mathrm{id} \in S_n$. Then a lift exists in the following diagram

$$\begin{array}{ccc} & & C(n) \\ & \overset{h_n^{\mathrm{id}}}{\nearrow} & \big\downarrow f_n \\ \mathcal{C}_1(n)^{\mathrm{id}} & \xrightarrow{g_n} & D(n) \end{array}$$

since f_n is a trivial fibration and $\mathcal{C}_1(n)^{\mathrm{id}}$ is a CW-complex (and therefore, cofibrant).

Noting that any point in another connected component $x \in \mathcal{C}_1(n)^\sigma$ can be written as σy for a unique $y \in \mathcal{C}_1(n)^{\mathrm{id}}$, we define $h_n^\sigma : \mathcal{C}_1(n)^\sigma \longrightarrow C(n)$ by $h_n^\sigma(x) = \sigma h_n^{\mathrm{id}}(y)$. These will be continuous, since the S_n action acts continuously, and

$$f_n(h_n^\sigma(x)) = f_n(\sigma h_n^{\mathrm{id}}(y)) = \sigma f_n(h_n^{\mathrm{id}}(y)) = \sigma g_n(y) = g_n(\sigma y) = g_n(x)$$

so each h_n^σ effectively defines a lift on its corresponding component.

Putting all these together, we get a function $h_n : \mathcal{C}_1(n) \to C(n)$, that will be continuous since it is so on every connected component. This will be a lift for our diagram, and a collection map (since we specifically defined it for this purpose).

This shows $\mathcal{C}_1$ is a Σ-cofibrant operad; verifying that $\mathcal{A}ss$ is Σ-cofibrant is identical, since its connected components $\{*\}$ are CW-complexes, and the point $*$ in the component indexed by σ is obtained as $*_\sigma = \sigma *_{\mathrm{id}}$.

The only thing left to see in order to use Theorem 35 is that there exists a weak equivalence

$$\mathcal{C}_1 \xrightarrow{\sim} \mathcal{A}ss$$

For each n, let $f_n : \mathcal{C}_1(n) \longrightarrow \mathcal{A}ss(n) = \coprod_{S_n} \{*\}$ be the map that takes an embedding of n little intervals to the point $\{*\}$ in the coproduct indexed by the permutation corresponding to the numbering of the embedding, read from left to right. Do these define an operad map?

It's clear that the maps will respect the S_n-action and the unit. Drawing a picture for a particular case illustrates how this will also respect compositions.

According to the model structure on operads, this map will be a weak equivalence if and only if each f_n is, so we need to show that all f_n's induce isomorphisms on the homotopy groups, for all choices of basepoints. But $\mathcal{A}ss(n)$ is a discrete space, so $\pi_0(\mathcal{A}ss(n)) = S_n$, and $\pi_i(\mathcal{A}ss(n)) = 0$ for every n and $i \geq 0$. Then, it suffices to show that the maps f_n induce bijections on π_0 and that $\pi_i(\mathcal{C}_1(n)) = 0$ for every n and $i \geq 0$.

For each n, the space of embeddings $\mathcal{C}_1(n)$ will have one connected component for each possible ordering of $\{1, 2, \ldots, n\}$, so $\pi_0(\mathcal{C}_1(n)) = S_n$ and we can see that $f_n = \mathrm{id}$. Furthermore, each of these components will be contractible, so no matter the choice of basepoint we will have $\pi_i(\mathcal{C}_1(n)) = 0$ for every n and $i \geq 0$.

Therefore, Theorem 35 ensures us that the induced adjunction

$$\varphi_! : \mathcal{C}_1\text{-}\mathbf{Alg} \rightleftarrows \mathcal{A}ss\text{-}\mathbf{Alg} : \varphi^*$$

is a Quillen equivalence, which implies that for every cofibrant C_1-algebra X, the unit map $X \to \varphi^*\varphi_! X$ is a weak equivalence in $\mathcal{C}_1$-$\mathbf{Alg}$. But $\varphi_! X$ is an $\mathcal{A}ss$-$\mathbf{Alg}$, i.e. a topological monoid, and $\varphi^*\varphi_! X$ is equal to $\varphi_! X$ as a space, with a $\mathcal{C}_1$ action that's constant in each connected component of $\mathcal{C}_1$, and acts like $\mathcal{A}ss$ on each of them, that is, $\varphi^*\varphi_! X$ is also a monoid as a $\mathcal{C}_1$-algebra.

We conclude that *cofibrant $\mathcal{C}_1$-algebras can be rectified to topological monoids*; that is, for every cofibrant C_1-algebra X, there exists a topological monoid M and a weak equivalence

$$X \xrightarrow{\sim} M$$

that respects the $\mathcal{C}_1$-algebra structures. Furthermore, we have an explicit description of M as $M = \varphi^*\varphi_! X$.

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