EXERCISES

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This work is not a part of *The Hitchhiker's Guide To The Incompleteness Theorem*, but please read that first before you continue.

We've talked about arithmetic in HGIT, and let us switch to geometry this time.

Theorem 1. *Two triangles that have equal corresponding sides are congruent.*

How many terms do we need to define to write out the previous theorem in a formal sentence? Maybe the first question to ask is, what are the elements of the world we are talking about? Triangles? Could be. One can define two predicates $P(T_1, T_2)$, $Q(T_1, T_2)$ to mean that Triangles T_1 and T_2 have "equal corresponding sides" and "congruent". Then our theorem will be simply:

$$\forall x \forall y (P(x, y) \Rightarrow Q(x, y))$$

But in general, we want to have a uniformed way of describing objects in geometry. For example, how about circles, straight lines or quadrilaterals? Or even more complicated geometric objects? Here we provide a possible way to handle all these objects, but we shall all agree that this is **not** the unique way to deal with geometry.

Let's only talk about 2-dimension geometry here. Our world will consists of points (on a plane). We put relation symbols:

- **Eqlh(A,B,C,D):** means $\overline{AB} = \overline{CD}$, i.e. the two line segments have equal length.
- **Eqang(A,B,C,D,E,F):** means $\angle ABC = \angle DEF$, i.e. the two angles have the same degree.
- **SL(A,B,C):** means that A, B, C are points on one straight line, in this order.

Exercise 1. *Define $Cong(A, B, C, D, E, F)$ which means that $\triangle ABC$ is congruent to $\triangle DEF$ (i.e. all corresponding sides and angles are the same), then write out Theorem 1 in a formal sentence.*

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One can define a circle with two points (center and one point on the circle) and a quadrilaterals with four (vertices), so it is not that difficult to write out some classical theorems in this formal language.

Exercise 2. *Try to write out this formula: "$\angle ABC + \angle DEF$ equals the straight angle ($180°$)".*

Exercise 3. *Try to write out this theorem: "In a quadrilateral, if opposite angles add up to $180°$, then this quadrilateral is cyclic (i.e. all vertices all lie on one circle)".*

Exercise 4. *Try to write out this formula: "Straight line AB is tangent to the circle with center O and one point C (i.e. radius $\overline{OC}$)".*

One can similarly write out axioms and proofs, but usually they are too long to be good exercises. Here is a real example in Euclid's Elements:

Exercise 5. *Use Theorem 1 (this is actually an axiom in Elements) to prove that every isosceles triangle has two same angles (angles opposite to the corresponding equal sides).*

An interesting thing is that, we can actually code the natural numbers with addition into elementary geometry. Intuitively we will fix a sequence of points on a straight line with equal distance apart from the adjacent points.

Exercise 6. *Define a sentence $Add(A, B, C, D, E, F)$ which says $\overline{AB} = \overline{CD} + \overline{EF}$, also define $P_n(A, B, C, D)$ which means that the length of AB is n times of the length of CD, i.e. $\overline{AB} = n\overline{CD}$. (Define one sentence for each n.)*

Now if we fix a sequence $A_0, A_1, A_2, ...$ on one straight line s.t. $\overline{A_0A_1} = \overline{A_1A_2} = ...$ and code natural numbers as those points in this order, we then can talk about addition:

$$A_i + A_j = A_n := \overline{A_0A_i} + \overline{A_0A_j} = \overline{A_0A_n}$$

Exercise 7. *Define the successor operator and code "$1 + 1 = 2$" into elementary geometry.*

Exercise 8. *Code the order "$<$" into elementary geometry.*

However, we cannot code the multiplication of natural numbers into geometry at the same time. You need to following result of Tarski: elementary geometry is complete, i.e. you can find an effective list of axioms which can prove every true sentence.

Exercise 9. *Prove that multiplication cannot be coded into geometry with addition, order and successor relation. (Use the incompleteness theorem.)*

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