HOMEWORK 10

Math 1920 - Fall 2018

<u>SELECTED SOLUTIONS</u>

November 1, 2018

16.6

18) SOLUTION: We have $x = ue^{e^v}$, $y = e^u$, hence the Jacobian of Φ is

$$\mathrm{Jac}(\Phi) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^v & ue^v \\ e^u & 0 \end{vmatrix} = -ue^{u+v}.$$

29) SOLUTION: Changing variables, we have:

$$\iint_{\mathcal{D}} y\mathrm{d}A = \iint_{\mathcal{R}} (u+v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \mathrm{d}u\mathrm{d}v.$$

We compute the Jacobian of Φ. Since $x = u^2$ and $y = u + v$, we have

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 0 \\ 1 & 1 \end{vmatrix} = 2u.$$

Plugging this in the integral above, we get:

$$\iint_{\mathcal{D}} y\mathrm{d}A = \iint_{\mathcal{R}} (u+v) \cdot 2u\mathrm{d}u\mathrm{d}v = \int_0^6 \int_1^2 (2u^2 + 2uv)\mathrm{d}u\mathrm{d}v = 82,$$

after computing the iterated integral.

30) SOLUTION: For the given transformation, we have $x = \frac{u^2}{v}$ and $y = \frac{v^2}{u}$. The Jacobian of Φ is the following determinant:

$$\mathrm{Jac}(\Phi) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} \\ -\frac{v^2}{u^2} & \frac{2v}{u} \end{vmatrix} = 3.$$

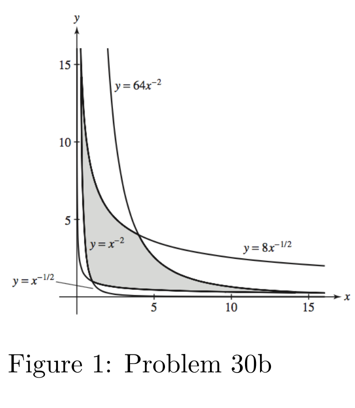

Figure 1: Problem 30b

The domain $\mathcal{D}$ is shown in Figure 1. To compute Area($\mathcal{D}$), use the change of variables formula. Since the area of the rectangle $\mathcal{R}$ is 9, we get

$$\text{Area}(\mathcal{D}) = \iint_{\mathcal{D}} 1 \cdot \mathrm{d}x\mathrm{d}y = \iint_{\mathcal{R}} 1 \cdot |\text{Jac}(\Phi)|\mathrm{d}u\mathrm{d}v = 3 \cdot \text{Area}(\mathcal{R}) = 27.$$

As for $f(x, y) = x + y$, since $x = \frac{u^2}{v}$ and $y = \frac{v^2}{u}$, the function in the uv-plane is

$$f(x, y) = x + y = \frac{u^2}{v} + \frac{v^2}{u}.$$

Using change of variables, we get

$$\iint_{\mathcal{D}} f(x, y)\mathrm{d}A = \iint_{\mathcal{R}} \left(\frac{u^2}{v} + \frac{v^2}{u}\right) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}u\mathrm{d}v = \int_1^4 \int_1^4 \left(\frac{u^2}{v} + \frac{v^2}{u}\right) \cdot 3\mathrm{d}u\mathrm{d}v = 126 \ln 4.$$

33) Solution: We show that the boundary of $\mathcal{D}_0$ is mapped to the boundary of $\mathcal{D}$ (Figure 2). We have:

$$x = u^2 - v^2, \qquad y = 2uv.$$

The line $v = u$ is mapped to the following set:

$$(x, y) = (u^2 - u^2, 2u^2) = (0, 2u^2) \Rightarrow x = 0, \quad y \geq 0.$$

That is, the image of the line $u = v$ is the positive y-axis. The line $v = 0$ is mapped to the following set:

$$(x, y) = (u^2, 0) \Rightarrow x = u^2, \quad y = 0 \Rightarrow y = 0, x \geq 0.$$

Thus, the line $v = 0$ is mapped to the positive x-axis. We now show that the vertical line $u = 1$ is mapped to the curve $y^2 + 4x = 4$. The image of the line $u = 1$ is the following set:

$$(x, y) = (1 - v^2, 2v) \Rightarrow x = 1 - v^2, \quad y = 2v.$$

We substitute $v = \frac{y}{2}$ in the equation $x = 1 - v^2$ to obtain

$$x = 1 - \left(\frac{y}{2}\right)^2 = 1 - \frac{y^2}{4} \Rightarrow 4x = 4 - y^2 \Rightarrow y^2 + 4x = 4.$$

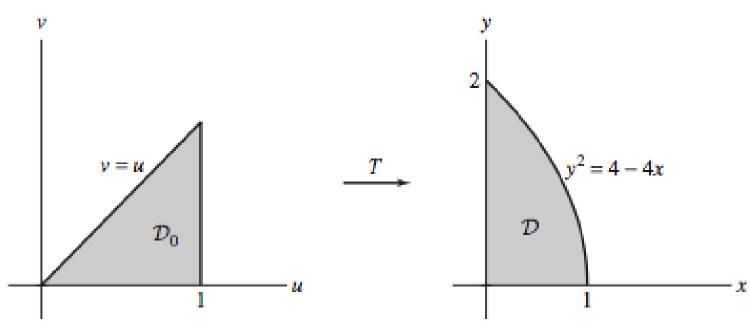

Figure 2: Problem 33

Since the boundary of $\mathcal{D}_0$ is mapped to the boundary of $\mathcal{D}$, we conclude that the domain D_0 is mapped by T to the domain $\mathcal{D}$ in the xy-plane. We now compute the integral $\iint_{\mathcal{D}} \sqrt{x^2+y^2}\mathrm{d}x\mathrm{d}y$. We have:

$$f(x,y) = \sqrt{(u^2-v^2)^2+(2uv)^2} = u^2+v^2.$$

We compute the Jacobian of T:

$$\mathrm{Jac}(\Phi) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2+v^2).$$

Using the change of variables formula gives:

$$\iint_{\mathcal{D}} \sqrt{x^2+y^2}\mathrm{d}x\mathrm{d}y = \iint_{\mathcal{D}_0} (u^2+v^2)\cdot 4(u^2+v^2)\mathrm{d}u\mathrm{d}v = 4\int_0^1\int_0^u (u^4+2u^2v^2+v^4)\mathrm{d}v\mathrm{d}u = \frac{56}{45}.$$

35) Solution: We define a map that sends the unit disk $u^2+v^2 \le 1$ onto the interior of the ellipse. That is:

$$x = 2u, \quad y = 3v \Rightarrow \Phi(u,v) = (2u, 3v).$$

Since $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 \le 1$ if and only if $u^2+v^2 = 1$, Φ is the map we need.

The function f in terms of u and v is $f(x,y) = e^{36(u^2+v^2)}$. The Jacobian of Φ is:

$$\mathrm{Jac}(\Phi) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6.$$

Using the change of variables formula gives

$$\iint_{\mathcal{D}} e^{9x^2+4y^2}\mathrm{d}A = \iint_{\mathcal{D}_0} e^{36(u^2+v^2)}\cdot 6\mathrm{d}u\mathrm{d}v.$$

After computing this last integral using polar coordinates, we get

$$\iint_{\mathcal{D}} e^{9x^2+4y^2}\mathrm{d}A = \frac{\pi(e^{36}-1)}{6}.$$

17.1

33) Solution: Let $\mathbf{F} = \langle F_1, F_2, F_3\rangle$. Then since the necessary partials are continuous, mixed partials are equal, so that

$$\begin{aligned} \mathrm{div}\ \mathrm{curl}(\mathbf{F}) &= \mathrm{div}\left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right\rangle \\ &= \frac{\partial}{\partial x}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\ &= \frac{\partial F_3}{\partial x\partial y} - \frac{\partial F_2}{\partial x\partial z} + \frac{\partial F_1}{\partial y\partial z} - \frac{\partial F_3}{\partial y\partial x} + \frac{\partial F_2}{\partial z\partial x} - \frac{\partial F_1}{\partial z\partial y} \\ &= 0. \end{aligned}$$

40) SOLUTION: This vector field is not conservative, since

$$\frac{\partial}{\partial z}(xz) = x \neq \frac{\partial}{\partial y}(y) = 1.$$

Hence no potential function exists.

51) SOLUTION: Since $\nabla\varphi = \mathbf{F}$, $\mathbf{F}$ is perpendicular to the level curves of φ, as seen in both the contour plots (A) and (B). The plot of $\mathbf{F}$ shows that $\mathbf{F}$ has the form $\mathbf{F} = \langle f(x), 0\rangle$ for an increasing function $f(x)$. Therefore, $\frac{\partial\varphi}{\partial x} = f(x)$ is increasing, implying that the rate of change of φ with respect to x is increasing. Hence, the density of the vertical lines is greater in the direction of growing x. We conclude that (A) is the contour plot of φ.

17.2

20) SOLUTION: The oriented path is parametrized by:

$$\mathbf{c}(t) = (\cos t, \sin t), \quad \pi \leq t \leq \frac{3\pi}{2}.$$

The integrand is:

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = -4\sin t + \frac{1}{2}\sin 2t.$$

The vector line integral can be computed as follows:

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\pi}^{\frac{3\pi}{2}} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\mathrm{d}t = \int_{\pi}^{\frac{3\pi}{2}} \left(-4\sin t + \frac{1}{2}\sin 2t\right)\mathrm{d}t = 4.5$$

28) SOLUTION: We have $\mathbf{c}(t) = (2+t^{-1}, t^3, t^2)$, $\mathbf{F}(\mathbf{c}(t)) = \langle y, z, x\rangle = \langle t^3, t^2, 2+t^{-1}\rangle$, $\mathbf{c}'(t) = \langle -t^{-2}, 3t^2, 2t\rangle$. The integrang is the dot product

$$\mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = 3t^4 + 3t + 2 \Rightarrow \int_{\mathcal{C}} y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z = \int_0^1 (3t^4 + 3t + 2)\mathrm{d}t = \frac{41}{10}.$$

42) SOLUTION:

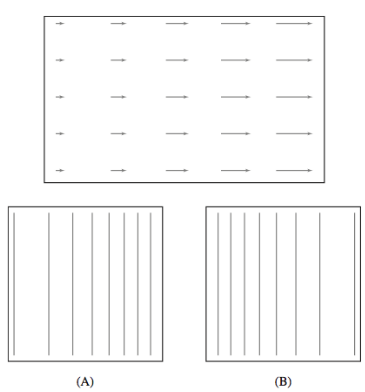

Figure 3: Problem 51

(a) **Positive:** The direction of the path is initially perpendicular to the vector field, becoming more and more oriented along the vector field.

(b) **Positive:** The path is oriented along the vector field when the vectors have a large magnitude. A short section of the path near the end is oriented against the vector field, but the magnitude of these vectors is small. The negative contribution of this section should not cancel out the earlier, strongly positive section.

(c) **Positive:** As before, the vector field has larger magnitude vectors in the section where the path is oriented along the vector field than the section where it is oriented against the vector field.

44) SOLUTION: The total mass is

$$M = \int_{\mathcal{C}} \sqrt{z} \mathrm{d}s.$$

We have

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 2t \rangle.$$

Using the Theorem on Scalar Line Integrals, we get

$$M = \int_{\mathcal{C}} \sqrt{z} \mathrm{d}s = \int_0^{2\pi} \rho(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \mathrm{d}t = \int_0^{2\pi} t\sqrt{1+4t^2} \mathrm{d}t.$$

We compute the integral using the substitution $u = 1 + 4t^2$, $\mathrm{d}u = 8t\mathrm{d}t$ and obtain

$$M \approx 166.86 \text{ g}.$$

61) SOLUTION: Parametrize $\mathcal{C}_R$ by:

$$\mathbf{c}(t) = (R\cos t, R\sin t), \quad 0 \le t \le 2\pi.$$

Then we have:

$$\mathbf{F}(\mathbf{c}(t)) = \frac{1}{R}\langle -\sin t, \cos t \rangle.$$
$$\mathbf{c}'(t) = R\langle -\sin t, \cos t \rangle.$$

Finally,

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) \mathrm{d}t = \int_0^{2\pi} 1 \mathrm{d}t = 2\pi.$$

The answer does not depend on R.