HOMEWORK 12 | SELECTED SOLUTIONS
Math 1920 | November 15, 2018

17.4

10) SOLUTION: The tangent vectors are:

$$\mathbf{T}_r = \frac{\partial \Phi}{\partial r} = \frac{\partial}{\partial r}\left(r\cos\theta, r\sin\theta, 1-r^2\right) = \langle \cos\theta, \sin\theta, -2r\rangle$$
$$\mathbf{T}_\theta = \frac{\partial \Phi}{\partial \theta} = \frac{\partial}{\partial \theta}\left(r\cos\theta, r\sin\theta, 1-r^2\right) = \langle -r\sin\theta, r\cos\theta, 0\rangle.$$

The normal vector is $N = (r,\theta) = \mathbf{T}_r \times \mathbf{T}_\theta = r\langle 2r\cos\theta, 2r\sin\theta, 1\rangle$.

Now we compute the tangency point and the normal vector at this point:

$$P = \Phi\left(\frac{1}{2}, \frac{\pi}{4}\right) = \left(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{3}{4}\right)$$
$$\mathbf{N}\left(\frac{1}{2}, \frac{\pi}{4}\right) = \frac{1}{2}\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right\rangle.$$

The equation of the plane through P, with normal vector $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\rangle$ is:

$$\left\langle x - \frac{1}{2\sqrt{2}}, y - \frac{1}{2\sqrt{2}}, z - \frac{3}{4}\right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right\rangle = 0.$$

Simplifying,

$$2\sqrt{2}x + 2\sqrt{2}y + 4 = 5.$$

21) SOLUTION: We let $z = g(x,y) = 1 - x - y$ and use the formula for the surface integral over the graph of $z = g(x,y)$, where $\mathcal{D}$ is the parameter domain in the xy-plane. That is:

$$\iint_S f(x,y,z)\mathrm{d}S = \iint_{\mathcal{D}} f(x,y,g(x,y))\sqrt{1 + g_x^2 + g_y^2}\mathrm{d}x\mathrm{d}y.$$

We have $g_x = -1$ and $g_y = -1$ therefore:

$$\sqrt{1 + g_x^2 + g_y^2} = \sqrt{3}.$$

We express the function $f(x,y,z) = z$ in terms of the parameters x and y:

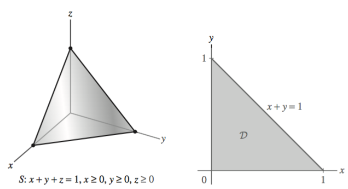

Figure 1: Problem 21

$$f(x, y, g(x, y)) = z = 1 - x - y.$$

The domain of integration is the triangle $\mathcal{D}$ in the xy-plane shown in the figure.

This way,

$$\iint_S f(x, y, z)\mathrm{d}S = \int_0^1 \int_0^{1-y} (1 - x - y)\sqrt{3}\mathrm{d}x\mathrm{d}y = \frac{\sqrt{3}}{6}.$$

24) Solution: We can use spherical coordinates to parametrize the cap S.

$$\Phi(\theta, \phi) = (2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi), \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \phi_0,$$

where ϕ_0 is determined by $\cos\phi_0 = \frac{1}{2}$, that is, $\phi_0 = \frac{\pi}{3}$. The length of the normal vector in spherical coordinates is:

$$\|\mathbf{n}\| = R^2 \sin\phi = 4\sin\phi.$$

We express the function $f(x, y, z) = z^2(x^2 + y^2 + z^2)^{-1}$ in terms of the parameters:

$$f(\Phi(\theta, \phi)) = (2\cos\phi)^2 4^{-1} = \cos^2\phi.$$

Using the theorem on computing the surface integral we get:

$$\iint_S f(x, y, z)\mathrm{d}S = \iint_{\mathcal{D}} f(\Phi(\theta, \phi))\|\mathbf{n}\|\mathrm{d}\phi\mathrm{d}\theta = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} (\cos^2\phi) \cdot 4\sin\phi\mathrm{d}\phi\mathrm{d}\theta = \frac{7\pi}{3},$$

after computing the iterated integral.

30) Solution: The sphere of radius R centered at the origin has the following parametrization in spherical coordinates:

$$\Phi(\theta, \phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi), \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi.$$

The length of the normal vector is

$$\|\mathbf{N}\| = R^2 \sin\phi.$$

Using the integral for the surface area gives:

$$\text{Area}(S) = \iint_{\mathcal{D}} \|\mathbf{N}\|\mathrm{d}\theta\mathrm{d}\phi = \int_0^{2\pi} \int_0^{\pi} R^2 \sin\phi\mathrm{d}\phi\mathrm{d}\theta = 4\pi R^2.$$

41) Solution: We compute the area of the portion of the sphere between the planes a and b. The portion S_1 of the sphere has the parametrization

$$\Phi(\theta, \phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi)$$

where

$$\mathcal{D}_1 : \quad 0 \le \theta \le 2\pi, \quad \phi_0 \le \phi \le \phi_1.$$

If we assume $0 < a < b$, then the angles ϕ_0 and ϕ_1 are determined by

$$\cos\phi_0 = \frac{b}{R} \Rightarrow \phi_0 = \cos^{-1}\frac{b}{R},$$
$$\cos\phi_1 = \frac{a}{R} \Rightarrow \phi_1 = \cos^{-1}\frac{a}{R}.$$

The length of the normal vector is $\|\mathbf{N}\| = R^2 \sin\phi$. We obtain the following integral:

$$\text{Area}(S_1) = \iint_{\mathcal{D}_1} \|\mathbf{N}\| \mathrm{d}\phi \mathrm{d}\theta = \int_0^{2\pi} \int_{\phi_0}^{\phi_1} R^2 \sin\phi \mathrm{d}\phi \mathrm{d}\theta = 2\pi R(b-a).$$

The area of the part S_2 of the cylinder of radius R between the planes $z = a$ and $z = b$ is:

$$\text{Area}(S_2) = 2\pi R(b-a).$$

We see that the two areas are equal:

$$\text{Area}(S_1) = \text{Area}(S_2).$$

17.5

11) Solution: We parametrize the surface by $\Phi(x,y) = (x, y, 1-x-y)$. The tangent and normal vectors are

$$\mathbf{T}_x = \frac{\partial\Phi}{\partial x} = \langle 1, 0, -1\rangle,$$
$$\mathbf{T}_y = \frac{\partial\Phi}{\partial y} = \langle 0, 1, -1\rangle,$$
$$\mathbf{N} = \mathbf{T}_x \times \mathbf{T}_y = \langle 1, 1, 1\rangle.$$

We also have:

$$\mathbf{F}(\Phi(x,y)) \cdot \mathbf{N} = -y^2 - 2 + x.$$

The we evaluate the surface integral as follows:

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_{\mathcal{D}} \mathbf{F}(\Phi(x,y)) \cdot \mathbf{N} \mathrm{d}x\mathrm{d}y = \int_0^1 \int_0^{1-y} (-y^2 - 2 + x)\mathrm{d}x\mathrm{d}y = -\frac{11}{12}.$$

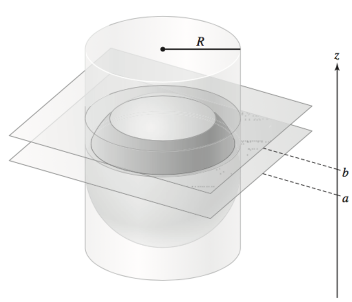

Figure 2: Problem 41 - Section 17.4

20) SOLUTION: We parametrize the sphere of radius R centered at the origin by

$$\Phi: \quad x = R\cos\theta\sin\phi, \quad y = R\sin\theta\sin\phi, \quad z = R\cos\phi, \quad 0 \le \theta < 2\pi, \quad 0 \le \phi \le \pi.$$

The outward pointing normal is $\mathbf{N} = (R^2\sin\phi)\mathbf{e}_r$. On the sphere $r = R$ we get $\mathbf{F}\cdot\mathbf{N} = \sin\phi$, hence

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathcal{D}} (\mathbf{F}\cdot\mathbf{N}) d\phi d\theta = \int_0^{2\pi}\int_0^{\pi} (\sin\phi) d\phi d\theta = 4\pi.$$

We see that the surface integral of $\mathbf{F}$ does not depend on the radius R of the sphere.

23) SOLUTION: We use spherical coordinates:

$$x = \cos\theta\sin\phi, \quad y = \sin\theta\sin\phi, \quad z = \cos\phi$$

with the parameter domain

$$0 \le \theta < 2\pi, \quad 0 \le \phi \le \frac{\pi}{2}.$$

The normal vector is

$$\mathbf{N} = \mathbf{T}_\phi \times \mathbf{T}_\theta = \sin\phi\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle.$$

We express the function in terms of the parameters:

$$\mathbf{v} = \langle 0, 0, z\rangle = \langle 0, 0, \cos\phi\rangle.$$

Hence,

$$\mathbf{v}\cdot\mathbf{N} = \sin\phi\cos^2\phi.$$

The flow rate of the fluid through the upper hemisphere S is equal to the flux of the velocity vector through S. That is,

$$\iint S\mathbf{v}\cdot d\mathbf{S} = \int_0^{\frac{\pi}{2}}\int_0^{2\pi} \sin\theta\cos^2\phi d\theta d\phi = \frac{2\pi}{3}\ \mathrm{m}^3/\mathrm{s}.$$

24) SOLUTION: We use the following parametrization for the surface:

$$\Phi: \quad x = 2r\cos\theta, \quad y = 3r\sin\theta, \quad z = 0$$
$$0 \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 1.$$

The tangent and normal vectors are:

$$\mathbf{T}_r = \langle 2\cos\theta, 3\sin\theta, 0\rangle, \quad \mathbf{T}_\theta = \langle -2r\sin\theta, 3r\cos\theta, 0\rangle.$$
$$\mathbf{T}_r \times \mathbf{T}_\theta = 6r\mathbf{k}.$$

Since the normal points to the positive z-direction, the normal vector is, $\mathbf{N} = 6r\mathbf{k} = \langle 0, 0, 6r\rangle$.

We also have $\mathbf{v}\cdot\mathbf{N} = 72r^4\cos^2\theta\sin\theta$. To compute the flux we proceed as follows:

$$\iint_S \mathbf{v}\cdot d\mathbf{S} = \int_0^{\frac{\pi}{2}}\int_0^1 72r^4\cos^2\theta\sin\theta dr d\theta = 4.8\ \mathrm{m}^3/\mathrm{s}.$$

29) Solution: The equation of the plane through the three vertices is $x + y + z = 1$, hence the upward pointing normal vector is:

$$\mathbf{N} = \langle 1, 1, 1 \rangle$$

and the unit normal vector:

$$\mathbf{e_n} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle.$$

We also have $\mathbf{v} \cdot \mathbf{e_n} = \frac{2}{\sqrt{3}}$. The flow rate through $\mathcal{T}$ is equal to the flux of $\mathbf{v}$ through $\mathcal{T}$. That is,

$$\iint_S \mathbf{v} \cdot \mathrm{d}\mathbf{S} = \iint_S \frac{2}{\sqrt{3}} \mathrm{d}S = \frac{2}{\sqrt{3}} \cdot \mathrm{Area}(S).$$

The area of the equilateral triangle $\mathcal{T}$ is $\frac{\sqrt{3}}{2}$, therefore $\iint_S \mathbf{v} \cdot \mathrm{d}\mathbf{S} = 1$.

Let $\mathcal{D}$ denote the projection of $\mathcal{T}$ onto the xy-plane. Then upward pointing normal is $\mathbf{N} = \langle 0, 0, 1 \rangle$. Observe that $\mathbf{v} \cdot \mathbf{N} = 2$ and hence:

$$\iint_D \mathbf{v} \cdot \mathrm{d}\mathbf{S} = \iint_{\mathcal{D}} (\mathbf{v} \cdot \mathbf{N}) \mathrm{d}S = \iint_{\mathcal{D}} 2 \mathrm{d}S = 1.$$