HOMEWORK 13 — SELECTED SOLUTIONS

Math 1920 — November 29, 2018

18.1

5) SOLUTION: In this function $P = x^2y$ and $Q = 0$, therefore $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -x^2$. We obtain the following integral:

$$I = \int_{\mathcal{C}} x^2 y \mathrm{d}x = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A = \iint_{\mathcal{D}} -x^2 \mathrm{d}A.$$

Converting to polar coordinates, this gives

$$I = \int_0^{2\pi} \int_0^1 -r^2 \cos^2\theta \cdot r\mathrm{d}r\mathrm{d}\theta = -\frac{\pi}{4},$$

after computing the iterated integral.

12) SOLUTION: We denote by $\mathcal{C}$ the path from A to B, and $\mathcal{D}$ is the region enclosed by $\mathcal{C}$ and the segment $\overline{BA}$. By Green's theorem,

$$\int_{\mathcal{C}+\overline{BA}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} + \int_{\overline{BA}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A$$

Parametrizing $\overline{AB}$ by $\langle -1, t\rangle$ with t from 0 to -1, we get

$$\int_{\overline{BA}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{-1} \langle -1, t\rangle \cdot \langle 0, 1\rangle \mathrm{d}t = 4$$

Since $Q = 4x$ and $P = x^3$, we have $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4$ and hence

$$\iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}A = \iint_{\mathcal{D}} 4\mathrm{d}A = 4 \cdot \mathrm{Area}(\mathcal{D}) = 16.$$

Substituting this into the first expression above,

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 20.$$

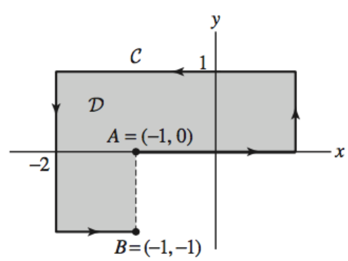

Figure 1: Problem 12

24) SOLUTION:

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{\mathcal{C}_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A$$

Substituting the given information gives

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} - 12 = \iint_{\mathcal{D}} -3\mathrm{d}x\mathrm{d}y = -3\mathrm{Area}(\mathcal{D}).$$

It is clear that Area($\mathcal{D}$)$= 60 - 4\pi$. Using this above,

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 12 - 3(60 - 4\pi) = 12\pi - 168.$$

26) SOLUTION: Let $R > 0$ be sufficiently small so that the circle $\mathcal{C}_R$ is contained in $\mathcal{C}$. Let $\mathcal{D}$ denote the region between $\mathcal{C}_R$ and $\mathcal{C}$. We apply Green's Theorem to the region $\mathcal{D}$, where the curves $\mathcal{C}$ and $\mathcal{C}_R$ is are both oriented counterclockwise as in the diagram. This gives

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} - \int_{\mathcal{C}_R} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_{\mathcal{D}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A.$$

Now,

$$\frac{\partial F_2}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$
$$\frac{\partial F_1}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

Since we are given that $\int_{\mathcal{C}_R} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 2\pi$, substituting above we obtain

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} - 2\pi = \iint_{\mathcal{D}} 0\mathrm{d}A = 0$$

or

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 2\pi.$$

27) SOLUTION: The vector field (A) does not have spirals nor it is a "shear flow". Therefore, the curl appears to be zero. The vector field (B) rotates in the counterclockwise direction, hence we expect the curl to be positive. The vector field (C) perhaps rotates more strongly clockwise than counterclockwise around the origin, indicating a negative curl (however, this is not completely clear, and concluding that it has a zero curl is also reasonable). Finally, in vector field (D) the fluid flows straight toward the origin without spiraling. We expect the curl to be zero.

32) SOLUTION:

a) We parametrize the segment from (x_1, y_1) to (x_2, y_2) by

$$x = tx_2 + (1-t)x_1, \quad y = ty_2 + (1-t)y_1, \quad 0 \le t \le 1.$$

Then, $\mathrm{d}x = (x_2 - x_2)\mathrm{d}t$ and $\mathrm{d}y = (y_2 - y_1)\mathrm{d}t$. Therefore,

$$-y\mathrm{d}x + x\mathrm{d}y = (x_1y_2 - x_2y_1)\mathrm{d}t.$$

We obtain the following integral:

$$\frac{1}{2}\int_{\mathcal{C}} -y\mathrm{d}x + x\mathrm{d}y = \frac{1}{2}\int_0^1 (x_1y_2 - x_2y_1)\mathrm{d}t = \frac{1}{2}(x_1y_2 - x_2y_1).$$

b) Let $A_i = (x_i, y_i)$, $i = 1, 2, \ldots, n$, and let $\mathcal{C}$ be the closed curve determined by the polygon. By the formula for the area enclosed by a simple close curve, the area of the polygon is

$$A = \frac{1}{2} \int_{\mathcal{C}} -y\mathrm{d}x + x\mathrm{d}y$$

We use additivity of the line integrals and the result in part (a) to write the integral as follows:

$$\begin{aligned} A &= \frac{1}{2} \left(\sum_{i=1}^{n-1} \int_{\overline{A_i A_{i+1}}} -y\mathrm{d}x + x\mathrm{d}y + \int_{\overline{A_n A_1}} -y\mathrm{d}x + x\mathrm{d}y \right) \\ &= \frac{1}{2} \left(\sum_{1}^{n-1} (x_i y_{i+1} - x_{i+1} y_i) + (x_n y_1 - x_1 y_n) \right) \end{aligned}$$

If we define $(x_{n+1}, y_{n+1}) = (x_1, y_1)$, we obtain the sum

$$A = \frac{1}{2} \sum_{i}^{n} (x_i y_{i+1} - x_{i+1} y_i).$$

34) Solution: Let $\mathcal{D}$ be the circle $x^2 + y^2 = 9$ together with its interior. The divergence of $\mathbf{F}$ is

$$\mathrm{div}(\mathbf{F}) = 5,$$

so that the flux is

$$\oint \mathbf{F} \cdot \mathbf{n}\mathrm{d}s = \iint_{\mathcal{D}} \mathrm{div}(\mathbf{F})\mathrm{d}s = 5 \iint_{\mathcal{D}} 1\mathrm{d}s.$$

This integral is five times the area of a circle of radius 3, so the answer is $5 \cdot \pi \cdot 3^2 = 45\pi$.

40) Solution: Using the result

$$\text{flux} = \iint_{\mathcal{D}} \mathrm{div}(\mathbf{F})\mathrm{d}A$$

we have, using polar coordinates:

$$\begin{aligned} \text{flux} = \iint_{\mathcal{D}} \mathrm{div}(\mathbf{F})\mathrm{d}A = \iint_{\mathcal{D}} x^2\mathrm{d}A &= \int_0^{2\pi} \int_0^2 (r\cos\theta)^2 (r)\mathrm{d}r\mathrm{d}\theta \\ &= 4\pi. \end{aligned}$$

18.2

1) Solution: We must show that

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S}$$

We first compute the line integral around the boundary curve. This curve is the unit circle oriented in the counterclockwise direction, so we parametrize it by

$$\gamma(t) = (\cos t, \sin t, 0), \quad 0 \leq t \leq 2\pi.$$

Then,

$$\mathbf{F}(\gamma(t)) \cdot \gamma'(t) = \langle 2\cos t \sin t, \cos t, \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle = -2\cos t \sin^2 t + \cos^2 t.$$

Then

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \left(-2\cos t \sin^2 t + \cos^2 t\right) \mathrm{d}t = \pi.$$

Now we compute the flux of the curl through the surface. We parametrize it by

$$\Phi(\theta, t) = (t\cos t, t\sin\theta, 1-t^2), \quad 0 \le t \le 1, \quad 0 \le \theta \le 2\pi.$$

Notive that $\mathbf{T}_\theta = \langle -t\sin\theta, t\cos\theta, 0\rangle$ and $\mathbf{T}_t = \langle \cos\theta, \sin\theta, -2t\rangle$, so

$$\mathbf{T}_\theta \times \mathbf{T}_t = \langle -2t^2\cos\theta, -2t^2\sin\theta, -t\rangle.$$

Since the normal is supposed to be pointing upward, the z-coordinate of the normal vector must be positive. Therefore, the normal vector is $\langle 2t^2\cos\theta, 2t^2\sin\theta, t\rangle$.

The curl in the parameters is:

$$\mathrm{curl}(\mathbf{F}) = \langle 1, 0, 1-2t\cos\theta\rangle.$$

This way,

$$\iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \int_0^{2\pi} \int_0^1 t\mathrm{d}t\mathrm{d}\theta = \pi.$$

The values of the integrals are equal, as stated in Stokes' Theorem.

6) Solution: curl($\mathbf{F}$) is

$$\mathrm{curl}(\mathbf{F}) = \left\langle \frac{xy}{\sqrt{y^2+1}} - 2z, -\sqrt{y^2+1}, -1 \right\rangle.$$

We compute the flux of the curl through the surface by using Stokes' Theorem and computing the line integral around the boundary circle. The oriented boundary of this surface is the triangle at height $z = 2$, oriented clockwise (when viewed from above). By Stokes' Theorem, the flux of the curl is equal to the line integral of $\mathbf{F}$ around the oriented boundary. The restriction of $\mathbf{F}$ to the boundary, where $z = 2$ is

$$\mathbf{F} = \left\langle x + y, 0, x\sqrt{y^2+1} \right\rangle.$$

Next, parametrize the three sides of this triangle for $0 \le t \le 1$:

$$\langle 1-t, 0, 2\rangle, \quad \langle 0, t, 2\rangle, \quad \langle t, 1-t, 2\rangle.$$

Thus $\mathrm{d}\mathbf{s}$ on the three sides is

$$\langle -1, 0, 0\rangle \mathrm{d}t, \quad \langle 0, 1, 0\rangle \mathrm{d}t, \quad \langle 1, -1, 0\rangle \mathrm{d}t,$$

and the dot products are (the z-components of $\mathbf{F}$ is not relevant because $\mathrm{d}\mathbf{s}$ has zero z-component):

$$\mathbf{F} \cdot \mathrm{d}\mathbf{s} = \langle 1-t, 0, *\rangle \cdot \langle -1, 0, 0\rangle \mathrm{d}t = (t-1)\mathrm{d}t$$
$$\mathbf{F} \cdot \mathrm{d}\mathbf{s} = \langle t, 0, *\rangle \cdot \langle 0, 1, 0\rangle \mathrm{d}t = 0$$
$$\mathbf{F} \cdot \mathrm{d}\mathbf{s} = \langle 1, 0, *\rangle \cdot \langle 1, -1, 0\rangle \mathrm{d}t = \mathrm{d}t$$

Therefore, the line integral around the oriented boundary is equal to

$$\int_0^1 (t-1)\mathrm{d}t + 0 + \int_0^1 \mathrm{d}t = \frac{1}{2},$$

and thus the flux of $\mathrm{curl}(\mathbf{F})$ through the surface is $\frac{1}{2}$.

15) SOLUTION:

(a) The induced orientation is defined so that as the normal vector travels along the boundary curve, the surface lies to its left. Therefore, the boundary circles on top and bottom have opposite orientations, which are shown in the figure below.

(b) We first compute the integral around the boundary circles using the following parametrizations:

$$\mathcal{C}_1 : \gamma_1(t) = (2\cos t, 2\sin t, 6), \quad t \text{ from } 2\pi \text{ to } 0$$
$$\mathcal{C}_1 : \gamma_1(t) = (2\cos t, 2\sin t, 1), \quad t \text{ from } 0 \text{ to } 2\pi$$

Observe that:

$$\mathbf{F}(\gamma_1(t)) \cdot \gamma_1'(t) = \langle 72\sin t, 0, 0\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle = -144\sin^2 t.$$
$$\mathbf{F}(\gamma_2(t)) \cdot \gamma_2'(t) = \langle 2\sin t, 0, 0\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle = -4\sin^2 t.$$

The line integral is thus

$$\int_{\mathcal{C}} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{\mathcal{C}_1} \mathbf{F}\cdot \mathrm{d}\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} = \int_{2\pi}^{0} (-144\sin^2 t)\mathrm{d}t + \int_0^{2\pi} (-4\sin^2 t)\mathrm{d}t = 140\pi.$$

The curl is

$$\mathrm{curl}(\mathbf{F}) = \langle 0, 2yz, -z^2\rangle.$$

We parametrize $\mathcal{S}$ by

$$\Phi(\theta, z) = (2\cos\theta, 2\sin\theta, z), \quad 0 \le \theta \le 2\pi, \quad 1 \le z \le 6.$$

The outward pointing normal is $\langle 2\cos\theta, 2\sin\theta, 0\rangle$, hence

$$\mathrm{curl}(\mathbf{F})(\Phi(\theta,z)) \cdot \mathbf{N} = \langle 0, 4z\sin\theta, -z^2\rangle \cdot \langle 2\cos\theta, 2\sin\theta, 0\rangle = 8z\sin^2\theta$$

We obtain the following integral:

$$\iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \int_1^6 \int_0^{2\pi} 8z\sin^2\theta \,\mathrm{d}\theta \mathrm{d}z = 140\pi.$$

The line integral and the flux have the same value. This verifies Stokes' Theorem.

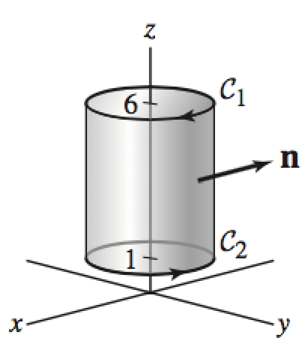

Figure 2: Problem 15a

18) Solution: We first compute the surface integral directly. The spherical cap is parametrized by

$$\Phi(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta\sin\phi, \cos\phi), \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{3}.$$

The outward pointing normal is $\sin\phi(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$, hence

$$\mathbf{F}(\Phi(\theta,\phi)) \cdot \mathbf{N} = \langle 0, -\cos\phi, 1\rangle \cdot \mathbf{N} = -\sin\theta\sin^2\phi\cos\phi + \sin\phi\cos\phi.$$

We obtain the following integral:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} (-\sin\theta\sin^2\phi\cos\phi + \sin\phi\cos\phi)\mathrm{d}\phi\mathrm{d}\theta = \frac{3\pi}{4}.$$

We now evaluate the flux using Stokes' Theorem. A straightforward computation shows that $\mathbf{F} = \mathrm{curl}(\mathbf{A})$, where $\mathbf{A} = \langle 0, x, xz\rangle$. By Stokes,

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_{\mathcal{S}} \mathrm{curl}(\mathbf{A}) \cdot \mathrm{d}\mathbf{S} = \int_{\mathcal{C}} \mathbf{A} \cdot \mathrm{d}\mathbf{r}.$$

To compute the line integral, we notice that the boundary curve is the circle $x^2 + y^2 = \frac{3}{4}$ in the plane $z = \frac{1}{2}$. We parametrize $\mathcal{C}$ by

$$\gamma(t) = \left(\frac{\sqrt{3}}{2}\cos t, \frac{\sqrt{3}}{2}\sin t, \frac{1}{2}\right), \quad 0 \le t \le 2\pi.$$

Hence,

$$\mathbf{A}(\gamma(t)) \cdot \gamma'(t) = \left\langle 0, \frac{\sqrt{3}}{2}\cos t, \frac{\sqrt{3}}{4}\cos t\right\rangle \cdot \left\langle -\frac{\sqrt{3}}{2}\sin t, \frac{\sqrt{3}}{2}\cos t, 0\right\rangle = \frac{3}{4}\cos^2 t.$$

We obtain the following line integral:

$$\int_{\mathcal{C}} \mathbf{A} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \frac{3}{4}\cos^2 t\mathrm{d}t = \frac{3\pi}{4},$$

which implies

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \frac{3\pi}{4}.$$

This agrees with the other computation of the flux, as expected.

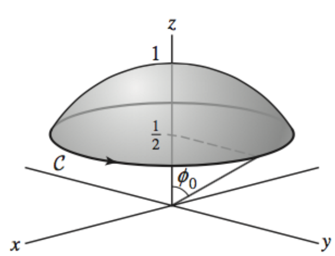

Figure 3: Problem 18

23) Solution: Since we are interested in $\oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{s}$, we can also consider $\iint \mathrm{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S}$, by Stokes' Theorem. The curl is $\langle 4y-2, 0, 1-2y\rangle$ anf the normal to the plane is $\langle a, b, c\rangle$. They are orthogonal if

$$\langle 4y-2, 0, 1-2y\rangle \cdot \langle a, b, c\rangle = 0,$$

which means

$$(4a-2c)y + (c-2a) = 0.$$

We conclude that $c = 2a$ and b arbitrary solves the problem. In other words, the plane $ax+by+2az = 0$ does the job.

27) Solution:

Let $\mathcal{C}$ be the unit circle centered at the origin in the xy-plane. Since $\mathbf{F}$ has a vector potential (say, $\mathbf{A}$), we have by Stokes' Theorem:

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iint_{\mathcal{S}} \mathrm{curl}(\mathbf{A}) \cdot \mathrm{d}\mathbf{S} = \oint_{\mathcal{C}} \mathbf{A} \cdot \mathrm{d}\mathbf{r} = 25.$$