Math 1920 Homework 3 Selected Solutions

13.6

24)

We substitute h into the equation for the hyperboloid and re-arrange to find

$$4h^2 - 1 = x^2 + 4y^2$$

And so this only has solutions for $4h^2 - 1 \geq 0$. If $4h^2 - 1 = 0$ then $h = \pm\frac{1}{2}$, in these cases, the unique solution is when $x = y = 0$ and h determined, i.e. the intersection is a point. If $|h| < 1/2$ then the inequality has no solutions, and so there is no intersection. Otherwise, if $|h| > 1/2$ then the intersection is an ellipse as

$$c = x^2 + 4y^2$$

is an ellipse for $c > 0$.

13.7

PQ2)

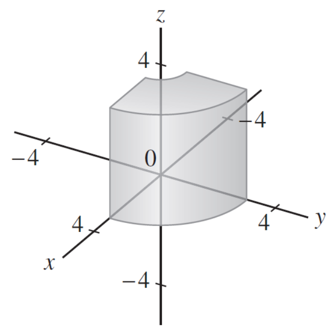
(b) is true. The z-axis is $r = 0$.

PQ4)

0 and π corresponding to the positive an negative parts of the z-axis.

24)

The inequality $1 \leq r \leq 3$ implies that the projection of the region onto the xy-plane is contained in the annulus $1 \leq \sqrt{x^2 + y^2} \leq 3$. The inequality $0 \leq \theta \leq \frac{\pi}{2}$ restricts our annulus to the first quadrant, and $0 \leq z \leq 4$ gives us height for:



28)

Since $x^2 + y^2 = r^2$ we get $r^2 + z^2 = 4$ and so $r = \sqrt{4 - z^2}$.

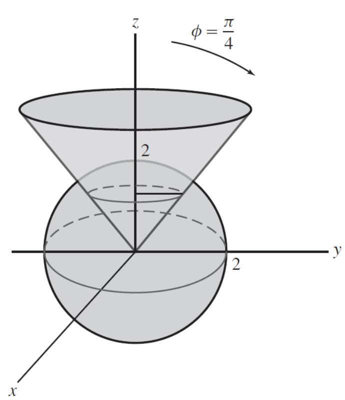
52)

$x^2 + y^2 + z^2 \le 1$ becomes $\rho^2 \le 1$. The inequalities $x \ge 0$ and $y \ge 0$ together determine that $0 \le \theta \le \frac{\pi}{2}$ and $y = x$ is equivalent to $\theta = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. Combining these we obtain

$$\left\{(\rho, \theta, \phi) : 0 \le \rho \le 1, \theta = \frac{\pi}{4}\right\}$$

58)

$\rho = 2$ is the sphere of radius 2 centered at the origin, and $\phi = \frac{\pi}{3}$ is a right circular cone with point at the origin as shown:



They intersect in a horizontal circle centered somewhere on the z-axis and with some radius. To find these values, we take a point on the circle for which we can easily compute the coordinates.

For instance, let P be the point in the intersection with y-coordinate 0 and positive x and coordinate, i.e., $P = (x_0, 0, z_0)$. We know $x_0^2 + z_0^2 = 4$ as P lies on the sphere radius two, and we know $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \frac{z_0}{2}$, as P lies in the cone. Consequently, the center of the circle is $(0, 0, \sqrt{2})$ and its radius is the x-coordinate of P: $\sqrt{2}$.

14.1

PQ2)

Projecting onto the xz-plane means setting your y-coordinate 0 so we get the curve $\langle t, 0, e^t\rangle$ which is the graph of $z = e^x$ in the xz-plane.

PQ4)

$(-2, 2, 3)$.

12)

(c) is a straight line, so it matches with (A). Each of (a)'s coordinates are bounded so it matches with (C). This leaves (b) with (B)

20)

We can write $\mathbf{r}(t) = \langle 6, 9, 4\rangle + \langle 3\sin t, 0, 3\cos t\rangle$ so the center is $(6, 9, 4)$ and the radius is 3. The y coordinate is constant and so the circle is in the plane $y = 9$.

32)

(b) and (c) are true, (a) is false.

34)

The do not collide, because if they did the y-coordinates would have to be equal, i.e., $t^2 = 4t^2$ and so $t = 0$, but the x and z coordinates are not the same for $t = 0$.

To check intersection, we try to solve $\mathbf{r}_1(t) = \mathbf{r}_2(s)$, i.e.,

$$\left\langle t, t^2, t^3\right\rangle = \left\langle 4s + 6, 4s^2, 7 - s\right\rangle.$$

From the first and second coordinates we find

$$\begin{aligned}(4s + 6)^2 &= 4s^2,\\ 16s^2 + 48s + 36 &= 4s^2,\\ 12s^2 + 48s + 36 &= 0,\\ s^2 + 4s + 3 &= 0,\\ (s + 3)(s + 1) &= 0,\end{aligned}$$

so $s=-3$ or $s=-1$ solves both the first and second coordinate, with $t=-6$ and $t=2$ as the corresponding t values. We check these work for the third equation:

$$t^3|_{t=-6} = -216 \neq 10 = (7-s)|_{s=-3}$$
$$t^3|_{t=2} = 8 = (7-s)|_{s=-1}$$

So $s=-1, t=2$ works while the other does not. Hence they intersect.

14.2

22)

a) First we compute $\mathbf{r}_1 \times \mathbf{r}_2$:

$$\mathbf{r}_1 \times \mathbf{r}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2 & 1 & 2t \\ 1 & 2 & e^t \end{vmatrix} = \left\langle e^t - 4t, 2t - t^2 e^t, 2t^2 - 1 \right\rangle$$

we differentiate it and find

$$\frac{d}{dt}(\mathbf{r}_1(t) \times \mathbf{r}_2(t)) = \left\langle e^t - 4, 2 - 2te^t - t^2 e^t, 4t \right\rangle$$

then plug in $t = 1$

$$\langle e-4, 2-3e, 4 \rangle .$$

b) Using the product rule we know $\frac{d}{dt}(\mathbf{r}_1(1) \times \mathbf{r}_2(1)) = \mathbf{r}_1(1) \times \mathbf{r}_2'(1) + \mathbf{r}_1'(1) \times \mathbf{r}_2(1)$ which is

$$\langle 1,1,2 \rangle \times \langle 0,0,e \rangle + \langle 2,0,2 \rangle \times \langle 1,2,e \rangle = \cdots = \langle e-4, 2-3e, 4 \rangle$$

52)

$\mathbf{r}''(t) = \left\langle e^{2t-2}, t^2 - 1, 1 \right\rangle$ so $r'(t) = \left\langle \frac{1}{2}e^{2t-2} + c_1, \frac{1}{3}t^3 - t + c_2, t + c_3 \right\rangle$. Using $\mathbf{r}'(1) = \langle 2,0,0 \rangle$ we find

$$\mathbf{r}'(t) = \left\langle \frac{1}{2}e^{2t-2} + \frac{3}{2}, \frac{1}{3}t^3 - t + \frac{2}{3}, t - 1 \right\rangle .$$

So then $\mathbf{r}(t) = \left\langle \frac{1}{4}e^{2t-2} + \frac{3}{2}t + d_1, \frac{1}{12}t^4 - \frac{1}{2}t^2 + \frac{2}{3}t + d_2, \frac{1}{2}t^2 - t + d_3 \right\rangle$. Using our other initial condition we find

$$\mathbf{r}(t) = \left\langle \frac{1}{4}e^{2t-2} + \frac{3}{2}t - \frac{7}{4}, \frac{1}{12}t^4 - \frac{1}{2}t^2 + \frac{2}{3}t - \frac{1}{4}, \frac{1}{2}t^2 - t + \frac{3}{2} \right\rangle .$$