Math 1920 Homework 5 Selected Solutions

15.3

PQ2)

As the denominator $y+1$ is constant in x we can just treat it as a constant when partially differentiating with respect to x. We are basically differentiating an expression of the form $\frac{x+c}{k}$ for constants c, k.

On the other hand, when you (partially) differentiate with respect to y, as both the numerator and denominator are changing in y, we need to use the quotient rule.

PQ4)

f_x is 0 for the given function, as the function does not depend on x in any way.

12)

f_y represents the rate of change when you move in the y-direction. At A when you move in the (positive) y-direction the function is decreasing, at B it is increasing, and at C we are increasing too (though not as quickly as at B, as can be seen by how close the level curves are). Hence the smallest value of f_y is at A, where it is negative.

22)

Let $z = \sin(u^2 v)$. Then $\frac{\partial z}{\partial u} = 2uv\cos(u^2 v)$ and $\frac{\partial z}{\partial v} = u^2 \cos(u^2 v)$.

76)

Let $u(x,t) = \sin(nx)e^{-n^2 t}$ where n is a constant. We compute $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$

$$\begin{aligned}\frac{\partial u}{\partial t} &= -n^2 \sin(nx)e^{-n^2 t}\\ \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(n\cos(nx)e^{-n^2 t}\right)\\ &= -n^2 \sin(nx)e^{n^2 t}\end{aligned}$$

which are equal, so u satisfies the heat equation.

82)

Recall that a function $f(x,y)$ is **harmonic** if $\Delta f = 0$ where Δ is the **Laplace operator**: $\Delta f = f_{xx} + f_{yy}$.

We compute Δu where $u(x,y) = \cos(ax)e^{by}$:

$$\begin{aligned}u_{xx} &= -a^2 \cos(ax)e^{by},\\ u_{yy} &= b^2 \cos(ax)e^{by},\\ \Delta u &= \cos(ax)e^{by}(b^2 - a^2).\end{aligned}$$

Δu is always 0 if and only if $b^2 = a^2$, i.e., if $b = a$ or $b = -a$.

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15.4

2)

The equation of the tangent plane to f at a point (a, b) is given by

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

We plug $a = 1$ and $b = 0.8$ into this for $f = 0.2x^4 + y^6 - xy$:

$$\begin{aligned} f(1, 0.8) &= 0.2 + (0.8)^6 - (0.2)(0.8) \\ &= 0.622144, \\ f_x(1, 0.8) &= 0.8x^3 - y|_{x=1,y=0.8} \\ &= 0, \\ f_y(1, 0.8) &= 6y^5 - x|_{x=1,y=0.8} \\ &= 0.96608. \end{aligned}$$

Therefore, the tangent plane is

$$z = 0.622144 + 0.96608(y - 0.8)$$

12)

The tangent plane to the graph of $z = f(x, y)$ has $(f_x, f_y, -1)$ as a normal vector. We want the planes to be parallel to $2x + 7y + 2z = 0$ which has normal vector $(2, 7, 2)$. The planes are parallel if and only if their normals are parallel, so we want points so that $f_x = -1$ and $f_y = -7/2$.

In this question $f(x, y) = xy^3 + 8y^{-1}$, so we compute the partials:

$$f_x = y^3, \quad f_y = 3xy^2 - 8y^{-2}.$$

We want $f_x = -1$ and so $y = -1$. We want $f_y = 3xy^2 - 8y^{-2} = 3x - 8 = -7/2$ and so $x = 3/2$. So the only values of x and y which works are $(3/2, -1)$. This corresponds to $z = -19/2$ for the point $(3/2, -1, -19/2)$ on the graph with tangent plane parallel to $2x + 7y + 2z = 0$.

14)

Let $f(x,y) = x(1+y)^{-1}$ and $(a,b) = (8,1)$. Then

$$f(a+h, b+k) \approx f(a,b) + f_x(a,b)h + f_y(a,b)k$$
$$= 4 + \frac{h}{2} + -2k$$

$\frac{7.98}{2.02} = \frac{8+(-0.02)}{2+(0.02)} = f(a+h, b+k)$ where $h = -0.02$ and $k = 0.02$ and so

$$\frac{7.98}{2.02} \approx 4 - 0.01 - 0.04 = 3.95$$

The true value is $3.950\overline{49504}$.

15.5

PQ2)

True.

PQ4)

You want to walk perpendicular to the gradient so NW or SE.

2)

Let $f(x,y) = e^{xy}$ and $\mathbf{r}(t) = \langle t^3, 1+t \rangle$.
a) $\nabla f = \langle ye^{xy}, xe^{xy} \rangle$ and $\mathbf{r}'(t) = \langle 3t^2, 1 \rangle$.
b) The chain rule for paths says

$$\begin{aligned}\frac{d}{dt} f(\mathbf{r}(t)) &= \nabla f_{\mathbf{r}(t)} \cdot \mathbf{r}'(t) \\ &= \left\langle (1+t)e^{t^3(1+t)}, t^3 e^{t^3(1+t)} \right\rangle \cdot \langle 3t^2, 1 \rangle \\ &= e^{t^3(1+t)}(3t^2 + 3t^3 + t^3) \\ &= e^{t^3+t^4}(3t^2 + 4t^3).\end{aligned}$$

c) Directly $f(\mathbf{r}(t)) = e^{t^3+t^4}$ and so

$$\frac{d}{dt} f(\mathbf{r}(t)) = (3t^2 + 4t^3)e^{t^3+t^4}$$

which is the same as part b).

6)

Let $g(x,y) = \frac{x}{x^2+y^2}$ then

$$\nabla g = \left\langle \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}, -\frac{2xy}{(x^2+y^2)^2} \right\rangle$$

32)

Let $f(x,y,z) = xy + z^3$ and $P = (3,-2,-1)$. Then $\nabla f = \left\langle y, x, 3z^2 \right\rangle$, $\nabla f_P = \langle -2, 3, 3 \rangle$, and so the direction of the origin is $-P/\left\|P\right\|$. We dot these together to compute the directional derivative:

$$\nabla f_P \cdot (-P/\left\|P\right\|) = \frac{\langle -2,3,3\rangle \cdot \langle -3,2,1\rangle}{\sqrt{15}} = \frac{15}{\sqrt{15}} = \sqrt{15}.$$

34)

I assume positive y is north and positive x is east. Let $z = x^2 + y^2 - y$ and we are at point $(1,2,3)$. a) The slope in the east direction is just the partial derivative with respect to x: $2x|_{x=1} = 2$. The slope is the tangent of the angle of inclination so $\tan^{-1} 2$ is the angle.

b) Similarly as in part a) but we compute the partial with respect to y: $2y - 1|_{y=2} = 3$ with angle $\tan^{-1} 3$.

c) In the north east direction we want the directional derivative in direction $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. We compute the gradient (of $f(x,y) = x^2 + y^2 - y$) and dot it with this vector

$$\nabla f_{(1,2)} \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \langle 2, 3\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{5}{\sqrt{2}}.$$

With corresponding angle $\tan^{-1} 5/\sqrt{2}$.

d) The direction of steepest slope is the direction of the gradient vector, and the slope is its length which is $\sqrt{2^2 + 3^2} = \sqrt{13}$.

52)

$f(x,y,z) = x^2/2 + y^3/3 + z^4/4$ works (as does f plus any constant).