HW 6 Selected Solutions

15.6

8)

Recall $\frac{\partial f}{\partial u} = \nabla f \cdot \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u} \right\rangle$. Here $f(x,y) = x^2 + y^2$, $x = e^{u+v}$, and $y = u + v$ so

$$\frac{\partial f}{\partial u} = \langle 2x, 2y \rangle \cdot \left\langle e^{u+v}, 1 \right\rangle = \left\langle 2e^{u+v}, 2u + 2v \right\rangle \cdot \left\langle e^{u+v}, 1 \right\rangle = 2e^{2(u+v)} + 2u + 2v$$

26)

Let z be defined by $z^4 + z^2x^2 - y - 8 = 0$. From the formula for implicit differentiation $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ where $F(x,y,z) = z^4 + z^2x^2 - y - 8$. So

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2z^2x}{4z^3 + 2zx^2} = -\frac{2xz}{4z^2 + 2x^2}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-1}{4z^3 + 2zx^2} = \frac{1}{4z^3 + 2zx^2}$$

We plug in the points $(3, 2, 1)$ and $(3, 2, -1)$ respectively for:

$$\frac{\partial z}{\partial x}(3, 2, 1) = -\frac{6}{22}, \frac{\partial z}{\partial y}(3, 2, 1) = \frac{1}{22}, \quad \frac{\partial z}{\partial x}(3, 2, -1) = \frac{6}{22}, \frac{\partial z}{\partial y}(3, 2, 1) = -\frac{1}{22}$$

44)

A function f is homogeneous of degree n if $f(\lambda x, \lambda y, \lambda z) = \lambda^n f(x, y, z)$ for all λ.

Suppose f is homogeneous of degree n. Define $F(t) = f(tx, ty, tz)$ for some choice of x, y, z. Then

$$F'(t) = \frac{d}{dt} f(tx, ty, tz) = \frac{d}{dt} t^n f(x, y, z) = nt^{n-1} f(x, y, z).$$

So $F'(1) = nf(x, y, z)$.

But we can compute F' another way. Write $\mathbf{r}(t) = \langle tx, ty, tz \rangle$, so that $F(t) = f(\mathbf{r}(t))$ then, using the chain rule,

$$F'(t) = \nabla f_{\mathbf{r}(t)} \cdot \mathbf{r}'(t) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle_{\mathbf{r}(t)} \cdot \langle x, y, z \rangle$$

As $\mathbf{r}(1) = \langle x, y, z \rangle$ we have

$$F'(1) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle_{(x,y,z)} \cdot \langle x, y, z \rangle = \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} y + \frac{\partial f}{\partial z} z.$$

And so $nf(x, y, z) = \frac{\partial f}{\partial x} x + \frac{\partial f}{\partial y} y + \frac{\partial f}{\partial z} z$ as required, as they are both equal to $F'(1)$.

Math 1920 Homework 6 Selected Solutions

15.7

PQ2)

The first point is a saddle. The second point is not even a critical point so can't be an extrema nor a saddle. The last two points are, respectively, a local min and local max.

20)

Let $f(x, y) = (x + y)\ln(x^2 + y^2)$, then

$$f_x = \ln(x^2 + y^2) + 2x\frac{x + y}{x^2 + y^2}, \quad f_y = \ln(x^2 + y^2) + 2y\frac{x + y}{x^2 + y^2}$$

These are not defined precisely when $x = y = 0$, but there f is not defined either. We set them equal to 0 and solve

$$(x^2 + y^2)\ln(x^2 + y^2) + 2x(x + y) = 0 = (x^2 + y^2)\ln(x^2 + y^2) + 2y(x + y).$$

From this we find $2y(x + y) = 2x(x + y)$ and so $(x + y)(2y - 2x) = 0$. Thus, $x = -y$ or $x = y$. If $x = y$ we find

$$0 = \ln(2x^2) + \frac{(2x)(2x)}{2x^2} = \ln(2x^2) + 2$$

which implies that $-2 = \ln(2x^2)$ and so $e^{-2} = 2x^2$ therefore $x = \pm\frac{1}{\sqrt{2}e}$.

On the other hand if $y = -x$ then

$$\ln(2x^2) + 2x\frac{0}{2x^2} = 0$$

and so $\ln(2x^2) = 0$ which implies $x = \pm\frac{1}{\sqrt{2}}$. So our critical points are

$$(\frac{1}{\sqrt{2}e}, \frac{1}{\sqrt{2}e}), (-\frac{1}{\sqrt{2}e}, -\frac{1}{\sqrt{2}e}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), \text{ and } (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$

We compute the second partials:

$$f_{xx} = \frac{4x}{x^2+y^2} + \frac{2(x+y)}{x^2+y^2} - \frac{4x^2(x+y)}{(x^2+y^2)^2}$$
$$f_{xy} = \frac{2y}{x^2+y^2} + \frac{2x}{x^2+y^2} - \frac{4xy(x+y)}{(x^2+y^2)^2}$$
$$f_{xy} = \frac{4y}{x^2+y^2} + \frac{2(x+y)}{x^2+y^2} - \frac{4y^2(x+y)}{(x^2+y^2)^2}$$

and apply the second derivative test

Critical point	f_{xx}	f_{yy}	f_{xy}	D	Type
$(\frac{1}{\sqrt{2e}}, \frac{1}{\sqrt{2e}})$	$2e\sqrt{2}$	$2e\sqrt{2}$	0	$8e^2$	Local min
$(-\frac{1}{\sqrt{2e}}, -\frac{1}{\sqrt{2e}})$	$-2e\sqrt{2}$	$2e\sqrt{2}$	0	$8e^2$	Local max
$(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$	$2\sqrt{2}$	$-2\sqrt{2}$	0	-8	Saddle point
$(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$	$-2\sqrt{2}$	$2\sqrt{2}$	0	-8	Saddle point

46)

The volume of such a box is $V(x,y,z) = xyz$ which we are trying to maximize with the constraints $x + \frac{1}{2}y + \frac{1}{3}z = 1$, $x, y, z \geq 0$. We can solve the constraint for z and we find $z = 1 - 3x - \frac{3}{2}y$ and substitute this into V for

$$V(x,y) = xy(1 - 3x - \frac{3}{2}y).$$

We find the critical points of this function in the first quadrant. The first partials are

$$V_x = 3y - 6xy - \frac{3}{2}y^2, \quad V_y = 3x - 3x^2 - 3xy$$

Setting the second equation equal to 0 we find $x = 0$ or $y = 1 - x$. The first we can ignore as $V(0,y) = 0$ and we are trying to maximize. Using $y = 1 - x$ in V_x and setting it to 0 we find $x = 1, \frac{1}{3}$. If $x = 1$ then $y = 0$, which gives 0 volume. Otherwise $x = 1/3$ and so $y = 2/3$. At $x = 1/3, y = 2/3$ we know $z = 1$.

This is the only critical point of the function that doesn't give is 0 and the value of V where is 2/9.

48)

We want to minimize the distance to a point from $(1,0,0) = P$. This is the same as minimizing the square distance from the point, so we want to minimize $f(x,y,z) = (x-1)^2 + y^2 + z^2$ subject to the constraint $z = x + y + 1$.

Plugging this into our function we find $f(x,y) = (x-1)^2 + y^2 + (x+y+1)^2$ is what we want to minimize. The partials of f are

$$f_x = 2(x-1) + 2(x+y+1), \quad f_y = 2y + 2(x+y+1)$$

If we set these equal to 0 we find $y = x - 1$ and $x = 1/3$ and so the point is $(1/3, -2/3, 2/3)$.

56)

We want to find the critical points of $E(m,b)$, we find the first partials

$$E_m = \sum_{j=1}^{n} 2(-x_j)(y_j - mx_j - b), \quad E_b = \sum_{j=1}^{n} -2(y_j - mx_j - b)$$

We set these equal to 0 and re-arrange

$$\begin{aligned} 0 &= E_m \\ &= \sum_{j=1}^{n} 2(-x_j)(y_j - mx_j - b) \\ &= -2\sum_{j=1}^{n} x_j y_j + 2m\sum_{j=1}^{n} x_j^2 + b\sum_{j=1}^{n} x_j \\ 0 &= E_b \\ &= \sum_{j=1}^{n} -2(y_j - mx_j - b) \\ &= -2\sum_{j=1}^{n} y_j + 2\sum_{j=1}^{n} mx_j + 2\sum_{j=1}^{n} b \end{aligned}$$

If we clear the 2s move the negative things over to one side, and note that the sum of b n-times is nb we derive

$$m\left(\sum_{j=1}^{n} x_j\right) + bn = \sum_{j=1}^{n} y_j, \quad m\sum_{j=1}^{n} x_j^2 + b\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} x_j y_j.$$

So there is a critical point here as required. Why is it a global min?

Well observe that for sufficiently large m, b that $E(m,b)$ is increasing as m or b tend to infinity. This means, for any point $E_0 = E(m_0, b_0)$ that there is an R such that $E(m,b) > E_0$ if $|m| > R$ and $|b| > R$. On the domain where $|m| \leq R$ and $|b| \leq R$ (which is closed and bounded) E has a minimal value. The observation before shows that that it must be a global minimum. As there is only one critical point, and we can show (say with the 2nd derivative test) that it is a local minimum, it must be the global minimum too.