HOMEWORK 9 — SELECTED SOLUTIONS

Math 1920 - Fall 2018 — October 25, 2018

16.4

3) SOLUTION: The domain $\mathcal{D}$ is a quarter circle of radius 2 in the first quadrant. We can describe it by $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq r \leq 2$ and f becomes $f(x,y) = \frac{1}{2}r^2 \sin 2\theta$. Using change of variables in polar coordinates,

$$\iint_{\mathcal{D}} xy\mathrm{d}A = \int_0^{\frac{\pi}{2}} \int_0^2 \left(\frac{1}{2}r^2 \sin 2\theta\right) r\mathrm{d}r\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \int_0^2 \frac{1}{2}r^3 \sin 2\theta \mathrm{d}r\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} 2\sin 2\theta \mathrm{d}\theta = 2.$$

21) SOLUTION: The region $\mathcal{W}$ can be written as follows in cylindrical coordinates:

$$\mathcal{W}: \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq r \leq 3, \quad 0 \leq z \leq r\cos\theta.$$

Its volume is the triple integral of the constant function 1 over itself:

$$\iiint_{\mathcal{W}} 1\mathrm{d}V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 \int_0^{r\cos\theta} r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 r^2 \cos\theta \mathrm{d}r\mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 9\cos\theta \mathrm{d}\theta = 18.$$

40) SOLUTION: The region $\mathcal{W}$ can be described as follows in cylindrical coordinates:

$$\mathcal{W}: \quad 0 \leq \theta \leq 2\pi, \quad b \leq r \leq a, \quad -\sqrt{a^2 - r^2} \leq z \leq \sqrt{a^2 - r^2}.$$

Then the volume is:

$$\iiint_{\mathcal{W}} 1\mathrm{d}V = \int_0^{2\pi} \int_a^b \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi} \int_a^b 2r\sqrt{a^2 - r^2}\mathrm{d}r\mathrm{d}\theta = \int_0^{2\pi} \frac{2}{3}(a^2 - b^2)^{\frac{3}{2}}\mathrm{d}\theta = \frac{4\pi}{3}(a^2 - b^2)^{\frac{3}{2}}.$$

58) SOLUTION: The triple integral over $\mathbb{R}^3$ can be computed as the limit as $R \to \infty$ of the triple integral over the ball of radius R. These balls have the following definition in spherical coordinates:

$$\mathcal{W}_R: \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \rho \leq R.$$

The function in spherical coordinates is $f(x,y,z) = (x^2 + y^2 + z^2 + 1)^{-2} = (\rho^2 + 1)^{-2}$. We obtain the following integral:

$$I_R = \iiint_{\mathcal{W}_R} (x^2 + y^2 + z^2 + 1)^{-2}\mathrm{d}V = \int_0^{2\pi} \int_0^{\pi} \int_0^R (1 + \rho^2)^{-2}\rho^2 \sin\phi \mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta = 4\pi \int_0^R (1 + \rho^2)^{-2}\rho^2 \mathrm{d}\rho.$$

This last integral can be computed using the trigonometric substitution $\rho = \tan u$, $\mathrm{d}\rho = \frac{1}{\cos^3 u}\mathrm{d}u$. This will give

$$I_R = 4\phi\left(\frac{\tan^{-1} R}{2} - \frac{\sin 2(\tan^{-1} R)}{4}\right).$$

We now let $R \to \infty$. Using the limit $\lim_{R\to\infty} \tan^{-1} R = \frac{\pi}{2}$, we obtain $I_R = \pi^2$.

60) SOLUTION: The improper integral $I = \iint_{\mathcal{D}} r^{-a} \mathrm{d}A$ is computed as the limit as $\varepsilon \to 0^+$ of the double integrals over the annulus $\mathcal{D}_\varepsilon$ defined by

$$\mathcal{D}_\varepsilon : \quad 0 \le \theta \le 2\pi, \quad \varepsilon \le r \le 1.$$

Using polar coordinates, we obtain:

$$\iint_{\mathcal{D}_\varepsilon} (\sqrt{x^2+y^2})^{-a} \mathrm{d}A = \int_0^{2\pi} \int_\varepsilon^1 r^{-a} r \mathrm{d}r \mathrm{d}\theta = 2\pi \int_\varepsilon^1 r^{1-a} \mathrm{d}r.$$

Taking the limit as $\varepsilon \to 0$, we conclude that $I = 2\pi \int_0^1 r^{-(a-1)} \mathrm{d}r$. This integral converges only if $a - 1 < 1$, or $a < 2$.

16.5

1) SOLUTION: The total mass M is obtained by integrating the mass density $\delta(x,y) = x^2 + y^2$ over the square $\mathcal{D}$ in the xy-plane. This gives

$$M = \iint_{\mathcal{D}} \delta(x,y) \mathrm{d}x \mathrm{d}y = \int_0^1 \int_0^1 (x^2+y^2) \mathrm{d}x \mathrm{d}y = \int_0^1 \left(\frac{1}{3} + y^2 \right) \mathrm{d}y = \frac{2}{3}.$$

9) SOLUTION: The given cone in cylindrical coordinates can be described by the following bounds:

$$0 \le r \le 3, \quad 0 \le \theta \le 2\pi, \quad r \le x \le 3.$$

The density function $\rho(x,y,z) = ae^{-bz}$ has the same expression in cylindrical coordinates. The total mass is:

$$M = \int_0^{2\pi} \int_0^3 \int_r^3 ae^{-bz} r \mathrm{d}z \mathrm{d}r \mathrm{d}\theta = -\frac{a}{b} \int_0^{2\pi} \int_0^3 (re^{-3b} - re^{-br}) \mathrm{d}r \mathrm{d}\theta = -\frac{a}{b} 2\pi \left(\frac{9}{2} e^{-3b} + \frac{3}{b} e^{-3b} + \frac{1}{b^2} e^{-3b} - \frac{1}{b^2} \right)$$

Since $a = 1.225 \times 10^9$ and $b = 0.13$, the total mass is $\approx 2.593 \times 10^{10}$.

27) SOLUTION: The octant $\mathcal{W}$ is defined by $0 \le \theta 2\pi$, $0 \le \phi \le \frac{\pi}{2}$, $0 \le \rho \le 1$, so we have

$$M_{xy} = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 (\rho \cos\phi)(\rho \sin\theta \sin\phi) \rho^2 \sin\phi \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta = \left(\int_0^{\frac{\pi}{2}} \sin\theta \mathrm{d}\theta \right) \left(\int_0^{\frac{\pi}{2}} \cos\phi \sin^2\phi \mathrm{d}\phi \right) \left(\int_0^1 \rho^4 \mathrm{d}\rho \right) = \frac{1}{15}.$$

and

$$M = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 (\rho \sin\theta \sin\phi) \rho^2 \sin\phi \mathrm{d}\rho \mathrm{d}\phi \mathrm{d}\theta = \left(\int_0^{\frac{\pi}{2}} \sin\theta \mathrm{d}\theta \right) \left(\int_0^{\frac{\pi}{2}} \sin^2\phi \mathrm{d}\phi \right) \left(\int_0^1 \rho^3 \mathrm{d}\rho \right) = \frac{\pi}{16}.$$

We conclude that

$$z_{CM} = \frac{1}{M} \iiint_{\mathcal{W}} z\rho(x,y,z) \mathrm{d}V = \frac{16}{15\pi} \approx 0.34.$$

51) SOLUTION:

$$P(X \geq 12, Y \geq 12) = \int_{12}^{18} \int_{12}^{48-2x} \frac{1}{9216}(48 - 2x - y)\mathrm{d}x\mathrm{d}y.$$

The result of the iterated integral is $P(X \geq 12, Y \geq 12) = \frac{1}{64}$.

54) SOLUTION: Since the probability density function is 1, the probability P is the integral of 1 over the region $\mathcal{W} = \{(x, y) : 0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad xy \geq \frac{1}{2}\}$, which is just the area of $\mathcal{W}$. Now, $\mathcal{W}$ is the area bounded by the curves $y = \frac{1}{2x}$ and $y = 1$ for $0 \leq x \leq 1$. Since these curves cross at $x = \frac{1}{2}$, the area is simply

$$P = \int_{\frac{1}{2}}^{1} \left(1 - \frac{1}{2x}\right) \mathrm{d}x = \frac{1}{2}(1 - \ln 2).$$