Final Exam | Math 1920 | December 12, 2011

8 questions; 150 points. For full credit, you must show your work! You may use anything that has been covered in class or in the book, as long as you *show clearly what you are using.* No calculators. **Please start each question on a new page in your booklet!** Good luck!

1. (15 points) Consider the function $f(x, y, z) = x^2y + 2y^2z - xz^3$.

   (a) Find the directional derivative of f at the point $(1, -1, 0)$ in the direction $\mathbf{v} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

   (b) Find the direction of steepest descent of f at the point $(1, -1, 0)$. Give your answer as a unit vector. (The direction of steepest descent is the direction in which the rate of decrease of the function is the greatest).

   (c) Find the rate of steepest descent of f at the point $(1, -1, 0)$.

2. (15 points) Evaluate the integral

$$\int_0^1 \int_0^{\sqrt[3]{1-y^3}} y \sqrt[3]{1 - x^3}\, dx\, dy.$$

3. (20 points) Find the extreme values of $f(x, y, z) = x^2 + yz$ on the intersection of the sphere $x^2 + y^2 + z^2 = 8$ and the plane $y + z = 0$.

4. (20 points) Let D be the region bounded above by the sphere $x^2 + y^2 + z^2 = 4$ and below by the cone $z = (x^2 + y^2)^{1/2}$. Calculate the total outward flux of the vector field $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ through the boundary of this region.

5. (20 points) Consider the circulation of the vector field $\mathbf{F} = x\,\mathbf{j}$ about the curve of intersection of the cylinder $x^2 + (y - 1)^2 = 1$ and the plane $y + z = 2$ counter-clockwise when viewed from above.

   (a) Calculate the circulation directly, using its definition, and

   (b) Calculate the circulation using Stokes' Theorem.

6. (20 points) Consider the surface obtained by rotating about the z-axis the portion of the hyperbola $x = (1 + z^2)^{1/2}$ that is between $z = 0$ and $z = 1$.

   (a) Express the position vector to a point on this surface in terms of the parameters r and θ of the cylindrical coordinate system. Be certain to include the range of the parameters.

   (b) Determine the components of the unit outward (i.e. pointing downward) normal $\mathbf{n}$ to the surface and the relationship between the curved area element $d\sigma$ and the product $dr d\theta$.

   (c) Set up, but do not evaluate, an iterated double integral whose value is the area of this surface.

7. (20 points) Let S be the surface $x^2+y^2=9$, $-1 \le z \le 4$, oriented so that the normals are pointing outward. You are given the following information (some of which may be irrelevant) about a vector field $\mathbf{F}$ which is defined and has continuous partial derivatives at every point in space:

   (a) $\mathbf{F}\cdot\mathbf{k} = z-2$, (b) $\nabla\cdot\mathbf{F} = 0$, and (c) $\nabla\times\mathbf{F} = \mathbf{j}$.

   Given this information, but knowing nothing else about $\mathbf{F}$, compute the flux of $\mathbf{F}$ across S. Notice that the surface S does *not* contain the top or bottom of the cylinder.

8. (20 points) True or False. For each, simply answer "True" or "False". (No explanation is needed. Do not abbreviate these to "T" or "F"). Throughout this problem, $\mathbf{F}$ is a vector field and $f(x,y,z)$ is a function. You may assume that $\mathbf{F}$ and $f(x,y,z)$ have continuous partial derivatives of all orders where they are defined.

   (a) If two vectors $\mathbf{v}$ and $\mathbf{w}$ satisfy: $\mathbf{v}\cdot\mathbf{w} = 0$, and $\mathbf{v}\times\mathbf{w} = \mathbf{0}$, then one of them must be the zero vector.
   (b) Suppose that $\mathbf{F}$ is defined everywhere in space except the origin. If $\nabla\cdot\mathbf{F} = 0$, then there is a function $f(x,y,z)$ such that $\mathbf{F} = \nabla f$.
   (c) Suppose that $\mathbf{F}$ is defined everywhere in space except the origin. If $\nabla\times\mathbf{F} = \mathbf{0}$, then there is a function $f(x,y,z)$ such that $\mathbf{F} = \nabla f$.
   (d) Suppose that $\mathbf{F}$ is defined everywhere in space. Then it is always the case that $\nabla\cdot(\nabla\times\mathbf{F}) = 0$.
   (e) Suppose that $f(x,y,z)$ is defined everywhere in space. Then it is always the case that $\nabla\times(\nabla f) = \mathbf{0}$.
   (f) Suppose that $f(x,y,z)$ is defined everywhere in space. Then it is always the case that $\nabla\cdot(\nabla f) = 0$.

Potentially useful formulae:

$$d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\mathbf{p}|}dA, \qquad d\sigma = |\mathbf{r}_u\times\mathbf{r}_v|\,du\,dv.$$

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA, \qquad \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA.$$

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma, \qquad \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_D \nabla\cdot\mathbf{F}\,dV.$$

Spherical coordinates: $x=\rho\sin\phi\cos\theta,\ y=\rho\sin\phi\sin\theta,\ z=\rho\cos\phi.$

Cylindrical coordinates: $x=r\cos\theta,\ y=r\sin\theta,\ z=z.$

$$\sin(2t) = 2\sin t\cos t, \qquad \cos(2t) = \cos^2 t - \sin^2 t.$$

$$\int \sin^2 t\,dt = t/2 - \sin(2t)/4 + C, \qquad \int \cos^2 t\,dt = t/2 + \sin(2t)/4 + C.$$