MATH 1920 - Fall 2011 - Final exam solutions

1. (a) Let $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 1,2,1\rangle}{\sqrt{1^2+2^2+1^2}} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$ be the unit vector in the direction of $\mathbf{v}$. Next we compute

$$\nabla f = \langle 2xy - z^3, x^2 + 4yz, 2y^2 - 3xz^2 \rangle.$$

Then

$$\begin{aligned}
D_{\mathbf{u}} f(1,-1,0) &= \nabla f(1,-1,0) \cdot \mathbf{u} \\
&= \langle 2 \cdot 1 \cdot (-1) - 0^3, 1^2 + 4 \cdot -1 \cdot 0, 2 \cdot (-1)^2 - 3 \cdot 1 \cdot 0^2 \rangle \cdot \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle \\
&= \langle -2, 1, 2 \rangle \cdot \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle \\
&= \frac{-2}{\sqrt{6}} + \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{6}} \\
&= \boxed{\frac{2}{\sqrt{6}}}.
\end{aligned}$$

(b) The direction of steepest descent of f at $(1,-1,0)$ is the opposite direction of the gradient of f at $(1,-1,0)$. As a unit vector, the direction of steepest descent is

$$-\frac{\nabla f(1,-1,0)}{\|\nabla f(1,-1,0)\|} = -\frac{\langle -2,1,2\rangle}{\sqrt{(-2)^2+1^2+2^2}} = \boxed{\left\langle \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} \right\rangle}.$$

(c) The magnitude of steepest descent is $-\|\nabla f(1,-1,0)\| = -\sqrt{(-2)^2+1^2+2^2} = \boxed{-3}$.

2. We will change the order of integration. The region is described by $0 \le y \le 1$ and $0 \le x \le \sqrt[3]{1-y^3}$. Since $0 \le y \le 1$, we have $0 \le y^3 \le 1$, so $0 \le 1-y^3 \le 1$, and thus $0 \le \sqrt[3]{1-y^3} \le 1$. Hence $0 \le x \le \sqrt[3]{1-y^3} \le 1$.

Thus the region of integration can be described by $0 \le x \le 1$ and $x \le \sqrt[3]{1-y^3} \le 1$. We solve the second inequality for y:

$$\begin{aligned}
x &\le \sqrt[3]{1-y^3} \le 1 \\
x^3 &\le 1 - y^3 \le 1 \\
-1 &\le y^3 - 1 \le -x^3 \\
0 &\le y^3 \le 1 - x^3 \\
0 &\le y \le \sqrt[3]{1-x^3}.
\end{aligned}$$

So the integral becomes

$$
\begin{aligned}
\int_0^1 \int_0^{\sqrt[3]{1-y^3}} y\sqrt[3]{1-x^3}\ dx\ dy &= \int_0^1 \int_0^{\sqrt[3]{1-x^3}} y\sqrt[3]{1-x^3}\ dy\ dx \\
&= \int_0^1 \frac{y^2}{2}\sqrt[3]{1-x^3}\Big|_{y=0}^{\sqrt[3]{1-x^3}}\ dx \\
&= \frac{1}{2}\int_0^2 (\sqrt[3]{1-x^3})^2\sqrt[3]{1-x^3}\ dx \\
&= \frac{1}{2}\int_0^2 (1-x^3)\ dx \\
&= \frac{1}{2}\left(x-\frac{x^4}{4}\right)\Big|_{x=0}^1 \\
&= \frac{1}{2}\left(1-\frac{1}{4}\right) \\
&= \boxed{\frac{3}{8}}.
\end{aligned}
$$

3. We use the method of Lagrange multipliers with two constraints. Let $g(x, y, z) = x^2 + y^2 + z^2 - 8$ and $h(x, y, z) = y + z$, so the constraints are $g(x, y, z) = 0$ and $h(x, y, z) = 0$. We set

$$
\begin{aligned}
\nabla f &= \lambda \nabla g + \mu \nabla h \\
\langle 2x, z, y\rangle &= \lambda\langle 2x, 2y, 2z\rangle + \mu\langle 0, 1, 1\rangle
\end{aligned}
$$
$$
\left\{\begin{array}{rcl} 2x &=& \lambda 2x \\ z &=& \lambda 2y + \mu \\ y &=& \lambda 2z + \mu \end{array}\right.
$$

The first equation can be re-written as $x(\lambda - 1) = 0$, so $x = 0$ or $\lambda = 1$.

Case 1: $x = 0$. Using the constraint $h(x, y, z) = y + z = 0$, we have $z = -y$. We plug this and $x = 0$ into the constraint $g(x, y, z) = x^2 + y^2 + z^2 - 8 = 0$ to get

$$
\begin{aligned}
0^2 + y^2 + (-y)^2 - 8 &= 0 \\
2y^2 &= 8 \\
y^2 &= 4 \\
y &= \pm 2.
\end{aligned}
$$

Then $z = -y = \mp 2$. Thus we obtain two critical points: $(0, 2, -2)$ and $(0, -2, 2)$.

Case 2: $\lambda = 1$. Then

$$\begin{cases} z = 2y + \mu \\ y = 2z + \mu \end{cases}.$$

Solving for μ, we have $\mu = z - 2y = y - 2z$. Hence $3z = 3y$, so $y = z$. Using the constraint $h(x, y, z) = y + z = 0$, we obtain $y + y = 0$. Then $2y = 0$, so $y = 0$. Thus $z = 0$ as well. Finally, we solve for x using the constraint $g(x, y, z) = x^2 + y^2 + z^2 - 8 = 0$:

$$\begin{aligned} x^2 + 0^2 + 0^2 - 8 &= 0 \\ x^2 &= 8 \\ x &= \pm\sqrt{8}. \end{aligned}$$

Hence we obtain two more critical points: $(\sqrt{8}, 0, 0)$ and $(-\sqrt{8}, 0, 0)$.

Finally, we find the extreme values of f by plugging these critical points into f:

$$\begin{aligned} f(0, 2, -2) &= 0^2 + 2 \cdot (-2) = -4 \\ f(0, -2, 2) &= 0^2 + (-2) \cdot 2 = -4 \\ f(\sqrt{8}, 0, 0) &= (\sqrt{8})^2 + 0 \cdot 0 = 8 \\ f(-\sqrt{8}, 0, 0) &= (-\sqrt{8})^2 + 0 \cdot 0 = 8. \end{aligned}$$

Thus f has a maximum value of $\boxed{8}$ at $\boxed{(\sqrt{8}, 0, 0) \text{ and } (-\sqrt{8}, 0, 0)}$, and a minimum value of $\boxed{-4}$ at $\boxed{(0, 2, -2) \text{ and } (0, -2, 0)}$ on the constraints.

4. By the Divergence Theorem,

$$\iint_{\partial D} \mathbf{F} \cdot d\mathbf{S} = \iiint_D \text{div } \mathbf{F} \; dV.$$

We compute

$$\text{div } \mathbf{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3x^2 + 3y^2 + 3z^2.$$

We compute the integral in spherical coordinates. The sphere $x^2+y^2+z^2 = 4$ is described in spherical coordinates by $\rho^2 = 4$ or $\rho = 2$. The cone $z = (x^2+y^2)^{1/2}$

is described in spherical coordinates by

$$\begin{aligned}
\rho\cos\varphi &= ((\rho\cos\theta\sin\varphi)^2 + (\rho\sin\theta\sin\varphi)^2)^{1/2}\\
\rho\cos\varphi &= (\rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta))^{1/2}\\
\rho\cos\varphi &= \rho\sin\varphi \text{ (since } \sin\varphi\geq 0 \text{ on the cone)}\\
\cos\varphi &= \sin\varphi \text{ (or } \rho = 0)\\
1 &= \tan\varphi \text{ (or } \rho = 0)\\
\varphi &= \frac{\pi}{4} \text{ (or } \rho = 0).
\end{aligned}$$

The last line follows because we always have $0 \leq \varphi \leq \pi$. Hence the region D is described by $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \frac{\pi}{4}$, and $0 \leq \rho \leq 2$. Finally, the integrand is $3x^2 + 3y^2 + 3z^2 = 3\rho^2$. Thus

$$\begin{aligned}
\iint_{\partial D} \mathbf{F}\cdot d\mathbf{S} &= \iiint_D \text{div } \mathbf{F}\ dV\\
&= \int_0^{2\pi}\int_0^{\pi/4}\int_0^2 3\rho^2\cdot\rho^2\sin\varphi\ d\rho\ d\varphi\ d\theta\\
&= \left(\int_0^{2\pi} 3\ d\theta\right)\left(\int_0^{\pi/4}\sin\varphi\ d\varphi\right)\left(\int_0^2 \rho^4\ d\rho\right)\\
&= 6\pi\cdot -\cos\varphi\Big|_0^{\pi/4}\cdot\frac{\rho^5}{5}\Big|_0^2\\
&= \boxed{\pi(2-\sqrt{2})\left(\frac{96}{5}\right)}.
\end{aligned}$$

5. (a) The intersection of the cylinder $x^2 + (y-1)^2 = 1$ and the plane $y + z = 2$, when projected into the xy-plane, is a circle of radius 1 centered at $(0, 1, 0)$. That circle is parametrized by $x(t) = \cos t$ and $y(t) = 1 + \sin t$ for $0 \leq t \leq 2\pi$. Finally, on the intersection of the cylinder and plane we have $y+z = 2$, so we must have $z(t) = 2-y(t) = 2-(1+y(t)) = 1-\sin t$. Thus we have the parametrization

$$\begin{aligned}
\mathbf{r}(t) &= \langle\cos t, \sin t + 1, 1 - \sin t\rangle,\ 0 \leq t \leq 2\pi\\
\mathbf{r}'(t) &= \langle -\sin t, \cos t, -\cos t\rangle\\
\mathbf{F}(\mathbf{r}(t)) &= \langle 0, \cos t, 0\rangle.
\end{aligned}$$

So, if $\mathcal{C}$ is the intersection of the cylinder and the plane, we have

$$\begin{aligned}
\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt \\
&= \int_0^{2\pi} \langle 0, \cos t, 0 \rangle \cdot \langle -\sin t, \cos t, -\cos t \rangle\ dt \\
&= \int_0^{2\pi} \cos^2 t\ dt \\
&= \int_0^{2\pi} \frac{1 + \cos 2t}{2}\ dt \\
&= \left(\frac{1}{2} t + \frac{\sin 2t}{4} \right) \Bigg|_0^{2\pi} \\
&= \boxed{\pi}.
\end{aligned}$$

(b) Let $\mathcal{S}$ be the portion of the plane $y + z = 2$ which lies inside the cylinder $x^2 + (y-1)^2 = 1$, with upward-pointing normal vector. Then by Stokes' Theorem, $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{S}} \operatorname{curl} \mathbf{F}\ dS$. When the surface $\mathcal{S}$ is projected into the xy-plane, it is a disk of radius 1 centered at $(0,1)$. This is parametrized by $x(t) = r\cos t$ and $y(t) = 1 + r\sin t$ for $0 \le r \le 1$, $0 \le \theta \le 2\pi$. The surface $\mathcal{S}$ lies on the plane $y + z = 2$, so we must have $z(t) = 2 - y(t) = 2 - (1 + r\sin\theta) = 1 - r\sin\theta$. Thus $\mathcal{S}$ is parametrized by

$$\begin{aligned}
G(\theta, r) &= (r\cos\theta, 1 + r\sin\theta, 1 - r\sin\theta),\ 0 \le \theta \le 2\pi,\ 0 \le r \le 1 \\
\mathbf{T}_\theta &= \langle -r\sin\theta, r\cos\theta, -r\cos\theta \rangle \\
\mathbf{T}_r &= \langle \cos\theta, \sin\theta, -\sin\theta \rangle \\
\mathbf{T}_\theta \times \mathbf{T}_r &= \langle 0, -r - r \rangle \\
\mathbf{N}(\theta, r) &= \langle 0, r, r \rangle
\end{aligned}$$

where the last line is because $\mathcal{S}$ is oriented with an upward-pointing normal vector (positive z-component). We have

$$\operatorname{curl} \mathbf{F} = \operatorname{curl}(\langle 0, x, 0 \rangle) = \langle 0, 0, 1 \rangle.$$

Thus

$$\begin{aligned}
\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} \\
&= \int_0^{2\pi} \int_0^1 \operatorname{curl} \mathbf{F} \cdot \mathbf{N}(\theta, r)\ dr\ d\theta \\
&= \int_0^{2\pi} \int_0^1 \langle 0, 0, 1 \rangle \cdot \langle 0, r, r \rangle\ dr\ d\theta \\
&= \int_0^{2\pi} \int_0^1 r\ dr\ d\theta \\
&= 2\pi \cdot \left. \frac{r^2}{2} \right|_0^1 \\
&= \boxed{\pi}.
\end{aligned}$$

6. (a) First we note that because this curve is rotated around the z-axis, we can replace x by r and describe the surface as $r = \sqrt{1 + z^2}$, $0 \le z \le 1$. Solving for z,

$$\begin{aligned}
r &= \sqrt{1 + z^2} \\
r^2 &= 1 + z^2 \\
r^2 - 1 &= z^2 \\
z &= \sqrt{r^2 - 1}
\end{aligned}$$

since z is nonnegative. Since $0 \le z \le 1$, we have $1 \le \sqrt{1 + z^2} \le \sqrt{2}$. That is, $1 \le r \le \sqrt{2}$.

Hence the projection of this surface onto the xy-plane is an annulus (the region between two circles) of inner radius 1 and outer radius $\sqrt{2}$ centered at the origin. Thus we can describe the x and y coordinates by $x = r\cos\theta$ and $y = r\sin\theta$ for $0 \le \theta \le 2\pi$ and $1 \le r \le \sqrt{2}$. Finally, we found above that $z = \sqrt{r^2 - 1}$. Thus this surface is parametrized by

$$\boxed{(r\cos\theta, r\sin\theta, \sqrt{r^2 - 1}),\ 0 \le \theta \le 2\pi,\ 1 \le r \le \sqrt{2}}.$$

(b) We find the outward pointing normal vector for this parametrization:

$$\begin{aligned}
\mathbf{T}_r &= \left\langle \cos\theta, \sin\theta, \frac{2r}{2\sqrt{r^2 - 1}} \right\rangle = \left\langle \cos\theta, \sin\theta, \frac{r}{\sqrt{r^2 - 1}} \right\rangle \\
\mathbf{T}_\theta &= \langle -r\sin\theta, r\cos\theta, 0 \rangle \\
\mathbf{T}_\theta \times \mathbf{T}_r &= \left\langle \frac{r^2\cos\theta}{\sqrt{r^2 - 1}}, \frac{r^2\sin\theta}{\sqrt{r^2 - 1}}, -r \right\rangle.
\end{aligned}$$

The surface looks like a vase with the bottom cut off, so the outward pointing normal vector will have a negative z-component. Thus we have $\mathbf{N}(r,\theta) = \left\langle \frac{r^2\cos\theta}{\sqrt{r^2-1}}, \frac{r^2\sin\theta}{\sqrt{r^2-1}}, -r \right\rangle$. We compute

$$\begin{aligned}
\|\mathbf{N}(r,\theta)\| &= \sqrt{\left(\frac{r^2\cos\theta}{\sqrt{r^2-1}}\right)^2 + \left(\frac{r^2\sin\theta}{\sqrt{r^2-1}}\right)^2 + (-r)^2} \\
&= \sqrt{\frac{r^4(\cos^2\theta+\sin^2\theta)}{r^2-1} + r^2} \\
&= \sqrt{\frac{r^4}{r^2-1} + \frac{r^2(r^2-1)}{r^2-1}} \\
&= \sqrt{\frac{2r^4-r^2}{r^2-1}} \\
&= r\sqrt{\frac{2r^2-1}{r^2-1}}.
\end{aligned}$$

So,

$$\boxed{dS = r\sqrt{\frac{2r^2-1}{r^2-1}}\ dr\ d\theta}.$$

(c) Let $\mathcal{S}$ be the surface above. Then

$$\boxed{\text{Area}(\mathcal{S}) = \iint_{\mathcal{S}} 1\ dS = \int_2^{2\pi}\int_1^{\sqrt{2}} r\sqrt{\frac{2r^2-1}{r^2-1}}\ dr\ d\theta}$$

7. Let $\mathcal{S}_1$ be the top of the cylinder ($x^2+y^2 \le 9$, $z=4$) oriented with upward pointing normal vector $\mathbf{n} = \mathbf{k}$, and let $\mathcal{S}_2$ be the bottom of the cylinder ($x^2+y^2 \le 9$, $z=-1$) oriented with downard pointing normal vector $\mathbf{n} = -\mathbf{k}$. Then $\mathcal{S}+\mathcal{S}_1+\mathcal{S}_2$ is the boundary of the cylinder, so by the Divergence Theorem,

$$\iint_{\mathcal{S}+\mathcal{S}_1+\mathcal{S}_2} \mathbf{F}\cdot d\mathbf{S} = \iiint_{\mathcal{W}} \text{div } \mathbf{F}\ dV = \iiint_{\mathcal{W}} 0\ dV = 0$$

where $\mathcal{W}$ is the cylinder. We also know

$$\iiint_{\mathcal{S}+\mathcal{S}_1+\mathcal{S}_2} \mathbf{F}\cdot d\mathbf{S} = \iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} + \iint_{\mathcal{S}_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{\mathcal{S}_2} \mathbf{F}\cdot d\mathbf{S}.$$

So to solve for the flux of $\mathbf{F}$ through $\mathcal{S}$, we find the flux of $\mathbf{F}$ through $\mathcal{S}_1$ and

$\mathcal{S}_2$. We have

$$\begin{aligned}
\iint_{\mathcal{S}_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_{\mathcal{S}_1} \mathbf{F} \cdot \mathbf{n} \; dS \\
&= \iint_{\mathcal{S}_1} \mathbf{F} \cdot \mathbf{k} \; dS \\
&= \iint_{\mathcal{S}_1} (z-2) \; dS \\
&= \iint_{\mathcal{S}_1} (4-2) \; dS \\
&= 2 \cdot \text{Area}(\mathcal{S}_1) \\
&= 2 \cdot \pi \cdot 3^2 \\
&= 18\pi.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\iint_{\mathcal{S}_2} \mathbf{F} \cdot d\mathbf{S} &= \iint_{\mathcal{S}_2} \mathbf{F} \cdot \mathbf{n} \; dS \\
&= \iint_{\mathcal{S}_2} \mathbf{F} \cdot -\mathbf{k} \; dS \\
&= \iint_{\mathcal{S}_2} -(\mathbf{F} \cdot \mathbf{k}) \; dS \\
&= \iint_{\mathcal{S}_2} (2-z) \; dS \\
&= \iint_{\mathcal{S}_2} (2-(-1)) \; dS \\
&= 3 \cdot \text{Area}(\mathcal{S}_2) \\
&= 3 \cdot \pi \cdot 3^2 \\
&= 27\pi.
\end{aligned}$$

So,

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} + \iint_{\mathcal{S}_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{\mathcal{S}_2} \mathbf{F} \cdot d\mathbf{S} &= 0 \\
\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} + 18\pi + 27\pi &= 0 \\
\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} &= \boxed{-45\pi}.
\end{aligned}$$

8. (a) True. If $\mathbf{v} \cdot \mathbf{w} = 0$, then $\|\mathbf{v}\|\|\mathbf{w}\| \cos\theta = 0$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. Hence $\cos\theta = 0$ or $\|\mathbf{v}\| = 0$ or $\|\mathbf{w}\| = 0$.

   If $\mathbf{v} \times \mathbf{w} = \mathbf{0}$, then $\|\mathbf{v} \times \mathbf{w}\| = 0$, so $\|\mathbf{v}\|\|\mathbf{w}\| \sin\theta = 0$. Hence $\sin\theta = 0$ or $\|\mathbf{v}\| = 0$ or $\|\mathbf{w}\| = 0$.

   We cannot have both $\cos\theta = 0$ and $\sin\theta = 0$, so it must be the case that either $\|\mathbf{v}\| = 0$ or $\|\mathbf{w}\| = 0$, i.e., $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$.

   (b) False. We can modify the vortex field $\left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ to obtain a vector field $\mathbf{F} = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right\rangle$. We have

$$\operatorname{div} \mathbf{F} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} + 0 = 0$$

   but the line integral of $\mathbf{F}$ around the unit circle in the xy-plane is 2π, so $\mathbf{F}$ is not path independent and thus not conservative. That is, there does not exist f such that $\mathbf{F} = \nabla f$.

   (c) True. If $\mathbf{F}$ has zero curl (i.e., satisfies the cross-partial condition) and is defined on a simply-connected domain, such as $\mathbb{R}^3$ without the origin, then $\mathbf{F}$ is conservative. That is, there exists f such that $\mathbf{F} = \nabla f$.

   (d) True. The divergence of the curl of a vector field is always zero.

   (e) True. The curl of the gradient of a function is always zero.

   (f) False. Consider $f(x, y, z) = x^2 + y^2 + z^2$. Then $\nabla f = \langle 2x, 2y, 2z \rangle$ and $\operatorname{div}(\nabla f) = 2 + 2 + 2 = 6 \neq 0$.