Math 1920, Final Exam
December 13, 2013, 9:00 AM to 11:30 AM

You are NOT allowed to use calculators, cell phones, electronic devices, the text or any other book or notes (except the sheet mentioned below). SHOW ALL WORK!
Writing clearly and legibly will improve your chances of receiving the maximum credit that your solution deserves. Please label the questions in your answer booklet clearly. There are ***7 questions****. You may use one double-sided* $8\frac{1}{2}$*"-by-11" sheet in your own handwriting. Write your name and section number on each booklet you use. You may leave when you have finished, but if you have not handed in your exam booklet and left the room by 11:15 AM, please remain in your seat so as not to disturb others who are still working.*

1. (21 points) Let $f_1(x,y,z) = x^2 - y^2 + z^2$, and $f_2(x,y,z) = xy$. What is the angle between the tangent planes for the level surfaces of f_1 and f_2 at the point $(\pi, e, \sqrt{2})$?

   The tangent planes at level surfaces have normals given by the gradient, which we can compute as
   $$\nabla f_1 = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 2x, -2y, 2z \rangle$$
   and
   $$\nabla f_2 = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle y, x, 0 \rangle.$$
   Recall that the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$ is $\cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$ and that the angle between two planes is the angle between their normal vectors. Thus the angle between the two planes at any point (x,y,z) is
   $$\cos^{-1}\left(\frac{\langle 2x, -2y, 2z\rangle \cdot \langle y, x, 0\rangle}{|\langle 2x, -2y, 2z\rangle||\langle y, x, 0\rangle|}\right) = \cos^{-1}(0) = \boxed{\frac{\pi}{2}}.$$
   Thus the two planes are always perpendicular, and so must be perpendicular at the point $(\pi, e, \sqrt{2})$.

2. (21 points) Let $\mathcal{W}$ be the region inside the unit sphere between the cones $z = \sqrt{x^2+y^2}$ and $z = \sqrt{3(x^2+y^2)}$. Compute
   $$\iiint_{\mathcal{W}} \frac{1}{x^2+y^2+z^2} dV.$$
   We will use spherical coordinates. The two cones are given by equations which can be converted using $z = \rho\cos(\phi)$ and $x^2+y^2 = (\rho\cos(\theta)\sin(\phi))^2 + (\rho\sin(\theta)\sin(\phi))^2 = \rho^2\sin^2(\phi)$ to get
   $$z = \sqrt{x^2+y^2} \quad \Rightarrow \quad \rho\cos(\phi) = \sqrt{\rho^2\sin^2(\phi)} \quad \Rightarrow \quad 1 = \tan(\phi) \quad \Rightarrow \quad \phi = \frac{\pi}{4}$$
   and
   $$z = \sqrt{3(x^2+y^2)} \quad \Rightarrow \quad \rho\cos(\phi) = \sqrt{3\rho^2\sin^2(\phi)} \quad \Rightarrow \quad \frac{1}{\sqrt{3}} = \tan(\phi) \quad \Rightarrow \quad \phi = \frac{\pi}{6}$$

where we used the fact that ρ and $\sin(\phi)$ are always non-negative to remove them from the square root. In addition, we divided by ρ, where $\rho = 0$ is a solution of the equation, so note that this solution is still contained in the final spherical equations. The cones and bounding unit sphere are radially symmetric, so the integral becomes

$$\begin{aligned}
\iiint_{\mathcal{W}} \frac{1}{x^2+y^2+z^2}\, dV &= \int_0^{2\pi} \int_{\pi/6}^{\pi/4} \int_0^1 \frac{1}{\rho^2} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta \\
&= \left(\int_0^{2\pi} d\theta \right) \left(\int_{\pi/6}^{\pi/4} \sin(\phi)\, d\phi \right) \left(\int_0^1 d\rho \right) \\
&= (2\pi) \left[-\cos(\phi) \right]_{\pi/6}^{\pi/4} (1) \\
&= 2\pi \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \right) \\
&= \boxed{\pi(\sqrt{3} - \sqrt{2})}.
\end{aligned}$$

3. (21 points) Consider the function $f(x, y) = x^3/3 - xy + y^2/2$ defined on all of space.

   (a) Find all the local maximum and minimum points, and saddle points for f.

   Local minimums, local maximums, and saddles all occur where the partials are simultaneously 0 or undefined. Thus we first need to compute the partials

$$f_x(x, y) = x^2 - y$$

   and

$$f_y(x, y) = y - x.$$

   These partials are defined everywhere, so we must find where they both equal 0. If $f_y(x, y) = 0$ then $x = y$, and substituting this into the other partial gives that both partials are zero when $x^2 - x = x(x - 1)$ is zero, which is exactly $x = 0, 1$. Thus the critical points are $(0, 0)$ and $(1, 1)$.

   To determine if these points are mins, maxes, or saddles we first compute the second order partials

$$\begin{aligned}
f_{xx}(x, y) &= 2x \\
f_{yy}(x, y) &= 1 \\
f_{xy}(x, y) = f_{yx}(x, y) &= -1
\end{aligned}$$

   and compute the determinants

$$D_{(0,0)} = f_{xx}(0,0) f_{yy}(0,0) - f_{xy}(0,0) f_{yx}(0,0) = (0)(1) - (-1)(-1) = -1$$

$$D_{(1,1)} = f_{xx}(1,1) f_{yy}(1,1) - f_{xy}(1,1) f_{yx}(1,1) = (2)(1) - (-1)(-1) = 1$$

   so since $D_{(0,0)} < 0$ and $D_{(1,1)} > 1$ with $f_{xx}(1, 1) > 0$, the second derivative test gives that $(0, 0)$ is a saddle point and $(1, 1)$ is a local minimum.

(b) Find the global maximum and minimum points when f is restricted to the square $0 \le x \le 1,\ 0 \le y \le 1$.

Global maximums and minimums can occur at critical points or on the boundary. In part (a) we already found the critical points $(0,0)$ and $(1,1)$, so now we consider the boundary.

The boundary consists of four sides. Restricting to these sides we get

Edge	Restriction of $f(x,y)$	derivative	critical points
$x=0$	$y^2/2$	y	$(0,0)$
$x=1$	$y^2/2-y+1/3$	$y-1$	$(1,1)$
$y=0$	$x^3/3$	x^2	$(0,0)$
$y=1$	$x^3/3-x+1/2$	x^2-1	$(1,1)$

and we must also consider the corners $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$. This means the total set of points under consideration is $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$. Evaluating f at these points gives

$$f(0,0)=0$$

$$f(0,1)=\frac{1}{2}$$

$$f(1,0)=\frac{1}{3}$$

$$f(1,1)=\frac{1}{3}-1+\frac{1}{2}=-\frac{1}{6}$$

and thus f has global maximum of $\frac{1}{2}$ at $(0,1)$ and global minimum of $-\frac{1}{6}$ at $(1,1)$.

4. (24 points) Let $\mathbf{F}=\langle x+y, x, -z\rangle$, a vector field defined everywhere. Let $\mathbf{G}=\frac{1}{r^2}\mathbf{F}$, where $r^2=x^2+y^2+z^2$. Note that $\mathbf{G}$ is defined everywhere except the origin.

(a) Calculate the divergence and curl of $\mathbf{F}$.

Applying definitions, we compute

$$\operatorname{div}(\mathbf{F})=\frac{\partial}{\partial x}(x+y)+\frac{\partial}{\partial y}(x)+\frac{\partial}{\partial z}(-z)=1+0-1=\boxed{0}$$

and

$$\operatorname{curl}(\mathbf{F})=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & x & -z \end{vmatrix}=\boxed{\langle 0,0,0\rangle}.$$

(b) Calculate the outward flux of $\mathbf{F}$ across the sphere $x^2+y^2+z^2=4$.

Let $\mathcal{S}$ be the sphere and $\mathcal{W}$ the region bounded by $\mathcal{S}$. Then the Divergence Theorem gives that

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}=\iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\,dV=\iiint_{\mathcal{W}} 0\,dV=\boxed{0}.$$

(c) Find a potential function for $\mathbf{F}$, if it exists.

Integrating each component yields that a potential function $f(x, y, z)$ would have the form

$$\begin{aligned} f(x, y, z) &= g_1(y, z) + \int (x + y)\, dx = \frac{1}{2}x^2 + xy + g_1(y, z) \\ &= g_2(x, z) + \int x\, dy = xy + g_2(x, z) \\ &= g_3(x, y) + \int -z\, dz = -\frac{1}{2}z^2 + g_3(x, y) \end{aligned}$$

so that we can combine to find a valid potential function $\boxed{f(x, y, z) = \frac{1}{2}x^2 + xy - \frac{1}{2}z^2}$.

(d) Calculate the work done along the path $\mathbf{c}(t) = (2\sin t \cos t, 2\sin^2 t, 2\cos t)$, from $t = 0$ to $t = \pi/2$ for the vector field $\mathbf{F}$.

Let $\mathcal{C}$ denote the given path. The path starts at

$$\mathbf{c}(0) = (2\sin(0)\cos(0), 2\sin^2(0), 2\cos(0)) = (0, 0, 2)$$

and ends at

$$\mathbf{c}(\pi/2) = (2\sin(\pi/2)\cos(\pi/2), 2\sin^2(\pi/2), 2\cos(\pi/2)) = (0, 2, 0),$$

so we can use the potential from part (c) to compute

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= f(0, 2, 0) - f(0, 0, 2) \\ &= \left(\frac{1}{2}(0)^2 + (0)(2) - \frac{1}{2}(0)^2\right) - \left(\frac{1}{2}(0)^2 + (0)(0) - \frac{1}{2}(2)^2\right) \\ &= \boxed{2}. \end{aligned}$$

(e) The divergence and curl of $\mathbf{G}$ are both non-zero and are fairly complicated. Calculate the outward flux of $\mathbf{G}$ across the sphere $x^2 + y^2 + z^2 = 4$. (*Hint*: Notice how $\mathbf{F}$ and $\mathbf{G}$ are related.)

Let $\mathcal{S}$ denote the sphere. We cannot use the Divergence Theorem for $\mathbf{G}$, as it is not defined at the origin. However, notice that on the sphere $x^2 + y^2 + z^2 = 4$ we have constant $r = 2$. Thus the integral can be related to the integral from part (b) and solved as

$$\iint_{\mathcal{S}} \mathbf{G} \cdot d\mathbf{S} = \iint_{\mathcal{S}} \frac{1}{4}\mathbf{F} \cdot d\mathbf{S} = \frac{1}{4}\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \frac{1}{4}(0) = \boxed{0}.$$

(f) Calculate the work done along the path $\mathbf{c}(t) = (2\sin t \cos t, 2\sin^2 t, 2\cos t)$, from $t = 0$ to $t = \pi/2$ for the vector field $\mathbf{G}$. (*Hint*: Notice how $\mathbf{F}$ and $\mathbf{G}$ are related.)

Let $\mathcal{C}$ denote the given path. Since

$$\begin{aligned}(2\sin(t)\cos(t))^2 + (2\sin^2(t))^2 + (2\cos(t))^2 &= 4\sin^2(t)\cos^2(t) + 4\sin^4(t) + 4\cos^2(t)\\ &= 4\sin^2(t)\left(\cos^2(t)+\sin^2(t)\right) + 4\cos^2(t)\\ &= 4\sin^2(t) + 4\cos^2(t)\\ &= 4\end{aligned}$$

we have that r has constant value $r = 2$ along the path, so the integral can be written as

$$\int_{\mathcal{C}} \mathbf{G}\cdot d\mathbf{r} = \int_{\mathcal{C}} \frac{1}{4}\mathbf{F}\cdot d\mathbf{r} = \frac{1}{4}\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \frac{1}{4}(2) = \boxed{\frac{1}{2}}.$$

5. (21 points) Let $\mathcal{S}$ be the half-cylinder $x^2 + y^2 = 1$, $x \geq 0$, $0 \leq z \leq 1$. Assume that the z component of $\mathbf{F}$ is zero and that $\mathbf{F}(0, y, z) = \langle y^2 + z^2, 0, 0\rangle$. Let $\mathcal{W}$ be the *solid* half-cylinder $x^2 + y^2 \leq 1, x \geq 0$, $0 \leq z \leq 1$, and assume that

$$\iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})dV = 2.$$

Find the outward flux of $\mathbf{F}$ through $\mathcal{S}$.

The region $\mathcal{W}$ is bounded by four surfaces: the half-cylinder $\mathcal{S}$, the plane $x = 0$, the plane $z = 0$, and the plane $z = 1$. Let $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ be the portions of $x = 0$, $z = 0$, $z = 1$ respectively so that $\mathcal{S} + \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3$ is the boundary of $\mathcal{W}$. Note that as portions of their planes, outward pointing unit normals to $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ are $\mathbf{n}_1 = \langle -1, 0, 0\rangle$, $\mathbf{n}_2 = \langle 0, 0, -1\rangle$, and $\mathbf{n}_3 = \langle 0, 0, 1\rangle$ respectively. Then the Divergence Theorem and additivity of integrals gives

$$\begin{aligned}\iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\,dV &= \iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} + \iint_{\mathcal{T}_1} \mathbf{F}\cdot d\mathbf{S} + \iint_{\mathcal{T}_2} \mathbf{F}\cdot d\mathbf{S} + \iint_{\mathcal{T}_3} \mathbf{F}\cdot d\mathbf{S}\\ &= \iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} + \iint_{\mathcal{T}_1} \mathbf{F}\cdot \mathbf{n}_1\,dS + \iint_{\mathcal{T}_2} \mathbf{F}\cdot \mathbf{n_2}\,dS + \iint_{\mathcal{T}_3} \mathbf{F}\cdot \mathbf{n_3}\,dS\\ &= \iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} + \iint_{\mathcal{T}_1} \mathbf{F}\cdot \mathbf{n}_1\,dS + \iint_{\mathcal{T}_2} 0\,dS + \iint_{\mathcal{T}_3} 0\,dS\end{aligned}$$

since we are given that $\mathbf{F}$ has z-component 0. Continuing,

$$\begin{aligned}&= \iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} + \iint_{\mathcal{T}_1} \langle y^2 + z^2, 0, 0\rangle \cdot \langle -1, 0, 0\rangle\,dS\\ &= \iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} - \iint_{\mathcal{T}_1} (y^2 + z^2)\,dS\end{aligned}$$

since $\mathbf{F} = \langle y^2 + z^2, 0, 0 \rangle$ on $\mathcal{T}_1$. Since $\mathcal{T}_1$ is just a rectangle in the yz-plane this then becomes

$$\begin{aligned}
&= \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} - \int_0^1 \int_{-1}^1 (y^2 + z^2)\, dy\, dz \\
&= \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} - \int_0^1 \left[\frac{1}{3}y^3 + yz^2\right]_{-1}^1 dz \\
&= \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} - \int_0^1 \left(\frac{2}{3} + 2z^2\right) dz \\
&= \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} - \left[\frac{2}{3}z + \frac{2}{3}z^3\right]_0^1 \\
&= \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} - \frac{4}{3}.
\end{aligned}$$

Given that $\iiint_{\mathcal{W}} \operatorname{div}(\mathbf{F})\, dV = 2$ then allows us to finally substitute and solve

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = 2 + \frac{4}{3} = \boxed{\frac{10}{3}}.$$

6. (21 points) Let $\mathbf{F} = \langle z^2 - y^2, x^2 - z^2, x^2 + y^2 \rangle$. Consider the hyperboloid $g(x, y, z) = x^2 + y^2 - z^2 = 1$.

(a) Show that at every point on the hyperboloid, curl($\mathbf{F}$) is tangent to the surface.

First compute

$$\operatorname{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 - y^2 & x^2 - z^2 & x^2 + y^2 \end{vmatrix} = \langle 2y + 2z, 2z - 2x, 2x + 2y \rangle.$$

This vector will be tangent to the surface if it is perpendicular to the surface normal. A vector normal to the surface of the hyperboloid can be found as

$$\nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle = \langle 2x, 2y, -2z \rangle.$$

The dot product of these two vectors is

$$\begin{aligned}
\operatorname{curl}(\mathbf{F}) \cdot \nabla g &= \langle 2y + 2z, 2z - 2x, 2x + 2y \rangle \cdot \langle 2x, 2y, -2z \rangle \\
&= 4xy + 4xz + 4yz - 4xy - 4xz - 4yz \\
&= 0
\end{aligned}$$

and thus curl($\mathbf{F}$) and ∇g are perpendicular at every point, which means curl($\mathbf{F}$) is tangent to the hyperboloid at every point.

(b) Let $\mathcal{S}$ be any (bounded) region on the hyperboloid, and let $\mathcal{C}$ be the boundary curve of the surface $\mathcal{S}$. Prove that the circulation $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = 0$.

Applying Stokes' Theorem, we have

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) \cdot d\mathbf{S} = \iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) \cdot \mathbf{n}\, dS = \iint_{\mathcal{S}} 0\, dS = 0$$

where we use the fact from part (a) that the curl is always tangent to the surface to conclude that $\mathrm{curl}(\mathbf{F}) \cdot \mathbf{n} = 0$.