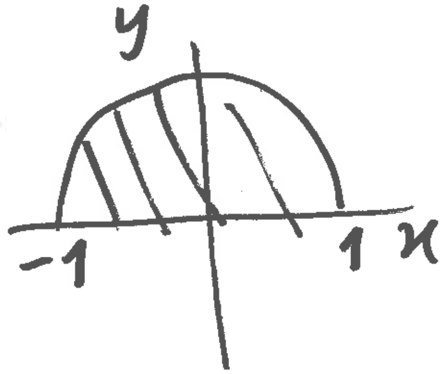
Math 1920 Fall 2016 Final Exam

Summary solutions

1.



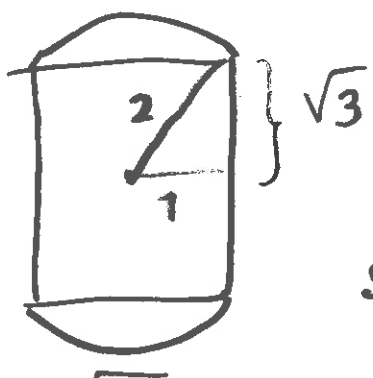
$$\begin{cases} -1 \le x \le 1 \\ 0 \le y \le \sqrt{1-x^2} \end{cases}$$

$$\begin{cases} 0 \le \theta \le \pi \\ 0 \le r \le 1 \end{cases}$$

$$\text{Integral} = \int_0^{\pi}\int_0^1 e^{-r^2}\, r\,dr\,d\theta$$

$$= \dots = \pi\left(1-\frac{1}{e}\right)$$

2. Vertical cross-section of solid:



cylinder has volume $2\pi\sqrt{3}$.

spherical cap has volume

$$\int_0^{2\pi}\int_0^1\int_{\sqrt{3}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \dots = 2\pi\left(\frac{8}{3}-\frac{3}{2}\sqrt{3}\right)$$

$$\text{Total} = 2\pi\sqrt{3} + 4\pi\left(\frac{8}{3}-\frac{3}{2}\sqrt{3}\right) = \frac{32\pi}{3} - 4\pi\sqrt{3}.$$

3. Let $f(x,y) = 2 \times \text{area} = (x+1)y = xy + y$.
Constraint: $g(x,y) = x^2 + y^2 = 1$.
Lagrange mult. $\nabla f = \lambda \nabla g$ gives
$x = \frac{1}{2}$, $y = \frac{1}{2}\sqrt{3}$.

4. $\frac{\partial f}{\partial x} = \frac{x}{x^2+y^2}$ gives $f(x,y) = \frac{1}{2}\ln(x^2+y^2) + g(y)$.
Then $f_y = \frac{y}{x^2+y^2} + g' = \frac{y}{x^2+y^2}$ so $g' = 0$, i.e.
$f = \frac{1}{2}\ln(x^2+y^2) + \text{const}$.
$f = \frac{1}{2}\ln(x^2+y^2)$ will do.
$\vec{F}$ incompressible:
$$\frac{\partial F_1}{\partial x} = \frac{-x^2+y^2}{(x^2+y^2)^2}, \quad \frac{\partial F_2}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2}$$
so $\mathrm{div}(\vec{F}) = 0$.

5. Denote box by W. By Gauss:
$$\iint_{\partial W} \vec{F} \cdot d\vec{S} = \iiint_W \mathrm{div}(\vec{F})\, dV.$$
$\mathrm{div}(\vec{F}) = 2x + 2y + 2z$

③

$\iiint_W 2x\,dV = a\,\mathrm{vol}(W)$

$\iiint_W 2y\,dV = b\,\mathrm{vol}(W)$

$\iiint_W 2z\,dV = c\,\mathrm{vol}(W)$

So flux $= (a+b+c)\,abc$.

6. (a) Parametrization of S:

$G(r,\theta) = (r\cos\theta, r\sin\theta, 2(1-r))$.

$0 \le r \le 1, \quad 0 \le \theta \le \frac{\pi}{2}$.

$$\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & -2 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle 2r\cos\theta, 2r\sin\theta, r\rangle.$$

$\vec{F}\cdot\vec{N} = \dots = 2r$.

$$\text{Flux} = \int_0^{\frac{\pi}{2}}\int_0^1 2r\,dr = \frac{\pi}{2}.$$

(b) Let W be the cone. Gauss:

$$\iint_{\partial W} \vec{F}\cdot d\vec{S} = \iiint_W \mathrm{div}(\vec{F})\,dV.$$

$\operatorname{div}(\vec{F}) = 3$, so $\iiint_{W} \operatorname{div}(\vec{F})\, dV = 3\,\mathrm{vol}(W)$

$= 3\left(\frac{1}{3}\right)\frac{\pi h}{4} = \frac{\pi}{2}.$

$\partial W = S \cup S_1 \cup S_2 \cup S_3,$

where $\begin{cases} S_1 = \text{intersection of } W \text{ with } yz\text{-plane} \\ S_2 = \quad " \quad " \quad " \quad " \quad xz\text{-} \quad " \\ S_3 = \quad " \quad " \quad " \quad " \quad xy\text{-} \quad " \end{cases}$

Since $\vec{F}$ is tangent to coordinate planes, flux of $\vec{F}$ through S_1, S_2, S_3 is 0.

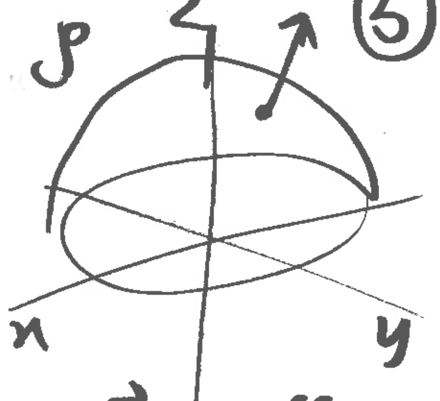
So $\iint_S \vec{F}\cdot d\vec{S} = \iint_{\partial W} \vec{F}\cdot d\vec{S} = \frac{\pi}{2}.$

7.(a) $\iint_S \operatorname{curl}(\vec{F})\cdot d\vec{S} = \int_{\partial S} \vec{F}\cdot d\vec{r}$

$\iiint_W \operatorname{div}(\vec{F})\, dV = \iint_{\partial W} \vec{F}\cdot d\vec{S}$

$(\partial W = S')$

(b)



$$\iint_S \vec{B}\cdot d\vec{S} = \iint_S \operatorname{curl}(\vec{A})\cdot d\vec{S} = \int_{\partial S} \vec{A}\cdot d\vec{r}.$$

$\partial S = \mathcal{C} =$ unit circle

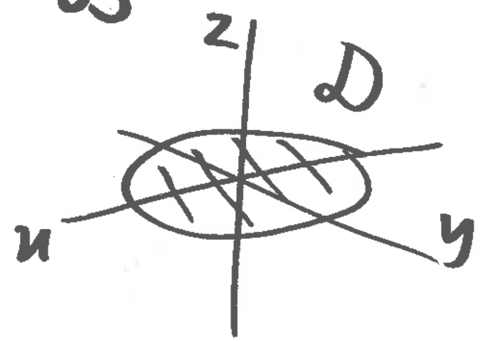
$\vec{r}(t) = \langle \cos t, \sin t, 0\rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 0\rangle$

$$\vec{A}(\vec{r}(t))\cdot \vec{r}\,'(t) = \langle \sin t, -\cos t, \ldots\rangle \cdot \langle -\sin t, \cos t, 0\rangle$$
$$= 1,$$

so $\int_{\partial S} \vec{A}\cdot d\vec{r} = \int_0^{2\pi} 1\, dt = 2\pi.$

(c)



$\iint_{S'} \vec{B}\cdot d\vec{S} = \iiint \operatorname{div}(\vec{B})\, dV = 0.$ Also

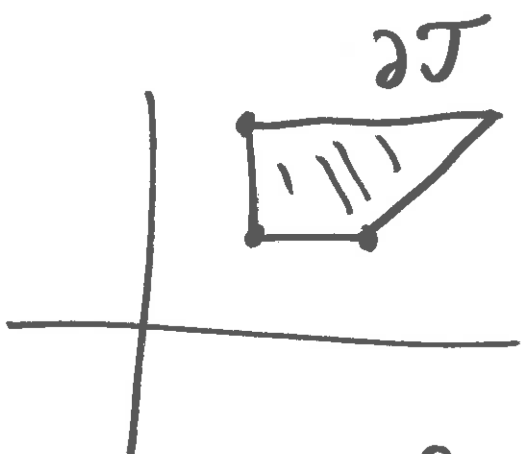
$\iint_{S'} \vec{B}\cdot d\vec{S} = \iint_D \vec{B}\cdot d\vec{S} + \iint_S \vec{B}\cdot d\vec{S}$, so

$$\iint_{S} \vec{B}\cdot d\vec{S} = -\iint_{D} \vec{B}\cdot d\vec{S}.$$

Now $\iint_D \vec{B}\cdot d\vec{S} = \iint_D \vec{B}\cdot\vec{k}\, dA = \iint_D (-2)\,dA$

$= -2\pi$, so $\iint_S \vec{B}\cdot d\vec{S} = 2\pi$.

8.



$$\int_{\partial T} \vec{F}\cdot d\vec{r} = \iint_T \mathrm{curl}_z(\vec{F})\,dA = \iint_T (1-(-1))\,dA$$

$= 2\,\mathrm{area}(T) = 2\left(\frac{3}{2}\right) = 3.$