Math 1920, Prelim 2
November 8, 2016, 7:30 PM to 9:00 PM

Problem 1. (20 points) Let $f(x,y) = x^2 + 2xy - y^2$ be defined in the domain $\mathcal{D} = \{(x,y) \mid x^2 + y^2 \le 1\}$.

(a) Find all the critical points f in the interior of $\mathcal{D}$ and determine whether they are relative maxima, minima, saddle points, or something else.

(b) In all of $\mathcal{D}$ what are the points where the maxima and minima are achieved?

Solution:

(a) Calculate the gradient $\nabla f = \langle 2x + 2y, 2x - 2y\rangle$. Then $\nabla f = 0$ only when $x = y = 0$. The discriminant at the only critical point $(0,0)$ is

$$D = f_{xx}f_{yy} - f_{xy}^2 = 2 \cdot (-2) - 2^2 = -8 < 0,$$

so the critical point is a saddle point and does not correspond to a local maximum or minimum.

(b) Since f is continuous and $\mathcal{D}$ is closed and bounded, all the maxima and minima must be on the boundary of $\mathcal{D}$, the unit circle. So we look for the maxima and minima of

$$f(\theta) = \cos^2\theta + 2\cos\theta\sin\theta - \sin^2\theta = \frac{1}{2}(1+\cos 2\theta) + \sin 2\theta - \frac{1}{2}(1-\cos 2\theta) = \cos 2\theta + \sin 2\theta.$$

So the critical points of $f(\theta)$ occur when $f'(\theta) = -2\sin 2\theta + 2\cos 2\theta = 0$, and this is when $\tan 2\theta = 1$. This is when $2\theta = \pi/4$ or $5\pi/4$. So $\theta = \pi/8, 5\pi/8, 9\pi/8, 13\pi/8$ represent the critical points as maximums and minimums alternately. $f(x,y) = f(-x,-y)$ implies that antipodal points on the circle have the same value, so the max's and min's occur in antipodal pairs.

Alternatively, one can use Lagrange multipliers. The circle is defined by $g(x,y) = x^2 + y^2 = 1$, and $\nabla g = \langle 2x, 2y\rangle$. So the critical points are where $\lambda\nabla g = \nabla f$, or $\lambda\langle 2x, 2y\rangle = \langle 2x+2y, 2x-2y\rangle$, and this means $2x(2x-2y) = 2y(2x+2y)$. Simplifying and using that the points lie on the unit circle, we need to solve:

$$\begin{aligned} x^2 - xy &= yx + y^2 \\ x^2 + y^2 &= 1. \end{aligned}$$

Substituting for y^2 in the first equation, we get $y = \frac{2x^2-1}{2x}$, and $x^2 + (\frac{2x^2-1}{2x})^2 = 1$ simplifies to $8x^4 - 8x^2 + 1 = 0$. Using the quadratic formula, $x = \pm\sqrt{2 \pm \sqrt{2}}/2$, and of course $y = \frac{2x^2-1}{2x}$.

Problem 2. (15 points) Let $\mathbf{F} = \langle y, z, x \rangle$ be a vector field, and $\mathbf{r}(t) = \langle \cos t, \sin t, c \rangle$, where $0 \leq t \leq 2\pi$ and c is a constant, determine a curve $\mathcal{C}$. Compute the line integral $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$. Does it depend on the value of c?

Solution: $\mathbf{F} = \langle \sin t, c, \cos t \rangle$, and $d\mathbf{r}(t) = \langle -\sin t, \cos t, 0 \rangle dt$. So,

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-\sin^2 t + c \cos t) dt = \int_0^{2\pi} (-\frac{1}{2}(1 - \cos 2t) + c \cos t) dt = -\pi.$$

No, the integral does not depend on the constant c.

Problem 3. (15 points) Find the point on the sphere $x^2 + y^2 + z^2 = 1$ which is the closest to the plane $3x + 4y + 12z = 26$.

Solution: The nearest point on the sphere is along the line from the center of the sphere, the origin, in the direction of the gradient to the plane which is $\langle 3, 4, 12\rangle$. The norm of $\langle 3, 4, 12\rangle$ is $\sqrt{3^2 + 4^2 + 12^2} = \sqrt{169} = 13$. So the point on the sphere is $\pm\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\rangle$. But this is $\langle \frac{3}{13}, \frac{4}{13}, \frac{12}{13}\rangle$, since everything happens in the first octant.

Alternatively, one can use Lagrange multipliers. The sphere has a gradient $2\langle x, y, z\rangle$, and the plane has gradient $\langle 3, 4, 12\rangle$. So you get the same answer.

Problem 4. (15 points) For what value of b is the vector field $\mathbf{F} = yz\mathbf{i} + (xz + z)\mathbf{j} + (xy + by + 1)\mathbf{k}$ conservative? In this case, determine a potential function for $\mathbf{F}$.

Solution: The curl of $\mathbf{F}$ is $\nabla \times \mathbf{F} = \langle x + b - x - 1, -y + y, z - z \rangle = \langle b - 1, 0, 0 \rangle$. So we must have $b = 1$ in order for $\mathbf{F}$ to be conservative. When $b = 1$, then $f = xyz + yz + z$ is easily seen to be the potential function, up to an additive constant, since $\nabla f = \mathbf{F}$.

Problem 5. (15 points) Given the vector field $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j}$, calculate the upward flux of F through the semi-circle $x^2 + y^2 = 1$, $y \geq 0$.

Solution: Parametrize the circle by $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \leq t \leq \pi$, so $\mathbf{F} = \langle \sin^2 t, \cos^2 t \rangle$, and the unit normal vector at $\mathbf{r}(t)$ is $\mathbf{n}(t) = \mathbf{r}(t)$. Then the upward flux is $\int_0^\pi \mathbf{F} \cdot \mathbf{n}\, dt = \int_0^\pi (\cos t \sin^2 t + \sin t \cos^2 t) dt = -\cos^3 t/3 + \sin^3 t/3]_0^\pi = 2/3$.

Problem 6. (20 points) Let $\mathcal{R}$ be the region inside the cone $z = (x^2 + y^2)^{1/2}$ and outside the sphere $x^2 + y^2 + z^2 = 1$ and bounded above by the plane $z = 2$. Set up the integral $\iiint_{\mathcal{R}} \frac{z}{(x^2+y^2+z^2)^{1/2}} dV$ in terms of spherical coordinates and cylindrical coordinates. You may leave your answer as an integral without evaluating it. But be sure to explicitly put in the limits of the integral.

Solution: Using spherical coordinates we get

$$\int_0^{2\pi} \int_0^{\pi/4} \int_1^{2/\cos\phi} \rho^2 \cos(\phi) \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

In cylindrical coordinates we split into two regions based on r and sum together to get

$$\int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \int_{\sqrt{1-r^2}}^{2} \frac{zr}{\sqrt{r^2+z^2}}\, dz\, dr\, d\theta + \int_0^{2\pi} \int_{\frac{1}{\sqrt{2}}}^{2} \int_r^{2} \frac{zr}{\sqrt{r^2+z^2}}\, dz\, dr\, d\theta.$$