Math 1920 *Solutions*

Prelim 2 7 November 2017 7:30–9:00 pm

1 (15 points). *Find the global minima and maxima of the function $f(x,y,z) = -x + 2y + z$ over the solid ellipsoid given by $x^2 + y^2 + \frac{1}{4}z^2 \le 1$.*

$\nabla f = \langle -1, 2, 1\rangle \neq \mathbf{0}$, so f has no critical points on the interior of the ellipsoid. On the boundary given by $g(x,y,z) = x^2 + y^2 + \frac{1}{4}z^2 = 1$ we use the Lagrange multiplier method: $\nabla g = \langle 2x, 2y, \frac{1}{2}z\rangle$, so setting $\nabla f = \lambda \nabla g$ gives

$$-1 = 2\lambda x, \qquad 2 = 2\lambda y, \qquad 1 = \frac{1}{2}\lambda z.$$

Hence $\lambda = -1/2x$, which yields $y = 1/\lambda = -2x$ and $z = 2/\lambda = -4x$. We look for a solution (x,y,z) on the boundary, so $x^2 + (-2x)^2 + \frac{1}{4}(-4x)^2 = 1$, i.e. $9x^2 = 1$, or $x = \pm\frac{1}{3}$. This gives $y = -2x = \mp\frac{2}{3}$ and $z = -4x = \mp\frac{4}{3}$. Therefore local extrema of f can only occur at $P_1 = (\frac{1}{3}, -\frac{2}{3}, -\frac{4}{3})$ or at $P_2 = (-\frac{1}{3}, \frac{2}{3}, \frac{4}{3})$. We have $f(P_1) = -3$ and $f(P_2) = 3$. The ellipsoid is closed and bounded, so f has at least one global maximum and minimum. Therefore the global minimum is attained at P_1 and the global maximum at P_2.

2 (15 points). *Let $\mathscr{W}$ be the solid enclosed between the surface $z = \sqrt{x^2+y^2}$ and the plane $z = h$, where h is a positive constant.*

(a) *Find the volume of $\mathscr{W}$.*

Inequalities for $\mathscr{W}$ in cylindrical coordinates are $0 \le \theta \le 2\pi$, $0 \le r \le h$, $r \le z \le h$. So

$$\begin{aligned}\operatorname{vol}(\mathscr{W}) &= \iiint_{\mathscr{W}} 1\,dV = \int_0^{2\pi}\int_0^h\int_r^h r\,dz\,dr\,d\theta = 2\pi\int_0^h [rz]_{z=r}^h\,dr \\ &= 2\pi\int_0^h (rh - r^2)\,dr = 2\pi\left[\frac{1}{2}r^2h - \frac{1}{3}r^3\right]_{r=0}^h = 2\pi\frac{1}{6}h^3 = \frac{1}{3}\pi h^3.\end{aligned}$$

(b) *Find the centroid of $\mathscr{W}$. (Recall that the centroid is the center of mass with respect to a constant density function equal to 1.) You may use the symmetries of $\mathscr{W}$ to shorten your calculations.*

By rotational symmetry centroid is $(\bar{x}, \bar{y}, \bar{z})$ with $\bar{x} = \bar{y} = 0$. To find $\bar{z}$ integrate z:

$$\begin{aligned}\iiint_{\mathscr{W}} z\,dV &= \int_0^{2\pi}\int_0^h\int_r^h rz\,dz\,dr\,d\theta = 2\pi\int_0^h\left[\frac{1}{2}rz^2\right]_{z=r}^h dr \\ &= \pi\int_0^h (rh^2 - r^3)\,dr = \pi\left[\frac{1}{2}r^2h^2 - \frac{1}{4}r^4\right]_{r=0}^h = \pi\frac{1}{4}h^4 = \frac{1}{4}\pi h^4.\end{aligned}$$

Hence $\bar{z} = \frac{1}{4}\pi h^4/\operatorname{vol}(\mathscr{W}) = \frac{1}{4}\pi h^4/(\frac{1}{3}\pi h^3) = \frac{3}{4}h$. Centroid is $(0, 0, \frac{3}{4}h)$.

(The answers will depend on h.)

3 (15 points). *Find the work done by the force $\mathbf{F}(x,y,z) = \langle yz, xz, xy\rangle$ along the path $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$ for $0 \le t \le T$. (The answer will depend on T.)*

$\mathbf{F}(\mathbf{r}(t)) = \langle t^5, t^4, t^3\rangle$ and $\mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$, so $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = t^5 + 2t^5 + 3t^5 = 6t^5$. Therefore

$$\text{work} = \int_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r} = \int_0^T \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^T 6t^5\,dt = T^6.$$

4 (15 points). *Let a be a constant and let $\mathbf{F}$ be the vector field*

$$\mathbf{F}(x, y) = \left\langle x \ln y + ye^x, \frac{1}{2}x^2 y^a + e^x + 1 \right\rangle.$$

The domain of $\mathbf{F}$ is the planar region $\mathscr{D}$ defined by $y > 0$.

(a) *Is $\mathscr{D}$ simply connected?*

Yes: $\mathscr{D}$ is an open half-plane, so every closed curve in $\mathscr{D}$ can be contracted within $\mathscr{D}$ to a point.

(b) *For what value of a is $\mathbf{F}$ conservative?*

Because $\mathscr{D}$ is simply connected, for $\mathbf{F}$ to be conservative it is necessary and sufficient that $\mathbf{F}$ satisfy the cross-partials condition $\partial F_1/\partial y = \partial F_2/\partial x$. We have

$$\frac{\partial F_1}{\partial y} = \frac{x}{y} + e^x, \qquad \frac{\partial F_2}{\partial x} = xy^a + e^x,$$

so the cross-partial condition holds if and only if $x/y = xy^a$ for all x and all $y > 0$. So $\mathbf{F}$ is conservative if and only if $a = -1$.

(c) *For the value of a you found in part* (b)*, find a potential of* $\mathbf{F}$.

Solve the system of partial differential equations

$$\frac{\partial f}{\partial x} = x \ln y + ye^x, \qquad \frac{\partial f}{\partial y} = \frac{1}{2}x^2 y^{-1} + e^x + 1$$

for f. The first equation yields $f = \frac{1}{2}x^2 \ln y + ye^x + k(y)$ for some function k of y. Then $f_y = \frac{1}{2}x^2 y^{-1} + e^x + k'(y)$. Plugging this into the second equation gives $\frac{1}{2}x^2 y^{-1} + e^x + k'(y) = \frac{1}{2}x^2 y^{-1} + e^x + 1$, i.e. $k'(y) = 1$, or $k = y + c$, where c is constant. So a potential of $\mathbf{F}$ is $f(x, y) = \frac{1}{2}x^2 \ln y + ye^x + y$.

5 (20 points). *Let $f(x, y) = x^3 + y^3 - 3xy$.*

(a) *Find all critical points of f in the plane $\mathbf{R}^2$.*

Setting $\nabla f = \langle 3x^2 - 3y, 3y^2 - 3x \rangle$ equal to $\mathbf{0}$ gives $x^2 = y$ and $y^2 = x$. Plugging first equation into second gives equations $x^4 = x$, i.e. $x(x^3 - 1) = 0$, i.e. $x = 0$ or $x = 1$. Corresponding values of y are $y = 0$ and $y = 1$, so critical points are $(0, 0)$ and $(1, 1)$.

(b) *For each critical point state whether it is a local minimum, local maximum, or saddle point.*

Apply second derivative test. The discriminant of f is $D(x, y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -3 \\ -3 & 6y \end{vmatrix} = 36xy - 9$. So $D(0, 0) = -9 < 0$: $(0, 0)$ is a saddle point. And $D(1, 1) = 27 > 0$, while $f_{xx}(1, 1) = 6 > 0$, so $(1, 1)$ is a local minimum.

(c) *Does f have global extrema? Explain your answer.*

No. Along the x-axis we have $f(x, 0) = x^3$ which goes to $\pm\infty$ as $x \to \pm\infty$, so f has neither global maximum nor global minimum.

6 (20 points). *Let $\mathscr{W}$ be the solid enclosed by the two spheres $x^2+y^2+z^2=R^2$ and $x^2+y^2+(z-R)^2=R^2$, where R is a positive constant.*

(a) *Sketch the solid $\mathscr{W}$. Include sufficient detail, such as coordinate axes and some indication of scale.*

(b) *Set up the integral $\iiint_{\mathscr{W}} xyz\,dV$ in cylindrical coordinates with the variables in the order $dz\,dr\,d\theta$. Write the answer as an integral with explicit limits of integration. Do not attempt to evaluate the integral.*

$\mathscr{W}$ is obtained by spinning the region enclosed by the circles $x^2+(z-R)^2=R^2$ and $x^2+z^2=R^2$ in the xz-plane about the z-axis. The limits for θ are therefore $0 \le \theta \le 2\pi$. The intersection points of the two circles are $(\pm\frac{1}{2}\sqrt{3}R, \frac{1}{2}R)$. This gives the limits for r: $0 \le r \le \frac{1}{2}\sqrt{3}R$. Given r, z ranges between the lower portion of the top circle $z = R - \sqrt{R^2-r^2}$ and the upper portion of the bottom circle $z=\sqrt{R^2-r^2}$. The integrand is $xyz = r^2 z\cos\theta\sin\theta$ and $dV = r\,dz\,dr\,d\theta$, so

$$\iiint_{\mathscr{W}} xyz\,dV = \int_0^{2\pi}\int_0^{\frac{1}{2}\sqrt{3}R}\int_{R-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r^3 z\cos\theta\sin\theta\,dz\,dr\,d\theta.$$

(c) *In the integral of part* (b) *change the order of integration to $dr\,dz\,d\theta$. Do not attempt to evaluate the answer.*

The limits for z are $0 \le z \le R$. On the interval $0 \le z \le \frac{1}{2}R$ the upper limit for r is given by the upper circle $r = \sqrt{R^2-(z-R)^2} = \sqrt{z(2R-z)}$. On the interval $\frac{1}{2}R \le z \le R$ the upper limit for r is given by the lower circle $r=\sqrt{R^2-z^2}$. Therefore

$$\iiint_{\mathscr{W}} xyz\,dV = \int_0^{2\pi}\int_0^{\frac{1}{2}R}\int_0^{\sqrt{z(2R-z)}} r^3 z\cos\theta\sin\theta\,dr\,dz\,d\theta + \int_0^{2\pi}\int_{\frac{1}{2}R}^{R}\int_0^{\sqrt{R^2-z^2)}} r^3 z\cos\theta\sin\theta\,dr\,dz\,d\theta.$$