Math 1920 *Solutions*

Final exam 13 December 2018 9:00–11:30 am

1 (10 points). *Suppose a function $g(u, v)$ satisfies $g(0, 1) = 3$ and $\nabla g(0, 1) = \langle 2, 1\rangle$. Define a function $f(x, y, z)$ by $f(x, y, z) = g\big(xe^{yz}, y\cos(x + z)\big)$. Let $P = (0, 1, 0)$.*

(a) *Find ∇f_P.*

$f(x, y, z) = g(u, v)$, where $u = xe^{yz}$, $v = y\cos(x + z)$. Use chain rule: $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial v}{\partial x}$ to get

$$\frac{\partial f}{\partial x} = e^{yz}\frac{\partial g}{\partial u} - y\sin(x + z)\frac{\partial g}{\partial v}.$$

If $x = 1$, $y = 1$, $z = 0$, then $u = 0$ and $v = 1$, so

$$\frac{\partial f}{\partial x}(0, 1, 0) = e^0\frac{\partial g}{\partial u}(0, 1) + 0 = 2.$$

Similarly,

$$\frac{\partial f}{\partial y} = xze^{yz}\frac{\partial g}{\partial u} + \cos(x + z)\frac{\partial g}{\partial v}, \qquad \frac{\partial f}{\partial z} = xye^{yz}\frac{\partial g}{\partial u} - y\sin(x + z)\frac{\partial g}{\partial v},$$

so

$$\frac{\partial f}{\partial y}(0, 1, 0) = 0 + \cos(0)(1) = 1, \qquad \frac{\partial f}{\partial z}(0, 1, 0) = 0 + 0 = 0.$$

Conclusion: $\nabla f_P = \langle 2, 1, 0\rangle$.

(b) *Find the equation of the tangent plane at P to the surface $f(x, y, z) = 3$.*

Normal vector is $\nabla f_P = \langle 2, 1, 0\rangle$, so equation is $2x + y = d$. Plane goes through $P = (0, 1, 0)$, so $d = 1$. Equation: $2x + y = 1$.

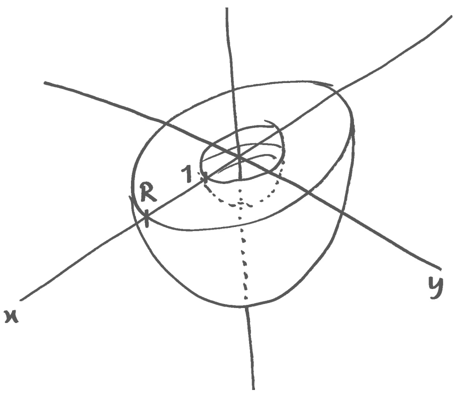
2 (8 points). *Evaluate the double integral $\displaystyle\int_0^1\int_{\sqrt[3]{x}}^1 x^2e^{y^{10}}\,dy\,dx$.*

Change order of integration: $\sqrt[3]{x} \le y \le 1$, $0 \le x \le 1$ becomes $0 \le x \le y^3$, $0 \le y \le 1$. So integral is

$$\int_0^1\int_0^{y^3} x^2e^{y^{10}}\,dx\,dy = \frac{1}{3}\int_0^1\Big[x^3e^{y^{10}}\Big]_{x=0}^{y^3}dy = \frac{1}{3}\int_0^1 y^9e^{y^{10}}\,dy = \frac{1}{3}\Big[\frac{1}{10}e^{y^{10}}\Big]_{y=0}^1 = \frac{1}{30}(e - 1).$$

3 (12 points). *Let $\mathscr{W}$ be the solid region given in spherical coordinates by $1 \le \rho \le R$, $\frac{1}{2}\pi \le \phi \le \pi$, $0 \le \theta \le 2\pi$, where $R > 1$ is a constant.*

(a) *Sketch the solid $\mathscr{W}$.*



(b) *Find the centroid of $\mathscr{W}$. (Recall that the centroid of a solid is the center of mass with respect to a constant mass density.)*

Solid is symmetric under rotation about z-axis, so centroid lies on z-axis, hence is of the form $(0,0,\bar{z})$ with $\bar{z} = \frac{1}{V}\iiint z\,dV$. Here $V = \operatorname{vol}(\mathscr{W})$. Volume of half ball is $\frac{2}{3}\pi R^3$. Scoop out little half ball to get $V = \frac{2}{3}\pi(R^3-1)$. The integral of z is

$$\iiint z\,dV = \int_0^{2\pi}\int_{\frac{1}{2}\pi}^{\pi}\int_1^R (\rho\cos\phi)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta = 2\pi\int_{\frac{1}{2}\pi}^{\pi}\cos\phi\sin\phi\,d\phi\int_1^R \rho^3\,d\rho$$
$$= 2\pi\left[\frac{1}{2}\sin^2\phi\right]_{\frac{1}{2}\pi}^{\pi}\left[\frac{1}{4}\rho^4\right]_1^R = 2\pi\left(-\frac{1}{2}\right)\frac{1}{4}(R^4-1) = -\frac{1}{4}\pi(R^4-1),$$

so $\bar{z} = -\dfrac{3}{8}\dfrac{R^4-1}{R^3-1}$ and centroid is $\left(0,0,-\dfrac{3}{8}\dfrac{R^4-1}{R^3-1}\right)$.

4 (12 points). *Determine the local minima, local maxima and saddle points of the function $f(x,y) = (x^2+y^2)^2 - 2c^2(x^2-y^2)$, where $c > 0$ is a constant. (The answers depend on c.)*
Locate critical points by solving $\nabla f = \mathbf{0}$: $f_x = 4x(x^2+y^2) - 4c^2x$, $f_y = 4y(x^2+y^2) + 4c^2y$, so (x,y) is critical if and only if

$$f_x = 4x(x^2+y^2-c^2) = 0, \qquad f_y = 4y(x^2+y^2+c^2) = 0.$$

Second equation gives $y = 0$, so from first equation we get $x = 0$ or $x = \pm c$. So critical points are $P = (0,0)$ and $Q_\pm = (\pm c, 0)$. Discriminant is

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 12x^2+4y^2-4c^2 & 8xy \\ 8xy & 4x^2+12y^2+4c^2 \end{vmatrix}.$$

So $D(P) = \begin{vmatrix} -4c^2 & 0 \\ 0 & 4c^2 \end{vmatrix} = -16c^4 < 0$; whereas $D(Q_\pm) = \begin{vmatrix} 8c^2 & 0 \\ 0 & 8c^2 \end{vmatrix} = 64c^4 > 0$ and $f_{xx}(Q_\pm) = 8c^2 > 0$. So by second derivative test P is saddle point and $Q_\pm$ are local minima.

5 (12 points). *Let $\mathscr{D}$ be the portion of the unit sphere $x^2+y^2+z^2 = 1$ contained in the octant $x \geq 0$, $y \geq 0$, $z \geq 0$. Let a, b, and c be positive constants. What is the maximum value of the function $f(x,y,z) = x^{2a}y^{2b}z^{2c}$ over the region $\mathscr{D}$? At what point of $\mathscr{D}$ is this maximum attained? (The answers depend on a, b, c.)*
$\mathscr{D}$ is closed and bounded and f is continuous, so f attains its global maximum somewhere on $\mathscr{D}$. The boundary $\partial\mathscr{D}$ of $\mathscr{D}$ is its intersection with each of the planes $x = 0$, $y = 0$, $z = 0$, so we have $f = 0$ on $\partial\mathscr{D}$. On the interior $\operatorname{int}\mathscr{D}$ of $\mathscr{D}$ the function f is positive, so the maximum must be attained on

int $\mathscr{D}$. To locate it, use Lagrange multiplier method: solve $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ for $(x,y,z) \in$ int $\mathscr{D}$, where $g(x,y,z) = x^2+y^2+z^2$. Since $\nabla f = \big\langle 2ax^{2a-1}y^{2b}z^{2c}, 2bx^{2a}y^{2b-1}z^{2c}, 2cx^{2a}y^{2b}z^{2c-1}\big\rangle$ and $\nabla g = 2\langle x,y,z\rangle$ we get

$$2ax^{2a-1}y^{2b}z^{2c} = 2\lambda x, \qquad 2bx^{2a}y^{2b-1}z^{2c} = 2\lambda y, \qquad 2cx^{2a}y^{2b}z^{2c-1} = 2\lambda z.$$

Multiplying these equations by x, y, resp. z gives

$$ax^{2a}y^{2b}z^{2c} = \lambda x^2, \qquad bx^{2a}y^{2b}z^{2c} = \lambda y^2, \qquad cx^{2a}y^{2b}z^{2c} = \lambda z^2.$$

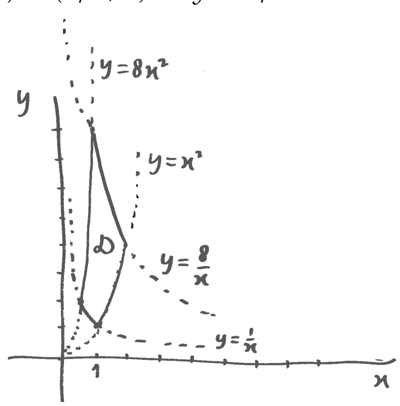
Adding equations and using constraint $g(x,y,z)=1$ gives $(a+b+c)x^{2a}y^{2b}z^{2c} = \lambda(x^2+y^2+z^2) = \lambda$, so $\lambda = (a+b+c)x^{2a}y^{2b}z^{2c}$. Substituting this into first equation gives $ax^{2a}y^{2b}z^{2c} = (a+b+c)x^{2a+2}y^{2b}z^{2c}$, so $x = \sqrt{a}/\sqrt{a+b+c}$. Similarly $y = \sqrt{b}/\sqrt{a+b+c}$, $z = \sqrt{c}/\sqrt{a+b+c}$. Conclusion: maximum is attained at the point $\big(\sqrt{a},\sqrt{b},\sqrt{c}\big)\big/\sqrt{a+b+c}$; maximum value is

$$f\left(\frac{\sqrt{a}}{\sqrt{a+b+c}}, \frac{\sqrt{b}}{\sqrt{a+b+c}}, \frac{\sqrt{c}}{\sqrt{a+b+c}}\right) = \frac{(2a)^a(2b)^b(2c)^c}{(2a+2b+2c)^{a+b+c}} = \frac{a^a b^b c^c}{(a+b+c)^{a+b+c}}.$$

6 (12 points). *Let $G(u,v) = (uv^{-1}, u^2v)$. Let $\mathscr{D}_0$ be the square in the uv-plane given by $1 \le u \le 2$, $1 \le v \le 2$.*

(a) *Sketch the region $\mathscr{D} = G(\mathscr{D}_0)$, i.e. the region in the xy-plane given by $x = uv^{-1}$, $y = u^2v$, $1 \le u \le 2$, $1 \le v \le 2$.*

Edges of $\mathscr{D}_0$ are given by $v=1$, $u=2$, $v=2$, $u=1$. So boundary curves of $\mathscr{D}$ are given by: $G(u,1) = (u,u^2)$ or $y = x^2$; $G(2,v) = (2/v, 4v)$ or $y = 8/x$; $G(u,2) = (u/2, 2u^2)$ or $y = 8x^2$; $G(1,v) = (1/v, v)$ or $y = 1/x$.



(b) *Use G to evaluate the integral $\displaystyle\iint_{\mathscr{D}} xy\,dx\,dy$.*

$\operatorname{Jac}(G) = \begin{vmatrix} 1/v & -u/v^2 \\ 2uv & u^2 \end{vmatrix} = u^2/v + 2u^2/v = 3u^2/v$, so change of variables formula with $f(x,y) = xy$ gives

$$\iint_{\mathscr{D}} f(x,y)\,dx\,dy = \iint_{\mathscr{D}_0} f(G(u,v))|\operatorname{Jac}(G)|\,du\,dv = \int_1^2\int_1^2 \frac{u}{v}u^2v\frac{3u^2}{v}\,du\,dv$$
$$= \int_1^2\int_1^2 3\frac{u^5}{v}\,du\,dv = \int_1^2 \frac{1}{2}\left[\frac{u^6}{v}\right]_{u=1}^2 dv = \frac{63}{2}\int_1^2 \frac{1}{v}\,dv = \frac{63}{2}\ln 2.$$

7 (16 points). *Let a, b, c, d be constants and let $\mathbf{F}$ be the vector field*

$$\mathbf{F}(x, y) = \frac{1}{x^2 + y^2}\langle ax + by, cx + dy\rangle.$$

The domain $\mathscr{D}$ of $\mathbf{F}$ is $\{ (x, y) \in \mathbf{R}^2 \mid (x, y) \neq (0, 0) \}$, the plane with the origin removed.

(a) *Is $\mathscr{D}$ simply connected? Explain your answer.*

A loop around the origin cannot be contracted within $\mathscr{D}$, so $\mathscr{D}$ is not simply connected.

(b) *Calculate* $\operatorname{curl}_z(\mathbf{F}) = \dfrac{\partial F_2}{\partial x} - \dfrac{\partial F_1}{\partial y}$.

$F_1 = \dfrac{ax + by}{x^2 + y^2}$ and $F_2 = \dfrac{cx + dy}{x^2 + y^2}$, so

$$\begin{aligned}
\frac{\partial F_2}{\partial x} &= \frac{c(x^2 + y^2) - (cx + dy)2x}{(x^2 + y^2)^2} = \frac{-cx^2 + cy^2) - 2dxy}{(x^2 + y^2)^2},\\
\frac{\partial F_1}{\partial y} &= \frac{b(x^2 + y^2) - (ax + by)2y}{x^2 + y^2} = \frac{bx^2 - by^2 - 2axy}{(x^2 + y^2)^2},\\
\operatorname{curl}_z(\mathbf{F}) &= \frac{-(b + c)x^2 + (b + c)y^2 + 2(a - d)xy}{(x^2 + y^2)^2}.
\end{aligned}$$

(c) *Let $\mathscr{C}$ be the circle of radius 1 about the origin oriented counterclockwise. Calculate $\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r}$. For what values of a, b, c, d is the integral equal to 0?*

Parametrize $\mathscr{C}$: $\mathbf{r}(t) = \langle \cos t, \sin t\rangle$ with $0 \le t \le 2\pi$; then $\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$, so

$$\begin{aligned}
\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) &= \langle a\cos t + b\sin t, c\cos t + d\sin t\rangle \cdot \langle -\sin t, \cos t\rangle\\
&= (-a + d)\cos t \sin t - b\sin^2 t + c\cos^2 t,
\end{aligned}$$

so

$$\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \int_0^{2\pi} \big((-a + d)\cos t \sin t - b\sin^2 t + c\cos^2 t\big)\, dt = \pi(-b + c).$$

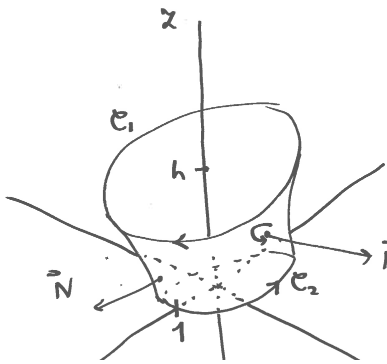
The integral is 0 if $b = c$ (a, d arbitrary).

(d) *For what values of a, b, c, d is $\mathbf{F}$ conservative? For those values, find a potential of $\mathbf{F}$.*

We need $\operatorname{curl}_z(\mathbf{F}) = 0$, i.e. $b = -c$ and $a = d$. But that is not enough, because $\mathscr{D}$ is not simply connected. We also need $\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r} = 0$ (path independence), so $b = c$. So we must have $a = d$, $b = c = 0$ for $\mathbf{F}$ to be conservative, i.e. $\mathbf{F} = \dfrac{a}{x^2 + y^2}\langle x, y\rangle$ with a arbitrary. This $\mathbf{F}$ has potential $f(x, y) = \dfrac{a}{2}\ln(x^2 + y^2)$, so is indeed conservative.

8 (20 points). *Let $\mathscr{S}$ be the surface in $\mathbf{R}^3$ defined by $x^2 + y^2 - z^2 = 1$ and $0 \le z \le h$, where h is a positive constant. Let $\mathscr{S}$ be oriented by the outward pointing normal vector.*

(a) *Sketch the surface $\mathscr{S}$. In your sketch indicate the orientation of the boundary of $\mathscr{S}$.*



(b) *Parametrize the surface $\mathscr{S}$. Specify the domain $\mathscr{D}$ of the parametrization.*

In cylindrical coordinates the surface is given by the equation $r^2 - z^2 = 1$, i.e. $r = \sqrt{z^2+1}$. So can use θ and z as parameters: $G(\theta, z) = \left(\sqrt{z^2+1}\cos\theta, \sqrt{z^2+1}\sin\theta, z\right)$. Domain: $0 \le \theta \le 2\pi, 0 \le z \le h$.

(c) *Calculate the outward pointing normal vector field* $\mathbf{N}$ *to the surface.*

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sqrt{z^2+1}\sin\theta & \sqrt{z^2+1}\cos\theta & -z \\ \frac{z}{\sqrt{z^2+1}}\cos\theta & \frac{z}{\sqrt{z^2+1}}\sin\theta & 1 \end{vmatrix} = \cdots = \left\langle \sqrt{z^2+1}\cos\theta, \sqrt{z^2+1}\sin\theta, -z \right\rangle.$$

(d) *Verify Stokes' Theorem for the surface $\mathscr{S}$ and the vector field* $\mathbf{F}(x,y,z) = \langle 2yz, 0, xy\rangle$.

Part 1: the surface integral. We have $\nabla \times \mathbf{F} = \langle x, y, -2z\rangle$, so

$$\nabla \times \mathbf{F}(G(\theta,z)) = \left\langle \sqrt{z^2+1}\cos\theta, \sqrt{z^2+1}\sin\theta, -2z \right\rangle \quad \text{and} \quad (\nabla\times\mathbf{F})\cdot\mathbf{N} = 3z^2+1,$$

and

$$\iint_{\mathscr{S}} (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \iint_{\mathscr{D}} (\nabla\times\mathbf{F})\cdot\mathbf{N}\,dA = \int_0^{2\pi}\int_0^h (3z^2+1)\,dz\,d\theta = 2\pi(h^3+h).$$

Part 2: the line integral. Two boundary pieces $\mathscr{C}_1$ and $\mathscr{C}_2$ parametrized by

$$\mathbf{r}_1(t) = \left\langle \sqrt{h^2+1}\cos t, \sqrt{h^2+1}\sin t, h \right\rangle, \quad \text{resp.} \quad \mathbf{r}_2(t) = \langle \cos t, \sin t, 0\rangle.$$

This parametrization of $\mathscr{C}_1$ goes the wrong way, so $\partial\mathscr{S} = -\mathscr{C}_1 + \mathscr{C}_2$. We have

$$\mathbf{r}_1'(t) = \left\langle -\sqrt{h^2+1}\sin t, \sqrt{h^2+1}\cos t, 0 \right\rangle, \qquad \mathbf{r}_2'(t) = \langle -\sin t, \cos t, 0\rangle,$$

so

$$\begin{aligned} \mathbf{F}(\mathbf{r}_1(t))\cdot\mathbf{r}_1'(t) &= \left\langle 2h\sqrt{h^2+1}\sin t, 0, (h^2+1)\cos t\sin t \right\rangle \cdot \left\langle -\sqrt{h^2+1}\sin t, \sqrt{h^2+1}\cos t, 0 \right\rangle \\ &= -2h(h^2+1)\sin^2 t, \\ \mathbf{F}(\mathbf{r}_2(t))\cdot\mathbf{r}_2'(t) &= 0, \end{aligned}$$

which yields

$$\int_{\mathscr{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-2h(h^2+1)\sin^2 t)\, dt = -2\pi h(h^2+1),$$
$$\int_{\mathscr{C}_2} \mathbf{F} \cdot d\mathbf{r} = 0.$$

Conclusion: $\int_{\partial\mathscr{S}} \mathbf{F} \cdot d\mathbf{r} = -\int_{\mathscr{C}_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathscr{C}_2} \mathbf{F} \cdot d\mathbf{r} = 2\pi h(h^2+1) = \iint_{\mathscr{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S}$.

9 (18 points). *Let* $\mathbf{F}$ *be the vector field defined by* $\mathbf{F} = \dfrac{1}{(x^2+y^2+z^2)^{3/2}}\langle x, y, z\rangle$.

(a) *Compute the upward flux of* $\mathbf{F}$ *through the surface* $\mathscr{S}_0$ *defined by* $x^2+y^2 \le R^2$, $z = h$, *where* $R > 0$ *and* $h > 0$ *are constant.*

Parametrize $\mathscr{S}_0$: $G(r,\theta) = (r\cos\theta, r\sin\theta, h)$ with domain $\mathscr{D}$ given by $0 \le r \le R, 0 \le \theta \le 2\pi$. Upward normal is

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle 0, 0, r\rangle.$$

So

$$\mathbf{F} \cdot \mathbf{N} = \frac{1}{(r^2+h^2)^{3/2}}\langle r\cos\theta, r\sin\theta, h\rangle \cdot \langle 0,0,r\rangle = \frac{hr}{(r^2+h^2)^{3/2}}$$

and upward flux is

$$\iint_{\mathscr{S}_0} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathscr{D}} \mathbf{F} \cdot \mathbf{N}\, dA = \int_0^{2\pi} \int_0^R \frac{hr}{(r^2+h^2)^{3/2}}\, dr\, d\theta = 2\pi \left[-\frac{h}{(r^2+h^2)^{1/2}} \right]_{r=0}^R$$
$$= 2\pi\left(1 - \frac{h}{(R^2+h^2)^{1/2}}\right).$$

(b) *Compute* $\nabla \cdot \mathbf{F}$.

We have $\mathbf{F} = \langle F_1, F_2, F_3\rangle$ with $F_1 = x/\rho^3$, $F_2 = y/\rho^3$, $F_3 = z/\rho^3$, where $\rho = \sqrt{x^2+y^2+z^2}$. Now $\partial\rho/\partial x = x/\rho$, so

$$\frac{\partial \rho^{-3} x}{\partial x} = -3\rho^{-4}\frac{\partial\rho}{\partial x}x + \rho^{-3} = -3\rho^{-5}x^2 + \rho^{-3}.$$

Similarly

$$\frac{\partial \rho^{-3} y}{\partial y} = -3\rho^{-5}y^2 + \rho^{-3}, \quad \frac{\partial \rho^{-3} y}{\partial y} = -3\rho^{-5}z^2 + \rho^{-3},$$

so $\nabla \cdot \mathbf{F} = -3\rho^{-5}(x^2+y^2+z^2) + 3\rho^{-3} = -3\rho^{-3} + 3\rho^{-3} = 0$.

(c) *Compute the upward flux of* $\mathbf{F}$ *through the surface* $\mathscr{S}$ *defined by* $z = e^{1-x^2-y^2}$, $x^2+y^2 \le 1$.

The boundary of $\mathscr{S}$ is the curve $x^2+y^2 = 1$, $z = 1$. Let $\mathscr{W}$ be the solid $x^2+y^2 \le 1$, $1 \le z \le e^{1-x^2-y^2}$. Then the boundary of $\mathscr{W}$ is $\partial\mathscr{W} = \mathscr{S} - \mathscr{S}_0$, where $\mathscr{S}_0$ is the surface of part (a) (for $R = h = 1$). Therefore by the divergence theorem

$$\iint_{\mathscr{S}} \mathbf{F} \cdot d\mathbf{S} - \iint_{\mathscr{S}_0} \mathbf{F} \cdot d\mathbf{S} = \iint_{\partial\mathscr{W}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathscr{W}} \nabla \cdot \mathbf{F}\, dV = 0,$$

because $\nabla \cdot \mathbf{F} = 0$. Therefore by part (a)

$$\iint_{\mathscr{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathscr{S}_0} \mathbf{F} \cdot d\mathbf{S} = 2\pi\left(1 - \frac{h}{(R^2 + h^2)^{1/2}}\right) = 2\pi\left(1 - \frac{1}{(1+1)^{1/2}}\right) = 2\pi\left(1 - \frac{1}{2}\sqrt{2}\right).$$