Math 1920 — *Solutions*

Prelim 1 — 2 October 2018 — 7:30–9:00 pm

1 (15 points). *Let* $\mathbf{u} = \langle 1, 0, 1\rangle$, $\mathbf{v} = \langle -2, 0, 3\rangle$, *and* $\mathbf{w} = \langle 1, 3, -1\rangle$.

(a) *Find the area of the parallelogram spanned by* $\mathbf{v}$ *and* $\mathbf{w}$.

Area is equal to magnitude of

$$\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 3 \\ 1 & 3 & -1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 0 & 3 \\ 3 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -2 & 0 \\ 1 & 3 \end{vmatrix} = -9\mathbf{i} + \mathbf{j} - 6\mathbf{k}.$$

So area is $\sqrt{81+1+36} = \sqrt{118}$.

(b) *Find the volume of the parallelepiped spanned by* $\mathbf{u}$, $\mathbf{v}$, *and* $\mathbf{w}$.

Volume is equal to absolute value of $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \langle 1,0,1\rangle\cdot\langle -9,1,-6\rangle = -15$, so volume is 15.

(c) *Find the projection of* $\mathbf{u}$ *along* $\mathbf{v}$.

Projection is $\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{-2+0+3}{4+0+9}\mathbf{v} = \frac{1}{13}\mathbf{v} = \left\langle -\frac{2}{13}, 0, \frac{3}{13}\right\rangle$.

2 (15 points). (a) *Write the equation of the cylinder* $x^2 + y^2 = 4$ *in spherical coordinates.*

$x = \rho\sin\phi\cos\theta$ and $y = \rho\sin\phi\sin\theta$ gives $x^2+y^2 = \rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) = \rho^2\sin^2\phi$, so $\rho^2\sin^2\phi = 4$. Taking square roots gives $|\rho\sin\phi| = 2$. Now ρ is distance to the origin, so $\rho \geq 0$, and ϕ is angle of declination, so $0 \leq \phi \leq \pi$, so $\sin\phi \geq 0$, so the equation is $\rho\sin\phi = 2$.

(b) *Write the equation* $x^2 + y^2 - z^2 = 5x$ *as an equation* $r = f(\theta, z)$ *in cylindrical coordinates.*

$x = r\cos\theta$ and $y = r\sin\theta$ gives $r^2 - z^2 = 5r\cos\theta$, so $r^2 - (5\cos\theta)r - z^2 = 0$. Solving this quadratic equation for r gives $r = \frac{1}{2}\left(5\cos\theta \pm \sqrt{25\cos^2\theta + 4z^2}\right)$. Since r is distance to the z-axis, we want $r \geq 0$. Since $\sqrt{25\cos^2\theta + 4z^2} \geq 5\cos\theta$, we must take the +-sign in the formula for r, so $r = \frac{5}{2}\cos\theta + \frac{1}{2}\sqrt{25\cos^2\theta + 4z^2}$.

3 (15 points). (a) *Find the arc length of the path* $\mathbf{r}(t) = \langle t^3, t^2\rangle$ *from time* 0 *to time* T.

Velocity is $\mathbf{r}'(t) = \langle 3t^2, 2t\rangle$, so arc length is equal to

$$s(T) = \int_0^T \|\mathbf{r}'(t)\|\,dt = \int_0^T \sqrt{9t^4 + 4t^2}\,dt = \int_0^T t\sqrt{9t^2+4}\,dt =$$
$$= \frac{2}{3}\frac{1}{18}(9t^2+4)^{3/2}\Big|_{t=0}^{t=T} = \frac{1}{27}(9T^2+4)^{3/2} - \frac{8}{27}.$$

(b) *For what value(s) of the constant* a *is the path* $\mathbf{r}(t) = \langle e^{at}, t\rangle$ *orthogonal to the level curves of the function* $f(x,y) = x^2 + y$?

The velocity vector $\mathbf{r}'(t) = \langle ae^{at}, 1\rangle$ is tangent to the path $\mathbf{r}(t)$. The gradient $\nabla f = \langle 2x, 1\rangle$ is orthogonal to the level curves of f. So we want $\mathbf{r}'(t)$ to be proportional to $\nabla f_{\mathbf{r}(t)}$ for all t. In other words, we want $\langle ae^{at}, 1\rangle = \lambda\langle 2e^{at}, 1\rangle$ for some scalar λ. This gives $ae^{at} = 2\lambda e^{at}$ and $\lambda = 1$, so $ae^{at} = 2e^{at}$, and hence $a = 2$.

4 (15 points). *Evaluate each of the following limits or state that it does not exist. Justify your answers.*

(a) $\displaystyle\lim_{(x,y)\to(2,1)} \frac{\ln(x^3-8y^2+1)}{x^3-8y^2}$.

Let $g(x,y) = x^3 - 8y^2$. Then g is continuous and $g(2,1) = 0$, so substituting $u = g(x,y)$ into the limit gives $\lim_{(x,y)\to(2,1)} \frac{\ln(g(x,y)+1)}{g(x,y)} = \lim_{u\to 0} \frac{\ln(u+1)}{u} = 1$. Here we used the standard one-variable limit $\lim_{u\to 0} \frac{\ln(u+1)}{u} = 1$ (which follows e.g. from l'Hôpital).

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$.

Let $f(x,y) = \frac{x^2y}{x^4+y^2}$. Then $f(x,0) = 0$ for all $x \neq 0$, so $\lim_{x\to 0} f(x,0) = 0$. On the other hand $f(x,x^2) = x^4/(x^4+x^4) = \frac{1}{2}$ for all $x \neq 0$, so $\lim_{x\to 0} f(x,x^2) = \frac{1}{2}$. So $f(x,y)$ has different limits along different paths through $(0,0)$, and therefore $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{5x^2+2y^2}$.

Let $f(x,y) = \frac{3x^2y}{5x^2+2y^2}$.

Method 1. The denominator satisfies $5x^2 + 2y^2 \geq 5x^2$. Therefore

$$|f(x,y)| = \frac{|3x^2y|}{|5x^2+2y^2|} = \frac{3x^2|y|}{5x^2+2y^2} \leq \frac{3x^2|y|}{5x^2} = \frac{3}{5}|y|.$$

The expression $\frac{3}{5}|y|$ goes to 0 as $(x,y) \to (0,0)$. Therefore $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ by the squeezing theorem.

Method 2. Substituting polar coordinates gives

$$f(x,y) = \frac{3r^3\cos^2\theta\sin\theta}{r^2(5\cos^2\theta+2\sin^2\theta)} = r\frac{3\cos^2\theta\sin\theta}{5\cos^2\theta+2\sin^2\theta}.$$

We have $5\cos^2\theta + 2\sin^2\theta = 3\cos^2\theta + 2\cos^2\theta + 2\sin^2\theta = 3\cos^2\theta + 2 \geq 2$, so

$$|f(x,y)| = r\frac{|3\cos^2\theta\sin\theta|}{5\cos^2\theta+2\sin^2\theta} \leq \frac{3}{2}r\cos^2\theta|\sin\theta| \leq \frac{3}{2}r.$$

Since $\lim_{(x,y)\to(0,0)} \frac{3}{2}r = 0$, we have $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ by the squeezing theorem.

5 (20 points). *Let* $f(x,y) = \sqrt{x^2+y}$.

(a) *Compute the partial derivatives* $f_x(x,y)$ *and* $f_y(x,y)$. *Show that* $f(x,y)$ *is differentiable at every point* (x,y) *with* $x^2+y>0$.

$f_x = \frac{x}{\sqrt{x^2+y}}$ and $f_y = \frac{1}{2\sqrt{x^2+y}}$, both of which are continuous functions of (x,y) when $x^2+y>0$. Therefore (by a theorem in the book) f is differentiable at every point (x,y) with $x^2+y>0$.

(b) *Find the equation of the tangent plane to the graph of* f *at* $(4,9)$.

Equation of tangent plane is $z = L(x, y)$, where $L(x, y)$ is the best linear approximation to f at $(4, 9)$. We have

$$L(x, y) = f(4, 9) + f_x(4, 9)(x-4) + f_y(4, 9)(y-9) = 5 + \frac{4}{5}(x-4) + \frac{1}{10}(y-9) = \frac{9}{10} + \frac{4}{5}x + \frac{1}{10}y,$$

so equation is $z = \frac{9}{10} + \frac{4}{5}x + \frac{1}{10}y$, or $8x + y - 10z = -9$.

(c) Use linear approximation to estimate $f(4.1, 8.8)$.

The linear approximation principle says that $f(a+h, b+k) \approx L(a+h, b+k)$ for small h and k. Taking $a = 4$, $b = 9$, $h = 0.1$, and $k = -0.2$ we get $f(4.1, 8.8) \approx L(4.1, 8.8) = 5 + \frac{4}{5}h + \frac{1}{10}k = 5 + 0.08 - 0.02 = 5.06$.

(d) *Find the unit vector* $\mathbf{u}$ *at* $(4, 9)$ *pointing in the direction along which* f *increases fastest.*

The direction in which f increases fastest at $(4, 9)$ is $\nabla f_{(4,9)} = \langle \frac{4}{5}, \frac{1}{10} \rangle$. The unit vector in this direction is $\mathbf{u} = \langle 8, 1 \rangle / \sqrt{8^2 + 1^2} = \frac{1}{\sqrt{65}} \langle 8, 1 \rangle$.

6 (20 points)**.** *Let* C *be the curve which is the intersection of two surfaces* $x^2 + y^2 = z^2$ *and* $x^2 + z^2 = 1$ *in* $\mathbf{R}^3$.

(a) *Sketch the two surfaces and the intersection curve* C*. (Show both surfaces and the curve in one single sketch). Include sufficient detail, such as coordinate axes and some indication of scale.*

The first surface is a vertical cone with a circular cross-section and the second surface is a cylinder along the y-axis. The intersection C consists of two closed curves.

(b) *Find a normal vector at the point* $P = \left(\frac{1}{2}\sqrt{2}, 0, \frac{1}{2}\sqrt{2}\right)$ *to each of the two surfaces.*

Normal vector to a surface $f(x, y, z) = 0$ at a point P is ∇f_P. Taking $f(x, y, z) = x^2 + y^2 - z^2$ gives $\nabla f = \langle 2x, 2y, -2z \rangle$ and $\nabla f_P = \langle \sqrt{2}, 0, -\sqrt{2} \rangle$. Taking $f(x, y, z) = x^2 + z^2 - 1$ gives $\nabla f = \langle 2x, 0, 2z \rangle$ and $\nabla f_P = \langle \sqrt{2}, 0, \sqrt{2} \rangle$.

(c) *Parametrize the curve* C.

The "top view" of the curve C (i.e. the projection of C onto the xy-plane) is a closed curve around the origin in the xy-plane, and so is of the form $\langle r(\theta)\cos\theta, r(\theta)\sin\theta \rangle$, where θ is the polar angle in the xy-plane. So our curve C is of the form $\mathbf{r}(\theta) = \langle r(\theta)\cos\theta, r(\theta)\sin\theta, z(\theta) \rangle$. Plugging $x = r(\theta)\cos\theta$, $y = r(\theta)\sin\theta$, $z = z(\theta)$ into the first surface gives $z(\theta)^2 = r(\theta)^2$. Plugging this into the second surface gives $r(\theta)^2\cos^2\theta + r(\theta)^2 = 1$, or $r(\theta) = 1/\sqrt{\cos^2\theta + 1}$. This gives the parametrization

$$\mathbf{r}(\theta) = \frac{1}{\sqrt{\cos^2\theta + 1}} \langle \cos\theta, \sin\theta, \pm 1 \rangle,$$

where the choice of sign corresponds to the top or bottom half of the curve C.