Math 1920 *Solutions*

Prelim 2 6 November 2018 7:30–9:00 pm

1 (15 points). (a) *Find the critical points of the function $f(x, y) = x^4 + y^2 - axy$ on $\mathbf{R}^2$, where a is a nonzero constant. (The answers depend on a.)*

Setting $\nabla f = \langle 4x^3 - ay, 2y - ax\rangle = \langle 0, 0\rangle$ gives $4x^3 - ay = 0$ and $2y - ax = 0$. Plugging $y = \frac{1}{2}ax$ into first equation gives $4x^3 - \frac{1}{2}a^2x = x(4x^2 - \frac{1}{2}a^2) = 0$, so $x = 0$ or $x = \pm a/\sqrt{8} = \pm\frac{1}{4}a\sqrt{2}$. For $x = 0$ we get $y = 0$, so $P = (0, 0)$ is critical. For $x = \pm\frac{1}{4}a\sqrt{2}$ we get $y = \pm\frac{1}{8}a^2\sqrt{2}$, so the two points $Q_\pm = \pm\big(\frac{1}{4}a\sqrt{2}, \frac{1}{8}a^2\sqrt{2}\big)$ are critical.

(b) *Classify the critical points from part* (a). *That is, determine which of them are local maxima, which are local minima, and which are saddle points.*

Discriminant is

$$D = \begin{vmatrix} 12x^2 & -a \\ -a & 2 \end{vmatrix} = 24x^2 - a^2.$$

So $D(P) = -a^2 < 0$: P is saddle point. And $D(Q_\pm) = \frac{24}{8}a^2 - a^2 = 2a^2 > 0$, while $f_{xx}(Q_\pm) = \frac{3}{2}a^2 > 0$: so $Q_\pm$ are local minima.

2 (15 points). *Find the center of mass of the cylinder of radius 2 and height 4 if its mass density is $\delta(x, y, z) = e^{-z}$.*

The cylinder $\mathscr{W}$ and the mass density δ are rotationally symmetric about the z-axis, so center of mass is on the z-axis: $P_{\mathrm{CM}} = (0, 0, z_{\mathrm{CM}})$. Limits for $\mathscr{W}$ in cylindrical coordinates: $0 \le r \le 2$, $0 \le \theta \le 2\pi$, $0 \le z \le 4$. Total mass is

$$\begin{aligned}\int_0^{2\pi}\int_0^2\int_0^4 e^{-z} r\,dz\,dr\,d\theta &= \int_0^{2\pi}\int_0^2 [-e^{-z}]_{z=0}^4 r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (1 - e^{-4}) r\,dr\,d\theta \\ &= \int_0^{2\pi}\Big[\frac{1}{2}(1 - e^{-4})r^2\Big]_{r=0}^2 d\theta = \int_0^{2\pi} 2(1 - e^{-4})\,d\theta = 4\pi(1 - e^{-4}).\end{aligned}$$

Integral of z is

$$\begin{aligned}\int_0^{2\pi}\int_0^2\int_0^4 ze^{-z} r\,dz\,dr\,d\theta &= \int_0^{2\pi}\int_0^2 [-ze^{-z} - e^{-z}]_{z=0}^4 r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (-4e^{-4} - e^{-4} + 1) r\,dr\,d\theta \\ &= \int_0^{2\pi}\Big[\frac{1}{2}(1 - 5e^{-4})r^2\Big]_{r=0}^2 d\theta = \int_0^{2\pi} 2(1 - 5e^{-4})\,d\theta = 4\pi(1 - 5e^{-4}).\end{aligned}$$

So $z_{\mathrm{CM}} = (1 - 5e^{-4})/(1 - e^{-4})$.

3 (15 points). *Use cylindrical coordinates to evaluate the integral*

$$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\frac{x^2+y^2}{2}} \frac{z}{\sqrt{x^2 + y^2}}\,dz\,dy\,dx.$$

The limits in Cartesian coordinates are $0 \le x \le 2$, $0 \le y \le \sqrt{4-x^2}$, $0 \le z \le \frac{1}{2}(x^2+y^2)$. The limits for x and y describe a quarter-disc $\mathscr{D}$ of radius 2 in the xy-plane given by $x^2+y^2 \le 4$, $x \ge 0$, $y \ge 0$. In polar coordinates $\mathscr{D}$ is given by $0 \le \theta \le \frac{1}{2}\pi$ and $0 \le r \le 2$. The limits for z are $0 \le z \le \frac{1}{2}(x^2+y^2) = \frac{1}{2}r^2$. The integrand in cylindrical coordinates is z/r and the volume element is $r\,dz\,dr\,d\theta$, so the integral is

$$\int_0^{\frac{1}{2}\pi}\int_0^2\int_0^{\frac{1}{2}r^2} \frac{z}{r} r\,dz\,dr\,d\theta = \int_0^{\frac{1}{2}\pi}\int_0^2\int_0^{\frac{1}{2}r^2} z\,dz\,dr\,d\theta = \int_0^{\frac{1}{2}\pi}\int_0^2 \left[\frac{1}{2}z^2\right]_{z=0}^{\frac{1}{2}r^2} dr\,d\theta =$$
$$\int_0^{\frac{1}{2}\pi}\int_0^2 \frac{1}{8}r^4\,dr\,d\theta = \int_0^{\frac{1}{2}\pi}\left[\frac{1}{40}r^5\right]_{r=0}^{2} d\theta = \frac{\pi}{2}\frac{32}{40} = \frac{2}{5}\pi.$$

4 (15 points). (a) *Let* $g(u,v) = f(u^3-v^3, v^3-u^3)$. *Show that* $v^2\dfrac{\partial g}{\partial u} + u^2\dfrac{\partial g}{\partial v} = 0$.

Apply chain rule to $f(x,y)$ into which we substitute $x = u^3 - v^3$, $y = v^3 - u^3$:

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} = 3u^2\frac{\partial f}{\partial x} - 3u^2\frac{\partial f}{\partial y} = 3u^2\Big(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\Big).$$
$$\frac{\partial g}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} = -3v^2\frac{\partial f}{\partial x} + 3v^2\frac{\partial f}{\partial y} = -3v^2\Big(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\Big).$$

Hence $v^2\dfrac{\partial g}{\partial u} = 3u^2v^2\Big(\dfrac{\partial f}{\partial x} - \dfrac{\partial f}{\partial y}\Big) = -u^2\dfrac{\partial g}{\partial v}$.

(b) *Is the vector field* $\mathbf{F} = \langle yz^2, xz^2, xyz\rangle$ *conservative? Explain your answer.*

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz^2 & xz^2 & xyz \end{vmatrix} = \langle -xz, yz, 0\rangle \ne \langle 0,0,0\rangle,$$

so $\mathbf{F}$ has nonzero curl, so $\mathbf{F}$ not conservative.

(c) *Evaluate the integral* $\int_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r}$, *where* $\mathbf{F}$ *is the vector field of part* (b) *and* $\mathscr{C}$ *is the line segment from the origin to the point* $(1,2,1)$.

Parametrize $\mathscr{C}$ in positive direction: $\mathbf{r}(t) = \langle t, 2t, t\rangle$. Then $\mathbf{r}'(t) = \langle 1,2,1\rangle$ and $0 \le t \le 1$, so

$$\int_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^1 \langle 2tt^2, tt^2, t(2t)t\rangle\cdot\langle 1,2,1\rangle\,dt =$$
$$\int_0^1 (2t^3+2t^3+2t^3)\,dt = \int_0^1 6t^3\,dt = \left[\frac{6}{4}t^4\right]_0^1 = \frac{3}{2}.$$

5 (20 points). *Let* $f(x,y,z) = xy + 2z$.

(a) *Find the maximal value of* f *on the sphere* $x^2+y^2+z^2 = 36$.

Use Lagrange multipliers with $g(x,y,z) = x^2+y^2+z^2$. We are looking for a point $P = (x,y,z)$ on the sphere satisfying $\nabla f_P = \lambda\nabla g_P$ for some λ. This gives $\langle y, x, 2\rangle = \lambda\langle 2x, 2y, 2z\rangle$, i.e.

$$y = 2\lambda x, \qquad x = 2\lambda y, \qquad 2 = 2\lambda z.$$

First two equations give $x = 2\lambda(2\lambda x) = 4\lambda^2 x$, so $(4\lambda^2 - 1)x = 0$, so $x = 0$ or $\lambda = \pm\frac{1}{2}$. For $x = 0$ we find $y = 0$ and (since P has to be on the sphere) $z = \pm 6$, so $P = P_\pm = (0, 0, \pm 6)$. For $\lambda = \frac{1}{2}$ we find $x = y$ and $z = 2$. Since P must be on the sphere this gives $x^2 + x^2 + 4 = 36$, so $x = \pm 4$, i.e. $P = Q_\pm = (\pm 4, \pm 4, 2)$. Similarly, $\lambda = -\frac{1}{2}$ gives $P = R_\pm = (\pm 4, \mp 4, -2)$. Compare values: $f(P_\pm) = \pm 12$, $f(Q_\pm) = 20$, $f(R_\pm) = -20$. So maximum value is $f(Q_\pm) = 20$.

(b) *Find the maximal value of f on the disc $y^2 + z^2 \le 36$, $x = 0$ in the yz-plane.*

On this disc we have $f(0, y, z) = 2z$, so the maximal value of f is attained for $y = 0$, $z = 6$: $f(0, 0, 6) = 12$.

(c) *Find the maximal value of f inside the half-ball $x^2 + y^2 + z^2 \le 36$, $x \ge 0$.*

Examine interior of half-ball for critical points: $\nabla f = \langle y, x, 2\rangle$ is never equal to $\mathbf{0}$, so no critical points in interior. So maximum value is attained on boundary, which consists of two pieces: a hemisphere and a disc. From parts (a) and (b) we see: maximal value is $f(Q_+) = 20$.

6 (20 points)**.** *Calculate the integral* $\displaystyle\int_{\mathscr{D}} xy\,dx\,dy$, *where $\mathscr{D}$ is the parallelogram with vertices at* $(0, 0)$, $(3, 1)$, $(1, 2)$ *and* $(4, 3)$.

The parallelogram $\mathscr{D}$ is spanned by the edge vectors $\langle 3, 1\rangle$ and $\langle 1, 2\rangle$. Define the linear map G by $G(u, v) = (3u + v, u + 2v)$. Then $G(1, 0) = (3, 1)$ and $G(0, 1) = (1, 2)$, so G sends the unit square $\mathscr{D}_0 = [0, 1] \times [0, 1]$ to $\mathscr{D}$. We have $\mathrm{Jac}(G) = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5$, so by change of variables formula

$$\int_{\mathscr{D}} xy\,dx\,dy = \int_{\mathscr{D}_0} (3u + v)(u + 2v)|\mathrm{Jac}(G)|\,du\,dv = \int_{\mathscr{D}_0} 5(3u^2 + 6uv + uv + 2v^2)\,du\,dv =$$

$$\int_{\mathscr{D}_0} (15u^2 + 35uv + 10v^2)\,du\,dv = \int_0^1 \left[5u^3 + \frac{35}{2}u^2 v + 10uv^2\right]_{u=0}^1 dv = \int_0^1 \left(5 + \frac{35}{2}v + 10v^2\right) dv =$$

$$\left[5v + \frac{35}{4}v^2 + \frac{10}{3}v^3\right]_{v=0}^1 = 5 + \frac{35}{4} + \frac{10}{3} = 5 + \frac{105 + 40}{12} = 17\frac{1}{12}.$$