Scissors Congruence and K-theory

by

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Abstract

In this thesis we develop a version of classical scissors congruence theory from the perspective of algebraic K-theory. Classically, two polytopes in a manifold X are defined to be scissors congruent if they can be decomposed into finite sets of pairwise-congruent polytopes. We generalize this notion to an abstract problem: given a set of objects and decomposition and congruence relations between them, when are two objects in the set scissors congruent? By packaging the scissors congruence information in a Waldhausen category we construct a spectrum whose homotopy groups include information about the scissors congruence problem. We then turn our attention to generalizing constructions from the classical case to these Waldhausen categories, and find constructions for cofibers, suspensions, and products of scissors congruence problems.

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Contents

Chapter 1

Introduction

1.0.1   A quick introduction to scissors congruence

The classical question of scissors congruence concerns the subdivisions of polyhedra. Given a polyhedron, when is it possible to dissect it into smaller polyhedra and rearrange the pieces into a rectangular prism? Or, more generally, given two polyhedra when is it possible to dissect one and rearrange it into the other?

In more modern language, we can express scissors congruence as a question about groups. Let $\mathcal{P}(E^3)$ be the free abelian group generated by polyhedra P, quotiented out by the two relations $[P] = [Q]$ if $P \cong Q$, and $[P \cup P'] = [P] + [P']$ if $P \cap P'$ has measure 0. Can we compute $\mathcal{P}(E^3)$? Can we construct a full set of invariants on it? More generally, let X be E^n (Euclidean space), S^n, or $\mathcal{H}^n$ (hyperbolic space). We define a simplex of X to be the convex hull of $n + 1$ points in X (restricted to an open hemisphere if $X = S^n$), and a polytope of X to be a finite union of simplices. Let G be any subgroup of the group of isometries of X. Then we can define a group $\mathcal{P}(X, G)$ to be the free abelian group generated by polytopes P of X, modulo the two relations $[P] = [g \cdot P]$ for any polytope P and $g \in G$, and $[P \cup P'] = [P] + [P']$ for any two polytopes P, P' such that $P \cap P'$ has measure 0. The goal of Hilbert's third problem is to classify these groups.

Consider the example when $X = E^n$ and G_n is the group of Euclidean isometries. We have a homomorphism

$$\mathcal{P}(E^n, G_n) \otimes \mathcal{P}(E^m, G_m) \longrightarrow \mathcal{P}(E^{n+m}, G_{n+m})$$

given on generators by

$$[P] \otimes [Q] \longmapsto [P \times Q],$$

where $P \times Q$ is the subset of $E^n \times E^m \cong E^{n+m}$ which projects to P in E^n and Q in E^m. Similarly, for $X = S^n$ and $G = O(n)$ we can construct a homomorphism

$$\mathcal{P}(S^n, O(n)) \otimes \mathcal{P}(S^m, O(m)) \longrightarrow \mathcal{P}(S^{n+m+1}, O(n+m+1))$$

in the following manner. Consider S^n to be embedded in $\mathbb{R}^{n+1}$, and let $\widetilde{P}$ be the solid cone generated by all points inside $P \subseteq S^n$. Then $\widetilde{P} \times \widetilde{Q} \subseteq \mathbb{R}^{n+1+m+1}$ is a solid cone in $\mathbb{R}^{n+m+2}$, and thus spans a polytope in S^{n+m+1}. We call this polytope $P * Q$, as it can also be constructed as the orthogonal join (inside S^{n+m+1}) of P and Q. Then the homomorphism $\mathcal{P}(S^n, O(n)) \otimes \mathcal{P}(S^m, O(m)) \longrightarrow \mathcal{P}(S^{n+m+1}, O(n+m+1))$ given by $[P] \otimes [Q] \longmapsto [P * Q]$ is the desired homomorphism.

Letting $\mathcal{P}(E^\infty) = \bigoplus_{n \geq 0} \mathcal{P}(E^n, G_n)$ and $\mathcal{P}(S^\infty) = \mathbb{Z} \oplus \bigoplus_{n \geq 0} \mathcal{P}(S^n, O(n))$ we see that the above homomorphisms assemble into graded homomorphisms

$$\mathcal{P}(E^\infty) \otimes \mathcal{P}(E^\infty) \longrightarrow \mathcal{P}(E^\infty) \qquad \text{and} \qquad \mathcal{P}(S^\infty) \otimes \mathcal{P}(S^\infty) \longrightarrow \mathcal{P}(S^\infty).$$

In order for the grading to work properly we need to give $\mathcal{P}(S^n, O(n))$ the grading $n+1$; the extra $\mathbb{Z}$ we added on will have grading 0 and should be considered to be the scissors congruence group of the empty polytope. (Note that $P * \emptyset = \emptyset * P = P$ for all polytopes P.) It is easy to see that these homomorphisms in fact give commutative ring structures on $\mathcal{P}(E^\infty)$ and $\mathcal{P}(S^\infty)$.

There is a fundamental difference between the scissors congruence groups of E^n and S^n, however. E^n is not compact and thus we have no "everything" polytope, whereas S^n is compact and contains itself as a polytope. If we think of polytopes of S^n as measuring solid angles, we may want to have a scissors congruence group that has $[S^n] = 0$. Note, howver, that $[S^n] = [S^0 * S^{n-1}]$, so if we want this canceling out to be ocnsistent with the ring structure on $\mathcal{P}(S^\infty)$ it suffices to quotient out by the ideal generated by $[S^0]$.[1] We will denote $\mathcal{P}(S^\infty)/([S^0])$ by $\widetilde{\mathcal{P}}(S^\infty)$; we denote the $n+1$-graded part of this by $\widetilde{\mathcal{P}}(S^\infty)_n$. (This will be the image of $\mathcal{P}(S^n, O(n))$ inside $\widetilde{\mathcal{P}}(S^\infty)$.) There will be an induced homomorphism $\widetilde{\mathcal{P}}(S^\infty) \otimes \widetilde{\mathcal{P}}(S^\infty) \longrightarrow \widetilde{\mathcal{P}}(S^\infty)$ which gives a commutative ring structure on $\widetilde{\mathcal{P}}(S^\infty)$.

Once we start thinking of $\widetilde{\mathcal{P}}(S^\infty)$ as being a scissors congruence group that measures angles we can construct another set of homomorphisms:

$$D_{n,m} \colon \mathcal{P}(S^n, O(n)) \longrightarrow \mathcal{P}(S^m, O(m)) \otimes \widetilde{\mathcal{P}}(S^\infty)_{n-m-1}.$$

These homomorphisms are constructed in the following manner. Consider a simplex x in S^n; this is the convex hull of $n+1$ points $x_0, \ldots, x_n \in S^n$. For any subset $I \subseteq \{0, \ldots, n\}$ let $x_I = \{x_i \mid i \in I\}$, A_I be the span of x_I inside $\mathbb{R}^{n+1}$, and

$$y_{\overline{I}} = \{\mathrm{Proj}(x_j, A_I^\perp)/\|\mathrm{Proj}(x_j, A_I^\perp)\| \mid j \notin I\},$$

where $\mathrm{Proj}(z, A)$ is the orthogonal projection of the point z (considered as a vector) into the subspace A, and $\|z\|$ is the length of the vector z. Note that the convex hull of x_I is a simplex in $S^{|I|-1}$ nad that the convex hull of $y_{\overline{I}}$ is a simplex in $S^{n-|I|}$. x_I measures the "volume" of the face spanned by the vertices in I, and $y_{\overline{I}}$ measures the "angle" at that face. We can then define

$$D_{n,m}(x) = \sum_{\substack{I \subseteq \{0,\ldots,n\} \\ |I|=m+1}} x_I \otimes y_{\overline{I}},$$

which extends to the desired homomorphism because simplices generate all polytopes. Note that we need the second coordinate to be reduced because otherwise this is not well-defined up to subdivision. The homomorphisms $D_{n,m}$ are called the *generalized Dehn invariants*, so called because the homomorphism $D_{3,1}$ is the Dehn invariant constructed to show that $\mathcal{P}(E^3, G_3) \not\cong \mathbb{R}$. (For more details, see [18].) These assemble into a *total Dehn invariant*

$$D \colon \mathcal{P}(S^\infty) \longrightarrow \mathcal{P}(S^\infty) \otimes \widetilde{\mathcal{P}}(S^\infty).$$

We can also reduce all of the $\mathcal{P}$'s to get a comultiplication $D \colon \widetilde{\mathcal{P}}(S^\infty) \longrightarrow \widetilde{\mathcal{P}}(S^\infty) \otimes \mathcal{P}(S^\infty)$. In fact, D makes $\widetilde{\mathcal{P}}(S^\infty)$ into a Hopf algebra, and $\mathcal{P}(E^\infty)$ is a comodule over this Hopf algebra. In addition, if we let $\mathcal{P}(\mathcal{H}^n, O(n; 1))$ be the scissors congruence group of n-dimensional hyperbolic space with its group of isometries, we can construct analogous homomorphisms $D_{n,m} \colon \mathcal{P}(\mathcal{H}^n) \longrightarrow \mathcal{P}(\mathcal{H}^m) \otimes \widetilde{\mathcal{P}}(S^\infty)_{n-m-1}$.

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[1] In [18], Sah quotients out by the ideal generated by a point so that there is no torsion in the resulting ring. We use S^0 instead, here, as torsion will not be a problem for our future discussion.

1.0.2 Connections to K-theory

The question of scissors congruence is highly reminiscent of a K-theoretic question. Algebraic K-theory classifies projective modules according to their decompositions into smaller modules; topological K-theory classifies vector bundles according to decompositions into smaller-dimensional bundles. Thus it is reasonable to ask whether scissors congruence can also be expressed as a K-theoretic question about polytopes being decomposed into smaller polytopes. As K-theory has a different array of computational tools than group homology it is possible that this new perspective would create new approaches for computing scissors congruence groups.

In addition, there are many seemingly-coincidental appearances of algebraic K-groups in the theory of scissors congruence. In [5] Dupont and Sah construct the following short exact sequence,

$$0 \longrightarrow (K_3(\mathbb{C})^{\text{indec}})^- \longrightarrow \mathcal{P}(\mathcal{H}^3) \xrightarrow{D_{3,1}} \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \longrightarrow K_2(\mathbb{C})^- \longrightarrow 0$$

(where the negative superscripts indicate the -1-eigenspace of complex conjugation, and "indec" indicates the indecomposable elements of K_3). For a detailed exploration of the techniques that lead to this result, see [4]. A more general result was obtained by Goncharov in [9], where he constructs a morphism from certain subquotients of the scissors congruence groups of spherical and hyperbolic space to certain subquotients of the Milnor K-theory of $\mathcal{C}$. Both of these results, however, are highly computational rather than conceptual; one goal of the current project is to find a more conceptual basis for these results.

If we had spaces X and Y such that $\pi_3(X) = \mathcal{P}(\mathcal{H}^3)$ and $\pi_3(Y) = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$ we might see the above short exact sequence as a fragment of the long exact sequence associated to a homotopy fiber sequence $F \longrightarrow X \longrightarrow Y$. If, in addition, each of X and Y were the space associated to the K-theory of some Waldhausen category and the map $X \longrightarrow Y$ were obtained as the map associated to an exact functor, we would have the desired homotopy fiber sequence, and would be able to extend the sequence to a long exact sequence. Thus our goal for this project is to construct a Dehn invariant $D_{3,1}$ between categories with an associated K-theory in such a way as to get the above short exact sequence. In order to construct this Dehn invariant we need the following ingredients:

1. a construction of K-theory spaces for scissors congruence problems and morphisms between them,

2. a construction for a quotient of a scissors congruence problem,

3. a construction for the tensor product of two scissors congruence problems, and

4. a construction of the Dehn invariant as a morphism between scissors congruence problems.

This thesis represents the first three of these steps.

In order to put scissors congruence into a K-theoretic framework we move away from the geometry of the problem and create an algebraic formulation of what it means to decompose polytopes. In his book [18], Sah defined a notion of "abstract scissors congruence", an abstract set of axioms that are sufficient to define a scissors congruence group. Inspired by this, we define a "polytope complex" to be a category which contains enough information to encode scissors congruence information. We then use a modified Q-construction to construct a category which encodes the movement of polytopes and which has a K-theory spectrum associated to it. We then prove that K_0 of this category is exactly $\mathcal{P}(X, G)$.

We start by defining an abstract object, which we call a *polytope complex*, which will encode the information of a scissors congruence problem: which objects are isomorphic to which others, and

what the allowed decompositions of objects are. The category of polytope complexes, **PolyCpx**, comes equipped with a functor SC to the category of Waldhausen categories, such that the following theorem holds:

Theorem 1.0.1. *For any polytope complex $\mathcal{C}$, $K_0\mathrm{SC}(\mathcal{C})$ is the free abelian group generated by the objects of $\mathcal{C}$, module the two relations*

$$[a] = \sum_{i \in I} [a_i] \quad \textit{for any decomposition of } a \textit{ into the pieces } \{a_i\}_{i \in I}$$

and

$$[a] = [b] \quad \textit{for any isomorphic objects } a \textit{ and } b.$$

For more details, see theorem 3.2.2. In particular, for the case of polytopes in a manifold X and a subgroup G of isometries of X, our construction gives us a polytope complex $\mathcal{C}_{X,G}$ such that $K_0\mathrm{SC}(\mathcal{C}_{X,G}) \cong \mathcal{P}(X,G)$ (see section 3.3.3).

The Waldhausen categories in the image of SC are constructed combinatorially, so are, on the surface, quite simple. However, most of the standard analytical tools for algebraic K-theory are not available for these categories, as the Waldhausen categories $\mathrm{SC}(\mathcal{C})$ do not come from an exact category (in the sense of Quillen, [17]), do not have a cylinder functor (as in [23], section 1.6) and are not good (in the sense of Toën, [21]). This means that very few computational techniques are directly available for analyzing this problem, as most approaches covered in the literature depend on one of these properties.

In order to construct a quotient of two scissors congruence problems we turn to a direct analysis of the structure of Waldhausen's $S_\bullet$-construction. It turns out that it is possible to duplicate this construction directly on polytope complexes: given a polytope complex $\mathcal{C}$ we can find a polytope complex $s_n\mathcal{C}$ such that $|wS_n\mathrm{SC}(\mathcal{C})| \simeq |w\mathrm{SC}(s_n\mathcal{C})|$. We can make this construction compatible with the simplicial structure maps from Waldhausen's $S_\bullet$-construction, and therefore construct an $S_\bullet$-construction directly on the polytope level.

However, as the $S_\bullet$-construction adds an extra simplicial dimension, it becomes necessary to be able to define the K-theory of a simplicial polytope complex $\mathcal{C}_\bullet$. (We consider a simplicial polytope complex to be a simplicial object in the category of polytope complexes; see section 4.1 for more details.) As the definition of K-theory relies on geometric realizations, we can define $K(\mathcal{C}_\bullet)$ to be the spectrum defined by

$$K(\mathcal{C}_\bullet)_n = |w \underbrace{S_\bullet \cdots S_\bullet}_{n} \mathrm{SC}(\mathcal{C}_\bullet)|.$$

By analyzing the $S_\bullet$-construction on $\mathrm{SC}(\mathcal{C}_\bullet)$ we obtain the following computation of the delooping of the K-theory of $\mathcal{C}_\bullet$:

Theorem 1.0.2. *Let $\mathcal{C}_\bullet$ be a simplicial polytope complex. Let $\sigma\mathcal{C}_\bullet$ be the simplicial polytope complex given by the bar construction. More concretely, we define*

$$(\sigma\mathcal{C}_\bullet)_n = \underbrace{\mathcal{C}_n \vee \mathcal{C}_n \vee \cdots \vee C_n}_{n},$$

with the simplicial structure maps defined in analogously to the usual bar construction. Then $\Omega K(\sigma\mathcal{C}_\bullet) \simeq K(\mathcal{C}_\bullet)$.

See section 4.4 and corollary 4.6.8 for more details. This allows us, among other things, to construct polytope complex models for all spheres $\mathbb{S}^n$ for $n \geq 0$.

The one computational tool for Waldhausen categories which does not depend in any way on extra assumptions is Waldhausen's cofiber theorem, which, given a functor $G : \mathcal{E} \to \mathcal{E}'$ between Waldhausen categories constructs a simplicial Waldhausen category $S.G$ whose K-theory is the cofiber of the map $K(G) : K(\mathcal{E}) \to K(\mathcal{E}')$. By passing this computation down through the polytope complex construction of $S.$ we find the following formula for the cofiber of a morphism of simplicial polytope complexes.

Theorem 1.0.3. *Let $g : \mathcal{C}. \to \mathcal{D}.$ be a morphism of simplicial polytope complexes. We define a simplicial polytope complex $(\mathcal{D}/g).$ by setting*

$$(\mathcal{D}/g)_n = \mathcal{D}_n \vee \underbrace{\mathcal{C}_n \vee C_n \vee \cdots \vee C_n}_{n}.$$

The simplicial structure maps are defined as for $\mathcal{D}. \vee \sigma\mathcal{C}.$, except that ∂_0 is induced by the three morphisms

$$\partial_0 : \mathcal{D}_n \to \mathcal{D}_{n-1} \qquad \partial_0 g_n : \mathcal{C}_n \to \mathcal{D}_{n-1} \qquad 1 : \mathcal{C}^{\vee n-1} \to \mathcal{C}^{\vee n-1}.$$

Then

$$K(\mathcal{C}.) \xrightarrow{K(g)} K(\mathcal{D}.) \to K((\mathcal{D}/g).)$$

is a cofiber sequence of spectra.

See section 4.5 and corollary 4.6.8 for more details. As another corollary, we also get the following result:

Proposition 1.0.4. *Let X and Y be homogeneous geodesic n-manifolds with a preferred open cover in which the geodesic connecting any two points in a single set is unique. If there exist preferred open subsets $U \subseteq X$ and $V \subseteq Y$ and an isometry $\varphi : U \to V$ then the scissors congruence spectra of X and Y are equivalent.*

In order to construct a tensor product of scissors congruence problems we construct a symmetric monoidal structure on the category of polytope complexes in such a way as to mirror the tensor product on the K_0-groups. This gives us the following proposition:

Proposition 1.0.5. *There exists a functor $\wedge : \mathbf{PolyCpx} \times \mathbf{PolyCpx} \longrightarrow \mathbf{PolyCpx}$ that makes the category of polytope complexes into a symmetric monoidal category. For all polytope complexes $\mathcal{C}$ and $\mathcal{D}$,*

$$K_0(\mathcal{C} \wedge \mathcal{D}) \cong K_0(\mathcal{C}) \otimes K_0(\mathcal{D}).$$

With respect to this structure, the functor K takes rings in $\mathbf{PolyCpx}$ to E_∞-ring spectra.

In particular, by applying this to Euclidean or spherical scissors congruence, we can produce ring structures on the spectra associated to these scissors congruence problems that specialize to the ring structures on $\mathcal{P}(E^\infty)$ and $\mathcal{P}(S^\infty)$. This gives us the third ingredient of the Dehn invariant. For more details, see section 5.2.

Chapter 2

Preliminaries

2.1 Notation

We will often be talking about "vertical" and "horizotnal" morphisms: different, non-composable category structures on the same set of objects. In a double category (or a polytope complex) we will denote vertical morphism by dashed arrows $A \dashrightarrow B$ and horizontal morphisms by solid arrows $A \longrightarrow B$. Note that "horizontal" morphisms are not necessarily drawn horizontally, and "vertical" morphisms are not necessarily drawn vertically.

We will often be discussing commutative squares. Sometimes, in order to save space, we will write

$$f, g : (A \longrightarrow B) \longrightarrow (C \longrightarrow D)$$

instead of the commutative square

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
f \downarrow & & \downarrow g \\
C & \longrightarrow & D
\end{array}
$$

Whenever we refer to an n-simplicial category we will always be referring to a functor $(\Delta^{\mathrm{op}})^n \longrightarrow \mathbf{Cat}$, rather than an enriched category. In order to distinguish simplicial objects from non-simplicial objects, we will add a dot as a subscript to a simplicial object; thus $\mathcal{C}$ is a polytope complex, but $\mathcal{C}_{\cdot}$ is a simplicial polytope complex. For any functor F we will write $F^{(n)}$ for the n-fold application of F.

Lastly, the category $\mathbf{Sp}$ will refer to the category of symmetric spectra.

2.2 Categorical Preliminaries

2.2.1 Grothendieck Twists

Definition 2.2.1. Given a category $\mathcal{D}$ we define the contravariant functor

$$F_{\mathcal{D}} : \mathbf{FinSet}^{\mathrm{op}} \longrightarrow \mathbf{Cat} \qquad \text{by} \qquad I \longmapsto \mathcal{D}^I.$$

The *Grothendieck twist of* $\mathcal{D}$, written $\mathrm{Tw}(\mathcal{D})$, is defined to be a Grothendieck construction applied to $F_{\mathcal{D}}$ as follows. We let the objects of $\mathrm{Tw}(\mathcal{D})$ be pairs $I \in \mathbf{FinSet}$, and $x \in \mathcal{D}^I$. A morphism $(I, x) \longrightarrow (J, y)$ in $\mathrm{Tw}(\mathcal{D})$ will be a morphism $I \longrightarrow J \in \mathbf{FinSet}$, together with a morphism $y \longrightarrow F_{\mathcal{D}}(f)(x) \in \mathcal{D}^I$. We will often refer to the function $I \longrightarrow J$ as the *set map* of a morphism.

$\mathrm{Tw}(\mathcal{D})$ is the Grothendieck construction $(\int_{\mathbf{FinSet}^{\mathrm{op}}} F_{\mathcal{D}}^{\mathrm{op}})^{\mathrm{op}}$. More concretely, an object of $\mathrm{Tw}(\mathcal{D})$ is a finite set I and a map of sets $f \colon I \longrightarrow \mathrm{ob}\,\mathcal{D}$; we will write an object of this form as $\{a_i\}_{i\in I}$, with the understanding that $a_i = f(i)$. A morphism $\{a_i\}_{i\in I} \longrightarrow \{b_j\}_{j\in J} \in \mathrm{Tw}(\mathcal{D})$ is a pair consisting of a morphism of finite sets $f \colon I \longrightarrow J$, together with morphisms $F_i \colon a_i \longrightarrow b_{f(i)}$ for all $i \in I$.

In general we will denote a morphism of $\mathrm{Tw}(\mathcal{D})$ by a lower-case letter. By an abuse of notation, we will use the same letter to refer to the morphism's set map, and the upper-case of that letter to refer to the $\mathcal{D}$-components of the morphism (as we did above). If a morphism $f \colon \{a_i\}_{i\in I} \longrightarrow \{b_j\}_{j\in J}$ has its set map equal to the identity on I we will say that f is a *pure $\mathcal{D}$-map*; if instead we have $F_i \colon a_i \longrightarrow b_{f(i)}$ equal to the identity on a_i for every i we will say that f is a *pure set map*.

If we consider an object $\{a_i\}_{i\in I}$ of $\mathrm{Tw}(\mathcal{D})$ to be a formal sum $\sum_{i\in I} a_i$ then we see that $\mathrm{Tw}(\mathcal{D})$ is the category of formal sums of objects in $\mathcal{D}$. Then we have a monoid structure on the isomorphism classes of objects of $\mathrm{Tw}(\mathcal{D})$ (with addition induced by the coproduct). In later sections we will investigate the group completion of this monoid, but for now we will examine the structures which are preserved by this construction.

Much of $\mathrm{Tw}(\mathcal{D})$'s structure comes from "stacking" diagrams of $\mathcal{D}$, so it stands to reason that much of $\mathcal{D}$'s structure would be preserved by this construction. The interesting consequence of this "layering" effect is that even though we have added in formal coproducts, computations with these coproducts can often be reduced to morphisms to singletons. Given any morphism $f \colon \{a_i\}_{i\in I} \longrightarrow \{b_j\}_{j\in J}$ we can write it as

$$\coprod_{j\in J} \left(\quad \{a_i\}_{i\in f^{-1}(j)} \xrightarrow{\ f|_{f^{-1}(j)}\ } \{b_j\} \quad \right)$$

Lemma 2.2.2. *If $\mathcal{D}$ has all pullbacks then $\mathrm{Tw}(\mathcal{D})$ has all pullbacks. The pullback of the diagram*

$$\{a_i\}_{i\in I} \xrightarrow{\ f\ } \{c_k\}_{k\in K} \xleftarrow{\ g\ } \{b_j\}_{j\in J}$$

is $\{a_i \times_{c_k} b_j\}_{(i,j)\in I\times_K J}$.

However, sometimes the categories we will be considering will not be closed under pullbacks. It turns out, however, that if we are simply removing some objects which are "sources" then $\mathrm{Tw}(\mathcal{D})$ will still be closed under pullbacks.

Lemma 2.2.3. *Let $\mathcal{C}$ be a full subcategory of $\mathcal{D}$ which is equal to its essential image, and let $\mathcal{D}'$ be the full subcategory of $\mathcal{D}$ consisting of all objects not in $\mathcal{C}$. Suppose that $\mathcal{C}$ has the property that for any $A \in \mathcal{C}$, if $\mathrm{Hom}(B, A) \neq \varnothing$ then $B \in \mathcal{C}$. Then if $\mathcal{D}$ has all pullbacks, so does $\mathrm{Tw}(\mathcal{D}')$.*

Proof. Let $U \colon \mathrm{Tw}(\mathcal{D}') \longrightarrow \mathrm{Tw}(\mathcal{D})$ be the inclusion induced by the inclusion $\mathcal{D}' \longrightarrow \mathcal{D}$. We define a projection functor $P \colon \mathrm{Tw}(\mathcal{D}) \longrightarrow \mathrm{Tw}(\mathcal{D}')$ by $P(\{a_i\}_{i\in I}) = \{a_i\}_{i\in I'}$, where $I' = \{i \in I \mid a_i \notin \mathcal{C}\}$.

Suppose that we are given a diagram $A \longrightarrow C \leftarrow B$ in $\mathrm{Tw}(\mathcal{D}')$. Let X be the pullback of $UA \longrightarrow UC \leftarrow UB$ in $\mathrm{Tw}(\mathcal{D})$; we claim that PX is the pullback of $A \longrightarrow C \leftarrow B$ in $\mathrm{Tw}(\mathcal{D}')$. Indeed, suppose we have a cone over our diagram with vertex D, then UD will factor through X, and thus $PUD = D$ will factor through PX. Checking that this factorization is unique is trivial. $\qquad\square$

We finish up this section with a quick result about pushouts. It's clear that $\mathrm{Tw}(\mathcal{D})$ has all finite coproducts, since we compute it by simply taking disjoint unions of indexing sets. However, it turns out that a lot more is true.

Lemma 2.2.4. *If $\mathcal{D}$ has all finite connected colimits then* $\mathrm{Tw}(\mathcal{D})$ *has all pushouts.*

Proof. Consider a morphism $f\colon \{a_i\}_{i\in I} \longrightarrow \{b_j\}_{j\in J} \in \mathrm{Tw}(\mathcal{D})$. We can factor f as a pure $\mathcal{D}$-map followed by a pure set map. Thus to show that $\mathrm{Tw}(\mathcal{D})$ contains all pushouts it suffices to show that $\mathrm{Tw}(\mathcal{D})$ contains all pushouts along pure set maps and pure $\mathcal{D}$-maps separately.

Now suppose that we are given a diagram

$$\{c_k\}_{k\in K} \xleftarrow{\quad g \quad} \{a_i\}_{i\in I} \xrightarrow{\quad f \quad} \{b_j\}_{j\in J}$$

It suffices to show that the pushout exists whenever g is a pure set-map or a pure $\mathcal{D}$-map. Suppose that g is a pure set map. For $x \in J \cup_I K$ we will write I_x (resp. J_x, K_x) for those elements in I (resp. J, K) which map to x under the pushout morphisms. The pushout of the above diagram in this case will be $\{d_x\}_{x\in J\cup_I K}$, where d_x is defined to be the colimit of the following diagram (if it exists in $\mathcal{D}$). The diagram will have objects a_i, b_j, c_k for all $i \in I_x$, $j \in J_x$ and $k \in K_x$. There will be an identity morphism $a_i \longrightarrow c_{g(i)}$ and a morphism $F_i\colon a_i \longrightarrow b_{f(i)}$ for all $i \in I_x$. Note that this colimit must be connected, since otherwise x wouldn't be a single element in $J \cup_I K$.

If g is a pure $\mathcal{D}$-map the pushout of this diagram will be $\{d_j\}_{j\in J}$, where d_j is defined to be the colimit of the diagram

$$\coprod_{i\in I_j} c_i \xleftarrow{\coprod_{i\in I_j} G_i} \coprod_{i\in I_j} a_i \xrightarrow{\coprod_{i\in I_j} F_i} b_j \qquad\qquad I_j = f^{-1}(j).$$

which exists as the diagram is connected. (Also, while we wrote the above diagrams using coproducts, they do not actually need to exist in $\mathcal{D}$. In that case, we just expand the coproduct in the diagram into its components to produce a diagram whose colimit exists in $\mathcal{D}$.) $\qquad\square$

Remark. In order for $\mathcal{D}$ to contain all finite connected colimits it suffices for it to contain all pushouts and all coequalizers. If $\mathcal{D}$ has all pushouts (but not necessarily all coequalizers) then examination of the proof above shows that $\mathrm{Tw}(\mathcal{D})$ must be closed under all pushouts along morphisms with injective set maps.

2.2.2 Double Categories

We will be using the notion of double categories originally introduced by Ehresmann in [6]; we follow the conventions used by Fiore, Paoli and Pronk in [7].

Definition 2.2.5. A *small double category* $\mathcal{C}$ is a set of objects $\mathrm{ob}\,\mathcal{C}$ together with two sets of morphisms $\mathrm{Hom}_v(A, B)$ and $\mathrm{Hom}_h(A, B)$ for each pair of objects $A, B \in \mathrm{ob}\,\mathcal{C}$, which we will call the *vertical* and *horizontal* morphisms. We will draw the vertical morphisms as dashed arrows, and the horizontal morphisms as solid arrows. $\mathcal{C}$ with only the morphisms from the vertical (resp. horizontal) set forms a category which will be denoted $\mathcal{C}_v$ (resp. $\mathcal{C}_h$).

In addition, a double category contains the data of "commutative squares", which are diagrams

$$\begin{array}{ccc} A & \xrightarrow{\sigma} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ C & \xrightarrow{\tau} & D \end{array}$$

which indicate that "$q\sigma = \tau p$". Commutative squares have to satisfy certain composition laws, which we omit here as they simply correspond to the intuition that they should behave just like commutative squares in any ordinary category.

Given two small double categories $\mathcal{C}$ and $\mathcal{D}$, a *double functor* $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a pair of functors $F_v\colon \mathcal{C}_v \longrightarrow \mathcal{D}_v$ and $F_h\colon \mathcal{C}_h \longrightarrow \mathcal{D}_h$ which takes commutative squares to commutative squares. We will denote the category of small double categories by **DblCat**.

Remark. A small double category is an internal category object in **Cat**. We do not use this definition here, however, since it obscures the inherent symmetry of a double category.

In general we will label vertical morphisms with Latin letters and horizontal morphisms with Greek letters. We will also say that a diagram consisting of a mix of horizontal and vertical morphisms commutes if any purely vertical (resp. horizontal) component commutes, and if all components mixing the two types of maps consists of squares that commute in the double category structure.

Now suppose that $\mathcal{C}$ is a small double category. We can define a double category $\mathrm{Tw}(\mathcal{C})$ by letting $\mathrm{ob}\,\mathrm{Tw}(\mathcal{C}) = \mathrm{ob}\,\mathrm{Tw}(\mathcal{C}_h)$ (which are the same as $\mathrm{ob}\,\mathrm{Tw}(\mathcal{C}_v)$ so there is no breaking of symmetry). We define the vertical morphisms to be the morphisms of $\mathrm{Tw}(\mathcal{C}_v)$ and the horizontal morphisms to be the morphisms of $\mathrm{Tw}(\mathcal{C}_h)$. In addition, we will say that a square

$$
\begin{array}{ccc}
\{a_i\}_{i\in I} & \xrightarrow{\ \sigma\ } & \{b_j\}_{j\in J} \\
\Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle q} \\
\{c_k\}_{k\in K} & \xrightarrow{\ \tau\ } & \{d_l\}_{l\in L}
\end{array}
$$

commutes if for every $i \in I$ the square

$$
\begin{array}{ccc}
a_i & \xrightarrow{\ \Sigma_i\ } & b_{\sigma(i)} \\
\Big\downarrow{\scriptstyle P_i} & & \Big\downarrow{\scriptstyle Q_{\sigma(i)}} \\
c_{p(i)} & \xrightarrow{\ T_{p(i)}\ } & d_{\tau(p(i))}
\end{array}
$$

commutes. It is easy to check that with this definition $\mathrm{Tw}(\mathcal{C})$ forms a double category as well, and in fact that Tw is a functor **DblCat** $\longrightarrow$ **DblCat**.

2.2.3 Multicategories

Definition 2.2.6. A *multicategory* $\mathcal{M}$ is the following information:

1. a class of objects $\mathrm{ob}\,\mathcal{M}$,

2. for each $k \geq 0$ and each $k+1$-tuple of objects $A_1, \ldots, A_k, B \in \mathrm{ob}\,\mathcal{M}$, written $\mathcal{M}(A_1, \ldots, A_k; B)$, called k-morphisms from $(A_1, \ldots, A_k)$ to B,

3. for each $k \geq 0$ and $n_1, \ldots, n_k \geq 0$ and objects $A_i, A_i^{(j)}, B \in \mathrm{ob}\,\mathcal{M}$ a function

$$
\mathcal{M}(A_1, \ldots, A_k; B) \times \prod_{i=1}^{k} \mathcal{M}(A_1^{(i)}, \ldots, A_{n_i}^{(i)}; A_i) \longrightarrow \mathcal{M}(A_1^{(1)}, \ldots, A_{n_k}^{(k)}; B)
$$

called *composition*,

4. for each $A \in \mathrm{ob}\,\mathcal{M}$ an identity element $1_A \in \mathcal{M}(A; A)$, and

5. for each $\mathcal{M}(A_1, \ldots, A_k; B)$ and each $\sigma \in \Sigma_k$ a "σ-action"

$$\sigma \cdot : \mathcal{M}(A_1, \ldots, A_k; B) \longrightarrow \mathcal{M}(A_{\sigma(1)}, \ldots, A_{\sigma(k)}; B).$$

The composition and action of Σ_k must satisfy certain associativity and coherence axioms. Given multicategories $\mathcal{M}$ and $\mathcal{M}'$, a multifunctor $F \colon \mathcal{M} \longrightarrow \mathcal{M}'$ is a function $f \colon \mathrm{ob}\,\mathcal{M} \longrightarrow \mathrm{ob}\,\mathcal{M}'$ together with a function

$$\mathcal{M}(A_1, \ldots, A_k; B) \longrightarrow \mathcal{M}(f(A_1), \ldots, f(A_k); f(B))$$

for each $k \geq 0$ and $k + 1$-tuple $A_1, \ldots, A_k, B$.

For more details on multicategories see, for example, [13], 2.1-2.2. (Note that what we term multicategories he calls *symmetric multicategories*.)

Example 2.2.7. Any symmetric monoidal category $(\mathcal{C}, \otimes, I)$ can be considered a multicategory by defining

$$\mathcal{C}(A_1, \ldots, A_k; B) = \mathrm{Hom}_{\mathcal{C}}(A_1 \otimes \cdots \otimes A_k, B).$$

Given two symmetric monoidal categories $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \boxtimes, I')$ the multifunctors between $\mathcal{C}$ and $\mathcal{D}$ are exactly the lax symmetric monoidal functors between $\mathcal{C}$ and $\mathcal{D}$. (See, for example, [13] example 2.1.10.)

2.3 Waldhausen K-theory

2.3.1 The K-theory of a Waldhausen category

This section contains a brief review of Waldhausen's $S.$ construction for K-theory, originally introduced in [23], as well as some results which are surely well-known to experts, but for which we could not find a reference.

Definition 2.3.1. A *Waldhausen category* is a small pointed category $\mathcal{W}$, together with two distinguished subcategories $c\mathcal{W}$ and $w\mathcal{W}$. The morphisms in $c\mathcal{W}$ are called the *cofibrations*, and the morphisms in $w\mathcal{W}$ are called the *weak equivalences*; these will be denoted $\hookrightarrow$ and $\xrightarrow{\sim}$. The category $\mathcal{W}$ satisfies the following extra conditions:

- the point in $\mathcal{W}$ is a zero object 0,

- both $c\mathcal{W}$ and $w\mathcal{W}$ contain all isomorphisms of $\mathcal{W}$,

- the morphism $0 \hookrightarrow A$ is a cofibration for all $A \in \mathcal{W}$,

- for any diagram $C \longleftarrow A \hookrightarrow B$ the pushout exists, and the induced morphism $C \hookrightarrow C \cup_A B$ is a cofibration, and

- for any diagram

$$
\begin{array}{ccccc}
C & \longleftarrow & A & \hookrightarrow & B \\
\downarrow{\scriptstyle \wr} & & \downarrow{\scriptstyle \wr} & & \downarrow{\scriptstyle \wr} \\
C' & \longleftarrow & A' & \hookrightarrow & B'
\end{array}
$$

the induced morphism $C \cup_A B \xrightarrow{\sim} C' \cup_{A'} B'$ is also a weak equivalence.

A functor $\mathcal{W} \longrightarrow \mathcal{W}'$ between Waldhausen categories is said to be *exact* if it preserves 0, cofibrations, weak equivalences and pushouts. We denote the category of Waldhausen categories by **WaldCat**.

Given a Waldhausen category $\mathcal{W}$, we define $S_n\mathcal{W}$ to be the category of commutative triangles defined as follows. An object A is a triangle of objects A_{ji} for pairs $0 \leq i < j \leq n$. The diagram consists of cofibrations $A_{ji} \hookrightarrow A_{(j+1)i}$ and morphisms $A_{ji} \longrightarrow A_{j(i+1)}$ such that for every $i < j < k$ the diagram

$$A_{ji} \hookrightarrow A_{ki} \longrightarrow A_{kj}.$$

is a cofiber sequence. A morphism $\varphi : A \longrightarrow B$ consists of morphisms $\varphi_{ji} : A_{ij} \longrightarrow B_{ji}$ making the induced diagram commute. Note that $S_0\mathcal{W}$ is the trivial category with one object and one morphism, and $S_1\mathcal{W} = \mathcal{W}$.

The $S_n\mathcal{W}$'s assemble into a simplicial object in categories by letting the k-th face map remove all objects A_{ij} with $i = k$ or $j = k + 1$, and the k-th degeneracy repeat a row and column appropriately. We can assemble the $S_n\mathcal{W}$'s into a simplicial Waldhausen category in the following manner. A morphism $\varphi : A \longrightarrow B \in S_n\mathcal{W}$ is a weak equivalence if φ_{ij} is a weak equivalence for all $i < j$. φ is a cofibration if for all $i < j$ the induced morphism

$$B_{ij} \cup_{A_{ij}} A_{(i+1)j} \longrightarrow B_{(i+1)j}$$

is a cofibration in $\mathcal{W}$. Note that this means that in particular for all $i < j$ the morphism φ_{ij} is a cofibration in $\mathcal{W}$.

We obtain the K-theory spectrum of a Waldhausen category $\mathcal{W}$ by defining

$$K(E)_n = \Omega \left| wS_{\cdot}^{(n)}\mathcal{W} \right|.$$

From proposition 1.5.3 in [23] we know that above level 0 this will be an Ω-spectrum.

We now turn our attention to some tools for computing with Waldhausen categories. An exact functor of Waldhausen categories $F : \mathcal{W} \longrightarrow \mathcal{W}'$, naturally yields a functor between $S_{\cdot}$ constructions, and therefore between the K-theory spectra. We are interested in several cases of such functors which produce equivalences on the K-theory level.

The first example we consider will be simply an inclusion of a subcategory. While a Waldhausen category can contain a lot of morphisms which are neither cofibrations nor weak equivalences, most of these are not important. We will say that a Waldhausen subcategory $\mathcal{W}'$ of a Waldhausen category $\mathcal{W}$ is a *simplification of $\mathcal{W}$* if it contains all objects, weak equivalences, and cofibrations of $\mathcal{W}$.

Lemma 2.3.2. *If $\mathcal{W}'$ is a simplification of $\mathcal{W}$ then the inclusion $\mathcal{W}' \longrightarrow \mathcal{W}$ induces the identity map on K-theory.*

Proof. This is true by simple observation of the definition of the K-theory of a Waldhausen category. On a Waldhausen category, the $S_{\cdot}$-construction uses only cofibrations in the definitions of the objects. As the cofibrations in $S_{\cdot}\mathcal{W}$ are in particular levelwise cofibrations, this means that for all $n \geq 0$, in $S_{\cdot}^{(n)}\mathcal{W}$ all morphisms in every diagram representing an object will be either cofibrations or cofiber maps. Thus all of the objects of $S_{\cdot}^{(n)}\mathcal{W}$ will be objects of $S_{\cdot}^{(n)}\mathcal{W}'$.

In order to obtain the n-th space of $K(\mathcal{W})$ we look at the geometric realization of $wS_{\cdot}^{(n)}\mathcal{W}$. Every k-simplex of this consists of a diagram, each of whose morphisms is either a cofibration, cofiber map, or weak equivalence. We know that all weak equivalences of $\mathcal{W}$ are in $\mathcal{W}'$, and thus every simplex of $K(\mathcal{W})_n$ is in $K(\mathcal{W}')_n$, which means that the natural inclusion is actually the identity morphism, as desired. $\qquad\square$

20

Note that there exists a minimal simplification of $\mathcal{W}$, as given any family of simplifications $\{\mathcal{W}_\alpha\}_{\alpha \in A}$, the category $\bigcap_{\alpha \in A} \mathcal{W}_\alpha$ will also be a simplification of $\mathcal{W}$.

Now we consider pairs of adjoint functors between Waldhausen categories. Suppose that we have an adjoint pair of exact functors $F \colon \mathcal{W} \rightleftarrows \mathcal{W}' \colon G$; these produce a pair of maps $K(F) \colon K(\mathcal{W}) \rightleftarrows K(\mathcal{W}') \colon K(G)$. Generally an adjoint pair of functors produces a homotopy equivalence on the classifying space level, so naively we might expect these to be inverse homotopy equivalences. Unfortunately, in the $S_{\cdot}$ construction we always restrict our attention to weak equivalences in the category, so we need more information than just an adjoint pair of exact functors. If both the unit and counit of our adjunction is a weak equivalence then we are fine, however, as the adjunction must also restrict to an adjunction on the subcategories of weak equivalences. We call an adjoint pair of exact functors satisfying this extra condition an *exact adjoint pair*, and we say that F is *exactly left adjoint* to G. Given any exact adjoint pair we get a pair of inverse equivalences on the K-theory level.

Lemma 2.3.3. *An exact adjunction induces an adjoint pair of functors $wF \colon w\mathcal{W} \rightleftarrows w\mathcal{W}' \colon wG$, and for all $n \geq 0$ the adjunction $S_n F \colon S_n \mathcal{W} \rightleftarrows S_n \mathcal{W}' \colon S_n G$ is also an exact adjunction.*

In such a case we sometimes say that F is *exactly adjoint* to G. When such an adjunction is an equivalence, we call it an *exact equivalence*. Note that any equivalence which is exact in both directions is an exact equivalence, as all isomorphisms are weak equivalences.

Proof. As F and G are exact we know that wF and wG are well-defined. In order to see that they are adjoint, note that the existence of a unit and counit are sufficient; as the unit and counit are natural weak equivalences they pass to natural transformations inside $w\mathcal{W}$ and $w\mathcal{W}'$, and thus give us the adjunction, as desired.

Now we need to show that an exact adjunction induces an exact adjunction on S_n. As an exact functor passes to an exact functor on the S_n-level all that we must show is that the two functors $S_n F$ and $S_n G$ will be adjoint. However, as both $S_n \mathcal{W}$ and $S_n \mathcal{W}'$ are diagram categories, with $S_n F$ and $S_n G$ defined levelwise, the adjunction follows directly from the adjunction between F and G. (The unit and counit will be defined levelwise. So we are done. $\qquad\square$

This lemma implies that for every n we get an induced pair of adjoint functors

$$wS_n^{(n)}\mathcal{W} \leftrightarrows wS_n^{(n)}\mathcal{W}',$$

and thus a levelwise homotopy equivalence between the K-theory spectra. Thus we can conclude the following corollary:

Corollary 2.3.4. *An exact adjunction induces a homotopy equivalence between the K-theory spectra of the Waldhausen categories.*

Now suppose that $\mathcal{W}'$ is a subcategory of $\mathcal{W}$ with the property that any morphism $f \in \mathcal{W}$ can be factored as hg, with h an isomorphism and $g \in \mathcal{W}'$, and such that $\mathcal{W}'$ contains the zero object of $\mathcal{W}$. Then $\mathcal{W}'$ is a Waldhausen category. Let $\widehat{S}_n \mathcal{W}$ be the full subcategory of $S_n \mathcal{W}$ containing all objects in $S_n \mathcal{W}'$. Then the following lemma shows that $\widehat{S}_n \mathcal{W}$ is an equivalent Waldhausen subcategory of $S_n \mathcal{W}$, and thus that for all $n \geq 1$,

$$\left| wS_{\cdot}^{(n-1)} \widehat{S}_{\cdot} \mathcal{W} \right| \simeq \left| wS_{\cdot}^{(n)} \mathcal{W} \right|.$$

Lemma 2.3.5. *Suppose that $\mathcal{W}'$ is a subcategory of $\mathcal{W}$ with the property that given any morphism $f : A \longrightarrow B$ in $\mathcal{W}$, there exists a factorization $f = hg$ where h is an isomorphism and $g \in \mathcal{W}'$. If we let $\widetilde{S}_n\mathcal{W}$ be the full subcategory of $S_n\mathcal{W}$ containing all those objects from $S_n\mathcal{W}'$ then $\widetilde{S}_n\mathcal{W}$ is exactly equivalent to $S_n\mathcal{W}$.*

Note that $\mathcal{W}'$ automatically inherits a Waldhausen structure from $\mathcal{W}$.

Proof. It suffices to show that every object of $S_n\mathcal{W}$ will be isomorphic to an object from $S_n\mathcal{W}'$. The condition on $\mathcal{W}'$ ensures that $\mathcal{W}'$ contains all objects of $\mathcal{W}$, as for any object $A \in \mathcal{W}$ if we factor the identity morphism into hg as given in the statement, $g \in \mathcal{W}'$ which means that $A \in \mathcal{W}'$.

Note that it suffices to show that we can replace the longest row of cofibrations by cofibrations in $\mathcal{W}'$, as any two objects of $S_n\mathcal{W}$ which are equal on the first line are isomorphic. As $\mathcal{W}'$ is a Waldhausen category, if we have an object of $S_n\mathcal{W}$ with first row from $S_n\mathcal{W}'$, we must have some object in $S_n\mathcal{W}'$ which is isomorphic to it. Thus it now remains to show that given a diagram

$$A_1 \stackrel{\iota_1}{\hookrightarrow} A_2 \stackrel{\iota_2}{\hookrightarrow} \cdots \stackrel{\iota_{n-1}}{\hookrightarrow} A_n$$

in $\mathcal{W}$ there exists an isomorphism of diagrams to such a diagram in $\mathcal{W}'$.

We will show that given such a diagram where $\iota_1, \ldots, \iota_{k-1} \in \mathcal{W}'$ there exists an isomorphic diagram where $\iota_1', \ldots, \iota_k'$ are in $\mathcal{W}'$. The base case where $k = 1$ is obvious. Assuming that we have the case for $k - 1$, factor ι_k into $g : A_k \longrightarrow A'$ and $h : A' \longrightarrow A_{k+1}$. The following diagram shows that we have the case for k:

$$
\begin{array}{ccccccccccccc}
A_1 & \stackrel{\iota_1}{\hookrightarrow} & \cdots & \stackrel{\iota_{k-1}}{\hookrightarrow} & A_k & \stackrel{g}{\hookrightarrow} & A' & \stackrel{\iota_{k+1}h}{\hookrightarrow} & A_{k+2} & \stackrel{\iota_{k+2}}{\hookrightarrow} & \cdots & \stackrel{\iota_{n-1}}{\hookrightarrow} & A_n \\[2pt]
\| & & & & \| & & \downarrow h & & \| & & & & \| \\[2pt]
A_1 & \stackrel{\iota_1}{\hookrightarrow} & \cdots & \stackrel{\iota_{k-1}}{\hookrightarrow} & A_k & \stackrel{\iota_k}{\hookrightarrow} & A_{k+1} & \stackrel{\iota_{k+1}}{\hookrightarrow} & A_{k+2} & \stackrel{\iota_{k+2}}{\hookrightarrow} & \cdots & \stackrel{\iota_{n-1}}{\hookrightarrow} & A_n
\end{array}
$$

So we are done. $\qquad\square$

We finish up this section with a short discussion of a simplification of the $S_\bullet$ construction. S_n can be defined more informally as the category whose objects are all choices of $n - 1$ composable cofibrations, together with the choices of all cofibers. As the cofiber of a cofibration $A \hookrightarrow B \in \mathcal{W}$ is a pushout, any object $A \in S_n\mathcal{W}$ is defined, up to isomorphism, by the diagram

$$A_{11} \hookrightarrow A_{21} \hookrightarrow \cdots \hookrightarrow A_{n1},$$

and any morphism φ by its restriction to this row. We will denote the category of such objects $F_n\mathcal{W}$. We can clearly make $F_n\mathcal{W}$ into a Waldhausen category in a way analogous to the way we made $S_n\mathcal{W}$ into a Waldhausen category. However, these do not assemble easily into a simplicial Waldhausen category, as ∂_0, the 0-th face map, must take cofibers, and this is only defined up to isomorphism. In order for these to assemble into a simplicial category we need, for every cofibration $A \hookrightarrow B$ a functorial choice of cofiber B/A in such a way that for any composition $A \hookrightarrow B \hookrightarrow C$ we have

$$(C/A)/(B/A) = C/B.$$

Then this choice of cofiber assembles the $F_\bullet\mathcal{W}$ into a well-defined simplicial category, and we can compute the K-theory of $\mathcal{W}$ using the $F_\bullet$ construction instead of the $S_\bullet$ construction.

2.3.2 K-Theory as a multifunctor

We want to show that K-theory is a symmetric monoidal functor **WaldCat** $\longrightarrow$ **Sp**; however, as **WaldCat** is not a monoidal category this is clearly not possible. However, it turns out that **WaldCat** can be given the structure of a multicategory in the following manner.

Definition 2.3.6. A functor $F\colon \mathcal{W}_1 \times \cdots \times \mathcal{W}_k \longrightarrow \mathcal{W}$ is k-exact if it satisfies the following conditions:

- For any $1 \leq i \leq k$ and any objects $A_1, \ldots, \widehat{A}_i, \ldots, A_k$ the functor

$$F(A_1, \ldots, \cdot, \ldots, A_k)\colon \mathcal{W}_i \longrightarrow \mathcal{W}$$

  given by fixing the given $k-1$ coordinates is exact.

- For any k cofibrations $A_i \hookrightarrow A_i' \in \mathcal{W}_i$ and any subset $S \subseteq \{1, \ldots, k\}$ define the object $A^S \in \mathcal{W}_1 \times \cdots \times \mathcal{W}_k$ to have the i-th coordinate be A_i if $i \notin S$ and A_i' if $i \in S$. Then for any subset $T \subseteq \{1, \ldots, k\}$ the induced morphism

$$\operatorname*{colim}_{S \subsetneq T} F(A^S) \longrightarrow F(A^T)$$

  is a cofibration.

We define the k-morphisms of **WaldCat** to be exactly the k-exact functors; we have Σ_k act on them by permuting the in-coordinates. For more details on this, see [2]. The goal of this section is to prove the following proposition:

Proposition 2.3.7. *The functor $K\colon$ **WaldCat** $\longrightarrow$ **Sp** is a multifunctor.*

In order to show that K is a multifunctor we need to show that any k-exact functor $F\colon \mathcal{W}_1 \times \cdots \times \mathcal{W}_k \longrightarrow \mathcal{W}$ gives rise to a morphism $K(\mathcal{W}_1) \wedge \cdots \wedge K(\mathcal{W}_k) \longrightarrow K(\mathcal{W})$. In the interest of simplifying the following analysis, we will restrict our attention to the case when $k = 2$; the other cases follow analogously. The data of a 2-morphism is, for every pair m_1, m_2, a morphism of spaces

$$\mu_{m_1, m_2}\colon K(\mathcal{W}_1)_{m_1} \wedge K(\mathcal{W}_2)_{m_2} \longrightarrow K(\mathcal{W})_{m_1+m_2}.$$

These spaces need to be coherent with respect to the spectral structure maps; in particular, we need the following diagram to commute:

For a Waldhausen category $\mathcal{W}$ and $0 \leq i \leq n$ we define a functor $\rho_{ni}\colon \mathcal{W} \longrightarrow S_n\mathcal{W}$, which is defined on objects by

$$\rho_{ni}(A)_{jk} = \begin{cases} * & \text{if } j \leq n - i \text{ or } k \geq i, \\ A & \text{otherwise} \end{cases}$$

and extends in the analogous manner to morphisms. Let S^1 be the pointed simplicial set which at level n is equal to the set $\{0, 1, \ldots, n\}$; we can also consider S^1 to be a pointed category with only trivial morphisms. Then we have a morphism of simplicial categories $P\colon \mathcal{W} \times S^1 \longrightarrow S.\mathcal{W}$ $(A, i) \longmapsto \rho_{ni}(A)$.

Lemma 2.3.8. *P a well-defined functor of simplicial categories.*

Proof. In order for P to be well-defined we need to show that the image of P is in $S.\mathcal{W}$, and that P is coherent with the simplicial maps. The first part of this is obvious, since ρ_{ni} is constructed to be a valid element of $S_n\mathcal{W}$. For the second part, note that we have

$$\partial_j \rho_{ni}(A) = \begin{cases} \rho_{(n-1)0}(A) & \text{if } j = 0 \text{ and } i = n \text{ or } j = n \text{ and } i = 1, \\ \rho_{(n-1)i}(A) & \text{if } j \leq n - i \text{ and } i \neq n \\ \rho_{(n-1)(i-1)}(A) & \text{if } j > n - i \end{cases} \tag{2.1}$$

$$= \rho_{(n-1)(\partial_j(i))}(A), \tag{2.2}$$

where in the right-hand side of the above, $i \in S_n^1$. Analogously,

$$s_j \rho_{ni}(A) = \begin{cases} \rho_{(n+1)i} & \text{if } j \leq n - i \\ \rho_{(n+1)(i+1)} & \text{if } j > n - i \end{cases} = \rho_{(n-1)(s_j(i))}(A),$$

so we are done $\qquad\qquad\square$

We thus have functors
$$P\colon S_.^{(m)}\mathcal{W} \times S^1 \longrightarrow S_.^{(m+1)}\mathcal{W}.$$

Note that by definition, if either $i = 0$ or $A = *$ $P(A, i) = *$, so P lifts to a map
$$P\colon NwS_.^{(m)}\mathcal{W} \wedge S^1 \longrightarrow NwS_.^{(m+1)}\mathcal{W}.$$

This is the spectral structure map of the K-theory of a symmetric spectrum.

Now consider a biexact functor $F\colon \mathcal{W}_1 \times \mathcal{W}_2 \longrightarrow \mathcal{W}$. We want to use F to construct morphisms $\mu_{m_1, m_2}\colon K(\mathcal{W}_1)_{m_1} \wedge K(\mathcal{W}_2)_{m_2} \longrightarrow K(\mathcal{W})_{m_1+m_2}$. To do this, we will first reexamine the $S.$ construction to make it easier to analyze.

Let $[n]$ be the ordered set $0 < 1 < \cdots < n$, considered as a pointed category (with 0 as the distinguished basepoint), and let $\text{Ar}\,[n]$ be the arrow category of $[n]$; we will denote an object in $\text{Ar}\,[n]$ as $j < i$. For a vector $\vec{n} = (n_1, \ldots, n_m)$ we will write $[\vec{n}] = [n_1] \times \cdots \times [n_m]$. We can think of an object of $S_n\mathcal{W}$ as a functor $X\colon \text{Ar}\,[n] \longrightarrow \mathcal{W}$ satisfying the extra conditions that $X(i = i) = *$ for all i, $X(i < j) \longrightarrow X(i < k)$ is a cofibration for all $i < j < k$ and any commutative square

$$
\begin{array}{ccc}
X(i < j) & \longrightarrow & X(i < k) \\
\downarrow & & \downarrow \\
X(j = j) & \longrightarrow & X(j < k)
\end{array}
$$

is a pushout square; from this perspective, $S_n\mathcal{W}$ is a full subcategory of the category of functors $\text{Ar}\,[n] \longrightarrow \mathcal{W}$. From this perspective, the simplicial structure on $S.\mathcal{W}$ is induced from the simplicial structure on $[\text{Ar}_., \mathcal{W}]$. More generally, $S_{n_1} \cdots S_{n_m}\mathcal{W}$ is naturally a full subcategory of the category of functors

$$X\colon \text{Ar}\,([n_1] \times \cdots \times [n_m]) \longrightarrow \mathcal{W},$$

24

satisfying conditions analogous to the condition above. (See [2], section 2, for more details.) The key fact we need about the objects of $S_{n_1} \cdots S_{n_m} \mathcal{W}$ is that they will be preserved by biexact functors in the following manner. Consider the composition

$$S_{\vec{n}_1}^{(m_1)} \mathcal{W}_1 \times S_{\vec{n}_2}^{(m_2)} \mathcal{W}_2 \longrightarrow \left[\mathrm{Ar}\,[\vec{n}_1], \mathcal{W}_1\right] \times \left[\mathrm{Ar}\,[\vec{n}_2], \mathcal{W}_2\right] \longrightarrow$$
$$\left[\mathrm{Ar}\,([\vec{n}_1] \times [\vec{n}_2]), \mathcal{W}_2 \times \mathcal{W}_2\right] \xrightarrow{\ F\circ\ } \left[\mathrm{Ar}\,([\vec{n}_1] \times [\vec{n}_2]), \mathcal{W}\right].$$

The key extra condition on the objects of $S_{\vec{n}_1 \vec{n}_2}^{(m_1+m_2)} \mathcal{W}$ is that this functor lands in $S_{\vec{n}_1 \vec{n}_2}^{(m_1+m_2)} \mathcal{W}$. By varying the coordinates of $\vec{n}_1$ and $\vec{n}_2$ these assemble into exact functors

$$S_{\cdot}^{(m_1)} \mathcal{W}_1] \times S_{\cdot}^{(m_2)} \mathcal{W}_2 \longrightarrow S_{\cdot}^{(m_1+m_2)} \mathcal{W}.$$

Applying $|Nw \cdot|$ to these and noting that any point with the basepoint as one of the coordinates gets mapped to the basepoint, we get maps

$$\mu_{m_1, m_2} \colon K(\mathcal{W}_1)_{m_1} \wedge K(\mathcal{W}_2)_{m_2} \longrightarrow K(\mathcal{W})_{m_1+m_2},$$

as desired.

In order to check these assemble into a map $K(\mathcal{W}_1) \wedge K(\mathcal{W}_2) \longrightarrow K(\mathcal{W})$ we that these satisfy the coherence conditions stated earlier. In order to show this, we will show that the following diagram commutes:

In fact, all of the morphisms except for the two horizontal morphisms are obtained by postcomposing functors $\mathrm{Ar}\,([\vec{n}_1] \times [\vec{n}_2]) \longrightarrow s\mathbf{Cat}$ with P or F. The horizontal morphisms, on the other hand, permute both source and target categories, and then permute the source categories back; everything in between is, once again, postcomposing with P or F. Thus in order for this diagram to commute it suffices to show that the diagram

commutes. Consider a triple $(A_1, A_2, i) \in W_1 \times W_2 \times S_n^1$. To check that the diagram commutes, we need to show that

$$\rho_{ni}(F(A_1, A_2)) = F(A_1, \rho_{ni}(A_2)) = F(\rho_{ni}(A_1), A_2).$$

25

Looking at each of these at spot jk we have that if $j \le n - i$ or $k \ge i$, the first is $*$, the second is $F(A_1, *)$ and the third is $F(*, A_2)$, which are all equal because F is biexact. Otherwise, these are all equal to $F(A_1, A_2)$, so are again all equal. So these diagrams commute on objects. Analogously, they commute on all morphisms.

This completes the proof of proposition 2.3.7.

Chapter 3

Scissors congruence spectra

3.1 Abstract Scissors Congruence

In this section we will deal with scissors congruence of abstract objects.

Definition 3.1.1. A *polytope complex* is a double category $\mathcal{C}$ satisfying the following properties:

(V) Vertically, $\mathcal{C}$ is a preorder which has a unique initial object and is closed under pullbacks. In addition, $\mathcal{C}$ has a Grothendieck topology.

(H) Horizontally, $\mathcal{C}$ is a groupoid.

(P) For any pair of morphisms $P\colon B' \dashrightarrow B$ and $\Sigma\colon A \longrightarrow B$, where P is vertical and Σ horizontal, there exists a unique commutative square

$$
\begin{array}{ccc}
\Sigma^* B' & \longrightarrow & B' \\
{\scriptstyle \Sigma^* P} \downarrow & & \downarrow {\scriptstyle P} \\
A & \xrightarrow{\ \Sigma\ } & B
\end{array}
$$

wheich we will call a *mixed pullback*. The functor $\Sigma^*\colon (\mathcal{C}_v \downarrow B) \longrightarrow (\mathcal{C}_v \downarrow A)$ is an equivalence of categories.

(C) If $\{X_\alpha \dashrightarrow X\}_{\alpha \in A}$ is a set of vertical morphisms which is a covering family of X, and $\Sigma\colon Y \longrightarrow X$ is any horizontal morphism, then $\{\Sigma^* X_\alpha \dashrightarrow Y\}_{\alpha \in A}$ is a covering family of Y.

(E) If $\{X_\alpha \dashrightarrow X\}_{\alpha \in A}$ is a covering family such that for some $\alpha_0 \in A$ we have $X_\alpha = \emptyset$, then the family $\{X_\alpha \dashrightarrow X\}_{\alpha \neq \alpha_0}$ is also a covering family.

A *polytope* is a non-initial object of $\mathcal{C}$. The full double subcategory of polytopes of $\mathcal{C}$ will be denoted $\mathcal{C}_p$. We will say that two polytopes $a, b \in \mathcal{C}$ are *disjoint* if there exists an object $c \in \mathcal{C}$ with vertical morphisms $a \dashrightarrow c$ and $b \dashrightarrow c$ such that the pullback $a \times_c b$ is the vertically initial object.

The main motivating example that we will refer to for intuition will be the example of Euclidean scissors congruence. Let the polytopes of $\mathcal{C}$ be polygons in the Euclidean plane, where we define a polygon to be a finite union of nondegenerate triangles. We define the vertical morphisms of $\mathcal{C}$

to be set inclusions (where we formally add in the empty set to be the vertically initial object). The topology on $\mathcal{C}_v$ will be the usual topology induced by unions; concretely, $\{P_\alpha \dashrightarrow P\}_{\alpha \in A}$ will be a cover if $\bigcup_{\alpha \in A} P_\alpha = P$. We define the set of horizontal morphisms $P \longrightarrow Q$ to be $\{g \in E(2) \mid g \cdot P = Q\}$.

Then axiom (V) simply says that the intersection of two polygons is either another polygon or else has measure 0 (and therefore we define it to be the empty set). Axiom (H) is simply the statement that $E(2)$ is a group. Axiom (P) says that if we have polygons P and Q and a Euclidean transformation g that takes P to Q, then any polygon sitting inside P is taken to a unique polygon inside Q. Axiom (C) says that Euclidean transformations preserve unions. Axiom (B) says that if you have a set of polygons $\{P_\alpha\}_{\alpha \in A}$ and sets $\{P_{\alpha\beta}\}_{\beta \in B_\alpha}$ such that

$$\bigcup_{\beta \in B_\alpha} P_{\alpha\beta} = P_\alpha \qquad \text{and} \qquad \bigcup_{\alpha \in A} \bigcup_{\beta \in B_\alpha} P_{\alpha\beta} = P$$

then we must have originally had $P = \bigcup_{\alpha \in A} P_\alpha = P$.

In order to define scissors congruence groups we want to look at the formal sums of polygons, and quotient out by the relations that $[P] = [Q]$ if $P \cong Q$, and if we have a finite set of polygons $\{P_i\}_{i \in I}$ which intersect only on the boundaries that cover P then $[P] = \sum_{i \in I}[P_i]$. Using a Grothendieck twist we can construct a category whose objects are exactly formal sums of polygons, and whose isomorphism classes will naturally quotient out the first of these relations. Thus we can now draw our attention to the second relation, which concerns ways of including smaller polygons into larger ones. In the language of polytope complexes, we want to understand the vertical structure of $\mathrm{Tw}(\mathcal{C})$.

We start with some results about how to move vertical information along horizontal morphisms. $\mathcal{C}$ has the property that "pullbacks exist", namely that if we have the lower-right corner of a commutative square consisting of a vertical and a horizontal morphism then we can complete it to a commutative square in a suitably universal fashion. It turns out that $\mathrm{Tw}(\mathcal{C}_p)$ has the same property.

Lemma 3.1.2. *Given any diagram*

$$A \xrightarrow{\ \sigma\ } B \xleftarrow{\ q\ } B'$$

in $\mathrm{Tw}(\mathcal{C}_p)$, let $(\mathrm{Tw}(\mathcal{C}_p) \downarrow (A, B'))$ be the category whose objects are commutative squares

$$
\begin{array}{ccc}
A' & \xrightarrow{\ \tau\ } & B' \\
{\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\
A & \xrightarrow{\ \sigma\ } & B
\end{array}
$$

and whose morphisms are commutative diagrams

28

Then $(\mathrm{Tw}(\mathcal{C}_p) \downarrow (A, B'))$ *has a terminal object.*

We will refer to this terminal object as the *pullback* of the diagram, and the square that it fits into a *pullback square*. We denote this terminal object by $\sigma^* B'$, and call the vertical morphism $\sigma^* q \colon \sigma^* B' \dashrightarrow A$ and the horizontal morphism $\widetilde{\sigma} \colon \sigma^* B' \longrightarrow B'$. Note that if σ (resp. q) is an isomorphism, then so is $\widetilde{\sigma}$ (resp. $\sigma^* q$).

Proof. Let $A' = \{\Sigma_i^* b'_{j'}\}_{(i,j') \in I \times_J J'}$. We also define morphisms $\tau \colon A' \longrightarrow B'$ by the set map $\pi_2 \colon I \times_J J' \longrightarrow J'$ and the horizontal morphisms $\Sigma_i \colon \Sigma_i^* b'_{j'} \longrightarrow b'_{j'}$ and $p \colon A' \dashrightarrow A$ by the set map $\pi_1 \colon I \times_J J' \longrightarrow I$ and the vertical morphisms $\Sigma_i^* (b'_{j'} \dashrightarrow b_{q(j)})$. Then these complete the original diagram to a commutative square by definition; the fact that it is terminal is simple to check. $\qquad\square$

Our second relation between polygons had the condition that we needed polygons to be disjoint, so we restrict our attention to vertical morphisms which have only disjoint polygons in each "layer."

Definition 3.1.3. Given a vertical morphism $p \colon \{a_i\}_{i \in I} \dashrightarrow \{b_j\}_{j \in J} \in \mathrm{Tw}(\mathcal{C}_p)_v$ we say that p is a *sub-map* if for every $j \in J$ and any two distinct $i, i' \in p^{-1}(j)$ we have $a_i \times_{b_j} a_{i'} = \emptyset$ in $\mathcal{C}_v$. We will say that a sub-map p is a *covering sub-map* if for every $j \in J$ the sets $\{a_i \dashrightarrow b_j\}_{i \in p^{-1}(j)}$ are covers according to the topology on $\mathcal{C}_v$.

We will denote the subcategory of sub-maps by $\mathrm{Tw}(\mathcal{C}_p)_v^{\mathrm{Sub}}$.

In the polygon example, a sub-map is simply the inclusion of several polygons which have measure-0 intersection into a larger polygon. A covering sub-map is such an inclusion which is in fact a tiling of the larger polygon. For example:

covering sub-map   sub-map
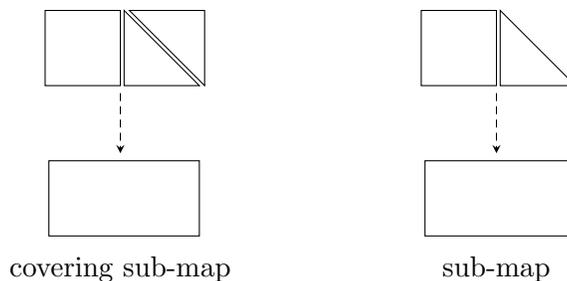
From this point onwards in the text all vertical morphisms of $\mathrm{Tw}(\mathcal{C}_p)$ will be sub-maps. If a sub-map is in fact a covering sub-map we will denote it by $A \dashrightarrow\!\!\!\!\rightarrow B$. We will also refer to horizontal morphisms as *shuffles* for simplicity. From lemma 2.2.3 we know that $\mathrm{Tw}(\mathcal{C}_p)_v$ has all pullbacks, and it is easy to see that the pullback of a sub-map will also be a sub-map. From axiom (B) we know that if $\{X_\alpha \dashrightarrow X\}_{\alpha \in A}$ is a covering family and $X_{\alpha_0} = \emptyset$ for an $\alpha_0 \in A$, then the family $\{X_\alpha \dashrightarrow X\}_{\alpha \in A \setminus \{\alpha_0\}}$ is also a covering family. Thus the pullback of a covering sub-map will be another covering sub-map, which means that not only is $\mathrm{Tw}(\mathcal{C}_p)_v^{\mathrm{Sub}}$ a category which has all pullbacks, but in fact the Grothendieck topology on $\mathcal{C}_v$ induces a Grothendieck topology on $\mathrm{Tw}(\mathcal{C}_p)_v$. It turns out that the pullback functor defined above acts continuously with respect to this topology.

Lemma 3.1.4. *Let $\sigma \colon A \longrightarrow B$ be a shuffle.*

1. *We have a functor $\sigma^* \colon (\mathrm{Tw}(\mathcal{C}_p)_v^{\mathrm{Sub}} \downarrow B) \longrightarrow (\mathrm{Tw}(\mathcal{C}_p)_v^{\mathrm{Sub}} \downarrow A)$ given by pulling back along σ. This functor preserves covering sub-maps.*

2. σ^* *has a right adjoint* σ_* *which also preserves covering sub-maps. If* σ *has an injective set map (in the sense of definition 2.2.1) then* $\sigma^*\sigma_* \cong 1$*; if* σ *has a surjective set map then* $\sigma_*\sigma^* \cong 1$*.*

Intuitively speaking, σ^* looks at how each polytope in the image is decomposed and decomposes its preimages accordingly. σ_* figures out what the minimal subdivision of the image that pulls back to a refinement of the domain is. In our polygon example, we have the following:

pulling back q along σ pushing forward p along σ
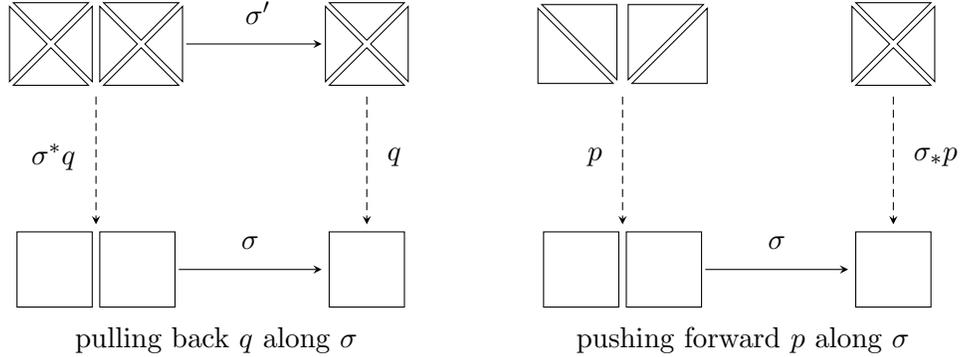
Proof.

1. From lemma 3.1.2 we know that σ^* is a functor $(\mathrm{Tw}(\mathcal{C}_p)_v^{\mathrm{Sub}} \downarrow B) \longrightarrow (\mathrm{Tw}(\mathcal{C}_p)_v \downarrow A)$, so it remains to show that σ^* maps sub-maps to sub-maps. This follows from the explicit computation of σ^* in the proof of lemma 3.1.2 and axiom (P) which gives us that pulling back along a horizontal morphism in $\mathcal{C}$ preserves pullbacks. The fact that σ^* preserves covers is true by axiom (C).

2. We will show that σ^* has a right adjoint by showing that the comma category $(\sigma^* \downarrow A')$ has a terminal object. We will write $A = \{a_i\}_{i\in I}$, $A' = \{a'_{i'}\}_{i'\in I'}$, etc. In addition, for any vertical morphism $f\colon Y = \{y_w\}_{w\in W} \dashrightarrow \{z_x\}_{x\in X}$ we will use the notation Y_x for the object $\{y_w\}_{w\in f^{-1}(x)}$.

   Suppose that we have a sub-map $q\colon B' \dashrightarrow B$ such that the pullback σ^*q factors through A'. For all $i \in I$ we have horizontal morphisms $\Sigma_i^{-1}\colon b_{\sigma(i)} \longrightarrow a_i$, so by the definition of pullback we have sub-maps $B'_{\sigma(i)} \dashrightarrow (\Sigma_i^{-1})^*A'_i$ and thus we must have a sub-map

$$B'_{\sigma(i)} \dashrightarrow \prod_{i\in\sigma^{-1}(j)} (\Sigma_i^{-1})^*A'_i.$$

(As vertically we are in a preorder, products and pullbacks are the same when they exist; we are omitting the object that we take the pullback over for conciseness of notation.) Thus the object

$$X = \coprod_{j\in J}\left(\prod_{i\in\sigma^{-1}(j)} (\Sigma_i^{-1})^*A'_i\right)$$

is clearly terminal in $(\sigma^* \downarrow A')$, and if $A' \dashrightarrow\!\!\!\rightarrow A$ was a cover, then $X \dashrightarrow\!\!\!\rightarrow B$ will also be one. (Note that if $\sigma^{-1}(j) = \emptyset$ then the product becomes $\{b_j\}$, as all of these products are in

the category of objects with sub-map to B.) If σ had an injective set map then $\sigma^{-1}(j)$ has size either 0 or 1 we must have $\sigma^*\sigma_* = 1$. If σ has a surjective set map then by definition $A_i' = \Sigma_i^* B_j$ for $i \in \sigma^{-1}(j)$ and all $j \in J$ will be represented, and so in fact $X \cong B'$ and $\sigma_*\sigma^* \cong 1$.

$\square$

We wrap up this section by defining the category of polytope complexes.

Definition 3.1.5. Let $\mathcal{C}$ and $\mathcal{D}$ be two polytope complexes. A double functor $F \colon \mathcal{C} \longrightarrow \mathcal{D}$ is called a *polytope functor* if it satisfies the two additional conditions

(FC) the vertical component $F_v \colon \mathcal{C}_v \longrightarrow \mathcal{D}_v$ is continuous and preserves pullbacks and the initial object, and

(FP) for any pair of morphisms $P \colon B' \dashrightarrow B$ and $\Sigma \colon A \longrightarrow B$, where P is vertical and Σ horizontal, we have $F(\Sigma^* P) = F(\Sigma)^* F(P)$. (In other words, F preserves mixed pullbacks.)

We denote the category of polytope complexes and polytope functors by **PolyCpx**.

3.2 Waldhausen Category Structure

Now that we have developed some machinery for looking at formal sums of polygons we can start constructing the group completion of our category $\mathrm{Tw}(\mathcal{C}_p)$. Our cofibrations will be inclusions of polygons which lose no information. Our weak equivalences will be the horizontal isomorphisms, together with vertical covering sub-maps (which will quotient out by both of the relations we are interested in). Since we now want to be able to mix sub-maps and shuffles we define our Waldhausen category by applying a sort of Q-construction to the double category $\mathrm{Tw}(\mathcal{C}_p)$.

Definition 3.2.1. The category $\mathrm{SC}(\mathcal{C})$ is defined to have $\mathrm{ob}\,\mathrm{SC}(\mathcal{C}) = \mathrm{ob}\,(\mathrm{Tw}(\mathcal{C}_p))$. The morphisms of $\mathrm{SC}(\mathcal{C})$ are equivalence classes of diagrams in $\mathrm{Tw}(\mathcal{C}_p)$

$$A \xdashleftarrow{\;p\;} A' \longrightarrow B$$

where two diagrams are considered equivalent if there is a vertical isomorphism $\iota \colon A_1' \dashrightarrow A_2' \in \mathrm{Tw}(\mathcal{C}_p)_v$ which makes the following diagram commute:

$$
\begin{array}{ccc}
 & A_1' & \\
p_1 \nearrow & \vdots\iota & \searrow \sigma_1 \\
A & & B \\
p_2 \searrow & \vdots & \nearrow \sigma_2 \\
 & A_2' & \\
\end{array}
$$

We will generally refer to a morphism as a *pure sub-map* (resp. *pure shuffle*) if in some representing diagram the shuffle (resp. sub-map) component is the identity.

We say that a morphism $A \xdashleftarrow{\;p\;} A' \xrightarrow{\;\sigma\;} B$ is a

cofibration if p is a covering sub-map and σ has an injective set map, and a

weak equivalence if p is a covering sub-map and σ has a bijective set map.

The composition of two morphisms $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ represented by

$$A \xleftarrow{\ p\ } A' \xrightarrow{\ \sigma\ } B \qquad \text{and} \qquad B \xleftarrow{\ q\ } B' \xrightarrow{\ \tau\ } C$$

is defined to be the morphism represented by the sub-map $p \circ \sigma^* q$ and the shuffle $\tau \circ \widetilde{\sigma}$.

Our goal is to prove the following theorem, which states that this structure gives us exactly the scissors congruence groups we were looking for.

Theorem 3.2.2. SC *is a functor* **PolyCpx** $\longrightarrow$ **WaldCat**. *Every Waldhausen category in the image of* SC *satisfies the Extension axiom, and has a canonical splitting for every cofibration sequence. In addition, for any polytope complex* $\mathcal{C}$, $K_0\mathrm{SC}(\mathcal{C})$ *is the free abelian group generated by the polytopes of* $\mathcal{C}$ *modulo the two relations*

$$[a] = \sum_{i \in I} [a_i] \qquad \text{for covering sub-maps } \{a_i\}_{i \in I} \dashrightarrow \{a\}$$

and

$$[a] = [b] \qquad \text{for horizontal morphisms } a \longrightarrow b \in \mathcal{C}.$$

We will defer most of the proof of this theorem until the last section of the paper, as it is largely technical and not very illuminating. Assuming the first part of the theorem, however, we will perform the computation of K_0 here.

Proof. In a small Waldhausen category $\mathcal{E}$ where every cofibration sequence splits, K_0 is the free abelian group generated by the objects of $\mathcal{E}$ under the relations that $[A \amalg B] = [A] + [B]$ for all $A, B \in \mathcal{E}$, and $[A] = [B]$ if there is a weak equivalence $A \xrightarrow{\sim} B$.

In SC$(\mathcal{C})$ an object $\{a_i\}_{i \in I}$ is isomorphic to $\coprod_{i \in I} \{a_i\}$, so $K_0\mathrm{SC}(\mathcal{C})$ is in fact generated by all polytopes of $\mathcal{C}$. Now consider any weak equivalence $f\colon A \xrightarrow{\sim} B \in \mathrm{SC}(\mathcal{C})$. We can write this weak equivalence as a pure covering sub-map followed by a pure shuffle with bijective set map (which will be an isomorphism of SC$(\mathcal{C})$). Any isomorphism of SC$(\mathcal{C})$ is a coproduct of isomorphisms between singleton objects; any isomorphism between singletons is simply a horizontal morphism of $\mathcal{C}$. Any pure covering sub-map is a coproduct of covering sub-maps of singletons. Thus the weak equivalences generate exactly the relations given in the statement of the theorem, and we are done. $\qquad\square$

3.3 Examples

3.3.1 The Sphere Spectrum

Consider the double category $\mathbb{S}$ with two objects, $\emptyset$ and $*$. We have one vertical morphism $\emptyset \dashrightarrow *$ and no other non-identity morphisms. There are no nontrivial covers. Then $\mathbb{S}$ is clearly a polytope complex.

Tw$(\mathbb{S})$ will be the category of pointed finite sets, where the cofibrations are the injective maps and the weak equivalences are the isomorphisms. By direct computation and the Barratt-Priddy-Quillen theorem ([1]) we see that $K(\mathrm{SC}(\mathbb{S}))$ is equivalent to the sphere spectrum (which justifies our notation).

3.3.2 G-analogs of Spheres

Consider the double category $\mathbb{S}_G$ with two objects, $\emptyset$ and $*$. We have one vertical morphism $\emptyset \dashrightarrow *$. In addition, $\mathrm{Aut}_h(*) = G$. This is clearly a polytope complex.

The objects of $\mathrm{SC}(\mathbb{S}_G)$ are the finite sets. As all weak equivalences are isomorphisms (and thus all cofibration sequences split exactly) we can compute the K-theory of $\mathrm{SC}(\mathbb{S}_G)$ by considering $\Omega BB(\mathrm{iso}\,\mathrm{SC}(\mathbb{S}_G))$. The automorphism group of a set $\{1, 2, \ldots, n\}$ is the group $G \wr \Sigma_n$, whose underlying set is $\Sigma_n \times G^n$, and where

$$(\sigma, (g_1, \ldots, g_n)) \cdot (\sigma', (g_1', \ldots, g_n')) = (\sigma\sigma', (g_{\sigma'(1)}g_1', g_{\sigma'(2)}g_2', \ldots, g_{\sigma'(n)}g_n')).$$

Thus the K-theory spectrum of this category will have $\coprod_{n\geq 0} B(G \wr \Sigma_n)$ as its 0-space, $B(\coprod_{n\geq 0} B(G \wr \Sigma_n))$ as its 1-space, and an Ω-spectrum above this.

3.3.3 Classical Scissors Congruence

Let $X = E^n, S^n$ or H^n (n-dimensional hyperbolic space), and let $\mathcal{G}_X$ be the poset of polytopes in X. (A polytope in X is a finite union of n-simplices of X; an n-simplex of X is the convex hull of $n + 1$ points (contained in a single hemisphere, if $X = S^n$).) The group G of isometries of X acts on $\mathcal{G}_X$; we define a horizontal morphism $P \longrightarrow Q$ to be an element $g \in G$ such that $g \cdot P = Q$. We say that $\{P_\alpha \dashrightarrow P\}_{\alpha \in A}$ is a covering family if $\bigcup_{\alpha \in A} P_\alpha = P$. Then $\mathcal{G}_X$ is a polytope complex.

Then by theorem 3.2.2 we obtain theorem 1.0.1: $K_0(\mathrm{SC}(\mathcal{G}_X)) = \mathcal{P}(X, G)$, the classical scissors congruence group of X. Thus it makes sense to call $K(\mathrm{SC}(\mathcal{G}_X))$ the *scissors congruence spectrum* of X.

3.3.4 Sums of Polytope Complexes

Suppose that we have a family of polytope complexes $\{\mathcal{C}_\alpha\}_{\alpha \in A}$. Then we can define the "wedge" of this family by defining $\mathcal{C}$ to be the double category where $\mathrm{ob}\,\mathcal{C} = \{\emptyset\} \cup \bigcup_{\alpha \in A} \mathrm{ob}\,\mathcal{C}_{\alpha p}$ (where $\emptyset$ will be the initial object), and all morphisms are just those from the $\mathcal{C}_\alpha$. We define $\mathrm{Aut}_{\mathcal{C}_h}(\emptyset) = \bigoplus_{\alpha \in A} \mathrm{Aut}_{h\mathcal{C}_\alpha}(\emptyset)$. Then $\mathcal{C}$ will be a polytope complex which represents the "union" of the polytope complexes in $\{\mathcal{C}_\alpha\}_{\alpha \in A}$. $\mathrm{SC}(\mathcal{C}) = \bigoplus_{\alpha \in A} \mathrm{SC}(\mathcal{C}_\alpha)$, where the summation means that all but finitely many of the objects of each tuple will be equal to the zero object. Then $K(\mathrm{SC}(\mathcal{C})) = \bigvee_{\alpha \in A} K(\mathrm{SC}(\mathcal{C}_\alpha))$.

3.3.5 Numerical Scissors Congruence

Suppose that K is a number field. Let $\mathcal{C}_K$ be the polytope complex whose objects are the ideals of $\mathcal{O}_K$. We have a vertical morphism $I \dashrightarrow J$ if $I | J$, and no nontrivial horizontal morphisms. (Note that $\mathcal{O}_K$ is the initial object.) We say that a finite family $\{J_\alpha \dashrightarrow I\}_{\alpha \in A}$ is a covering family if $I | \prod_{\alpha \in A} J_\alpha$, and an infinite family is a covering family if it contains a finite covering family. In this case it is easy to compute that $K(\mathcal{C}_K)$ is a bouquet of spheres, one for every prime power ideal of $\mathcal{O}_K$.

Now suppose that $K/\mathbb{Q}$ is Galois with Galois group $G_{K/\mathbb{Q}}$. We define $\mathcal{C}_{K/\mathbb{Q}}$ to be the polytope complex whose objects and vertical morphisms are the same as those of $\mathcal{C}_K$, but where the horizontal morphisms $I \longrightarrow J$ are $\{g \in G_{K/\mathbb{Q}} \,|\, g \cdot I = J\}$. Then the K-theory $K(\mathrm{SC}(\mathcal{C}_{K/\mathbb{Q}}))$ will be

$$\bigvee_{p^\alpha \in \mathbb{Z}} K(\mathrm{SC}(\mathbb{S}_{D_\mathfrak{p}})),$$

where for each p, $\mathfrak{p}$ is a prime ideal of K lying above p and $D_{\mathfrak{p}}$ is the decomposition group of $\mathfrak{p}$. (For more information about factorizations of prime ideals, see for example [20].) Thus this spectrum encodes all of the inertial information for each prime ideal of $\mathbb{Q}$.

From the inclusion $\mathbb{Q} \longrightarrow K/\mathbb{Q}$ we get an induced polytope functor $\mathcal{C}_{\mathbb{Q}} \longrightarrow \mathcal{C}_{K/\mathbb{Q}}$, and therefore an induced map $f \colon K(\mathrm{SC}(\mathcal{C}_{\mathbb{Q}})) \longrightarrow K(\mathrm{SC}(\mathcal{C}_{K/\mathbb{Q}}))$. To compute this map, it suffices to consider what this map does on every sphere in the bouquet $K(\mathrm{SC}(\mathcal{C}_{\mathbb{Q}}))$. Consider the sphere indexed by a prime power p^{α}. If we factor $p = \mathfrak{p}_1^e \cdots \mathfrak{p}_g^e$ then this sphere maps to g times the map $K(\mathbb{S}) \longrightarrow K(\mathbb{S}_{D_{\mathfrak{p}}})$ (induced by the obvious inclusion $\mathbb{S} \longrightarrow \mathbb{S}_{D_{\mathfrak{p}}}$), with target the copy of this indexed by $p^{e\alpha}$. Thus f encodes all of the splitting and ramification data of the extension $K/\mathbb{Q}$. In particular, we see that the map $K(\mathrm{SC}(\mathcal{C}_{\mathbb{Q}})) \longrightarrow K(\mathrm{SC}(\mathcal{C}_{K/\mathbb{Q}}))$ contains all of the factorization information contained in $K/\mathbb{Q}$.

Note that we can generalize the definition of $\mathcal{C}_{K/\mathbb{Q}}$ to any Galois extension L/K; with this definition $\mathcal{C}_K = \mathcal{C}_{K/K}$.

3.4   Proof of theorem 3.2.2

This section consists mostly of a lot of technical calculations which check that $\mathrm{SC}(\mathcal{C})$ satisfies all of the properties that theorem 3.2.2 claims. In order to spare the reader all of the messy details we isolate these results in their own section.

3.4.1   Some technicalities about sub-maps and shuffles

In this section all diagrams are in $\mathrm{Tw}(\mathcal{C}_p)$, and all vertical morphisms are sub-maps.

The following lemmas formalize the idea that we can often commute shuffles and sub-maps past one another. In particular, it is obvious that if we have a sub-map whose codomain is a horizontal pushout, then we can pull this sub-map back to the components of the pushout. However, it turns out that we can do this in the other direction as well: given suitably consistent sub-maps to the components of a pushout, we obtain a sub-map between pushouts.

Lemma 3.4.1. *Given a diagram*

$$
\begin{array}{ccccc}
C' & \xleftarrow{\ \tau'\ } & A' & \xrightarrow{\ \sigma'\ } & B' \\
{\scriptstyle r}\big\downarrow & & {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\
C & \xleftarrow{\ \tau\ } & A & \xrightarrow{\ \sigma\ } & B
\end{array}
$$

where the right-hand square is a pullback square and σ has an injective set map, there is an induced sub-map $C' \cup_{A'} B' \dashrightarrow C \cup_A B$. If p, q, r are all covering sub-maps then this map will be, as well.

Proof. Consider the right-hand square. Write $A = \{a_i\}_{i \in I}$, $B = \{b_j\}_{j \in J}$ (and analogously for A', B'). Write $J = I \cup J_0$ for $J_0 = J \backslash \mathrm{im} I$; and $J' = I' \cup J_0'$, analogously. We claim that q can be written as $q_I \cup q_0$, where $q_I \colon \{b_{j'}'\}_{j' \in I'} \dashrightarrow \{b_j\}_{j \in I}$ and $q_0 \colon \{b_{j'}'\}_{j' \in J_0'} \dashrightarrow \{b_j\}_{j \in J_0}$. Indeed, q_I is well-defined because the diagram commutes, and q_0 is well-defined because the right-hand square is a pullback. But then we can write the given diagram as the coproduct of two diagrams

$$C' \xleftarrow{\tau'} A' \xrightarrow{\sigma'} \{b'_{j'}\}_{j' \in I'} \qquad \emptyset \longleftarrow \emptyset \longrightarrow \{b'_{j'}\}_{j' \in J'_0}$$

with vertical maps r, p, q_I on the left and q_0 on the right, over

$$C \xleftarrow{\tau} A \xrightarrow{\sigma} \{b_j\}_{j \in I} \qquad \emptyset \longleftarrow \emptyset \longrightarrow \{b_j\}_{j \in J_0}$$

As the statement obviously holds in the right-hand diagram, it suffices to consider the case of the left-hand diagram, where σ is bijective. In this case, σ and σ' are both isomorphisms, and so the morphism we are interested in is r, for which the lemma clearly holds. $\qquad\square$

Suppose that we are given a diagram

$$
\begin{array}{ccccc}
A' & \xrightarrow{\sigma'} & B' & \xleftarrow{f'} & C' \\
\downarrow p & & \downarrow q & & \downarrow r \\
A & \xrightarrow{\sigma} & B & \xleftarrow{f} & C
\end{array}
$$

Then from the definition of σ^* and $(\sigma')^*$ it is easy to see that we get an induced sub-map $(\sigma')^* C' \dashrightarrow \sigma^* C$, which will be a covering sub-map if p, q, r are. The analogous statement for σ_* is more difficult to prove, but is also true.

Lemma 3.4.2. *Given a diagram*

$$
\begin{array}{ccccc}
C' & \dashrightarrow{f'} & A' & \xrightarrow{\sigma'} & B' \\
\downarrow r & & \downarrow p & & \downarrow q \\
C & \dashrightarrow{f} & A & \xrightarrow{\sigma} & B
\end{array}
$$

where the right-hand square is a pullback, the induced sub-map $\sigma'_* C' \dashrightarrow \sigma_* C$ exists and is a covering sub-map if q and f' are covering sub-maps.

Proof. We can assume that σ's set map is surjective, since otherwise we can write the right-hand square as the coproduct of two squares

$$
\begin{array}{ccc}
A' & \xrightarrow{\sigma'} & B'_0 \\
\downarrow & & \downarrow \\
A & \xrightarrow{\sigma} & B_0
\end{array}
\qquad
\begin{array}{ccc}
\emptyset & \longrightarrow & B'_1 \\
\downarrow & & \downarrow \\
\emptyset & \longrightarrow & B_1
\end{array}
$$

In the right-hand case the map we are interested in is just $B'_1 \dashrightarrow B_1$, so the result clearly holds. So we focus on the original question when σ has a surjective set-map. As $(\mathrm{Tw}(\mathcal{C}_p)_v^{\mathrm{Sub}} \downarrow B)$ is a preorder and both $\sigma'_* C'$ and $\sigma_* C$ sit over B it suffices to show that this morphism exists in $(\mathrm{Tw}(\mathcal{C}_p)_v^{\mathrm{Sub}} \downarrow B)$. We claim that it suffices to show that $\sigma'_* C' = \sigma_* C'$, as if this is the case then

$$\mathrm{Hom}_{(\mathrm{Tw}(\mathcal{C}_p)_v^{\mathrm{Sub}} \downarrow B)}(\sigma'_* C', \sigma_* C) = \mathrm{Hom}_{(\mathrm{Tw}(\mathcal{C}_p)_v^{\mathrm{Sub}} \downarrow B)}(\sigma_* C', \sigma_* C)$$
$$\supseteq \mathrm{Hom}_{(\mathrm{Tw}(\mathcal{C}_p)_v^{\mathrm{Sub}} \downarrow A)}(C', C) \neq \emptyset$$

so we will be done.

Write $A = \{a_i\}_{i \in I}$, $A' = \{a'_{i'}\}_{i' \in I'}$, $B = \{b_j\}_{j \in J}$, etc., and let $C'_i = \{c'_{k'}\}_{pf'(k')=i}$ for $i \in I$ and $C'_{i'} = \{c'_{k'}\}_{f'(k')=i'}$ for $i' \in I'$. Then we know that

$$\sigma_* C' = \coprod_{j \in J} \prod_{i \in \sigma^{-1}(j)} (\Sigma_i^{-1})^* C'_i \qquad \text{and} \qquad \sigma'_* C' = \coprod_{j' \in J'} \prod_{i' \in \sigma'^{-1}(j')} (\Sigma'^{-1}_{i'})^* C'_{i'}.$$

(Note that all of these products exist because $\mathcal{C}$ is vertically closed under pullbacks, and in a preorder a pullback is the same as a product.) Now

$$\prod_{i \in \sigma^{-1}(j)} (\Sigma_i^{-1})^* C'_i = \prod_{i \in \sigma^{-1}(j)} \coprod_{i' \in p^{-1}(i)} (\Sigma'^{-1}_{i'})^* C'_{i'}$$

Because the right-hand square is a pullback square we can associate I' to pairs $(i, j') \in I \times_J J'$. For any two such pairs $i'_1 = (i_1, j'_1)$ and $i'_2 = (i_2, j'_2)$ if $j'_1 \neq j'_2$ then $(\Sigma'^{-1}_{i'_1})^* C'_{i'_1} \times (\Sigma'^{-1}_{i'_2})^* C'_{i'_2} = \emptyset$; in particular we know that most of the crossterms in this product will be $\emptyset$. Thus we can reorder the indexing of the product and swap the coproduct and the product to get

$$\prod_{i \in \sigma^{-1}(j)} \coprod_{i' \in p^{-1}(i)} (\Sigma'^{-1}_{i'})^* C'_{i'} = \coprod_{j' \in q^{-1}(j)} \prod_{i' \in \sigma'^{-1}(j')} (\Sigma'^{-1}_{i'})^* C'_{i'}$$

$$= \coprod_{j' \in q^{-1}(j)} \prod_{i' \in \sigma'^{-1}(j')} (\Sigma'^{-1}_{i'})^* C'_{i'}.$$

Thus

$$\sigma_* C' = \coprod_{j \in J} \prod_{i \in \sigma^{-1}(j)} (\Sigma_i^{-1})^* C'_i = \coprod_{j \in J} \coprod_{j' \in q^{-1}(j)} \prod_{i' \in \sigma'^{-1}(j')} (\Sigma'^{-1}_{i'})^* C'_{i'} = \sigma'_* C'$$

and we have our desired sub-map $\sigma'_* C' \dashrightarrow \sigma_* C$. If q and f' are covering sub-maps then $\sigma'_* f'$ is a covering sub-map, which means that $\sigma'_* C' \dashrightarrow \sigma_* C$ is a covering sub-map (as it is the pullback of $q\sigma'_* f'$ along $\sigma_* f$), as desired. $\qquad \square$

Lastly we prove a couple of lemmas which show that covering sub-maps do not lose any information. The first of these shows that if two shuffles are related by covering sub-maps then they contain the same information; the second shows that pulling back a covering sub-map along a shuffle cannot lose information.

Lemma 3.4.3. *Suppose that we have the following diagram:*

$$
\begin{array}{ccc}
A' & \xrightarrow{\ \tau\ } & B' \\
{\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle q} \\
A & \xrightarrow{\ \sigma\ } & B
\end{array}
$$

Then this diagram is a pullback square. If q has a surjective set-map and τ is an isomorphism then σ must also be an isomorphism.

Proof. Pulling back q along σ gives us a commutative square

$$A' \xrightarrow{\ \tau\ } B'$$
$$\downarrow{\scriptstyle r} \qquad\qquad \downarrow{\scriptstyle \cong}$$
$$\sigma^*B' \xrightarrow{\ \sigma'\ } B'$$

so it suffices to show that in any such diagram r is an isomorphism. Suppose it were not. Then there would exist $a' \in A'$ and an $a \in \sigma^*B'$ such that we have a non-invertible vertical morphism $a' \dashrightarrow a$, and horizontal morphisms $F_a \colon a \longrightarrow b$ (for some $b \in B'$) such that the pullback of the vertical identity morphism on b is the non-invertible morphism $a' \dashrightarrow a$. Contradiction. Thus r must be an isomorphism, and we are done with the first part.

Now suppose that q has a surjective set map and τ is an isomorphism. As any shuffle with bijective set map is an isomorphism it suffices to show that σ has a bijective set map. However, as this is a pullback square on the underlying set maps we can just consider it there. As q has a surjective set map and the pullback of σ along q is a bijection σ must also be a bijection, and we are done. $\qquad\square$

3.4.2 Checking the axioms

We now verify that our definition of $\mathrm{SC}(\mathcal{C})$ works and then check the axioms for it to be a Waldhausen category. First we check that all of our definitions are well-defined.

Lemma 3.4.4. $\mathrm{SC}(\mathcal{C})$ *is a category, and the cofibrations and weak equivalences form subcategories of* $\mathrm{SC}(\mathcal{C})$.

Proof. We need to check that composition is well-defined. Suppose that we are given morphisms $f \colon A \longrightarrow B$ and $g \colon B \longrightarrow C$ in $\mathrm{SC}(\mathcal{C})$, and suppose that we are given two different diagrams representing each morphism. Then we have the following diagram, where the top and bottom squares are pullbacks:

As each vertical section represents the same map we have reindexings $\iota_A \colon A_1 \longrightarrow A_2$ and $\iota_B \colon B_1 \longrightarrow B_2$; we need to show that we therefore have a reindexing $\sigma_1^* B_1' \longrightarrow \sigma_2^* B_2'$. It is easy to see that pulling back a reindexing along either a sub-map or a shuffle produces another reindexing. Thus if we pull back ι_A along q_2' to get a morphism ι_A', and then pull back ι_B along $\iota_A' \sigma_2'$ to get ι_B' we get a diagram

where $\iota'_2\iota'_1\colon X \longrightarrow \sigma_2^*B'_2$. However, as both the upper and lower squares are pullbacks they are unique up to unique vertical isomorphism, so we obtain a vertical isomorphism $X \dashrightarrow \sigma_1^*B'_1$, and we are done.

It remains to show that weak equivalences and cofibrations are preserved by composition. Consider a composition of morphisms determined by the following diagram:

If q is a cover then so is q', so if both p and q are covers then $q'p$ is also a cover. From the formula in lemma 2.2.2 it is easy to see that if a shuffle has an injective (resp. bijective) set map then so will its pullback, so if both σ and τ are injective (resp. bijective) then $\tau\sigma'$ will be as well. Thus cofibrations and weak equivalences form subcategories, as desired. $\square$

Lemma 3.4.5. *Any isomorphism is both a cofibration and a weak equivalence.*

Proof. Suppose that $f\colon A \longrightarrow B$ is an isomorphism with inverse $g\colon B \longrightarrow A$. In $\mathrm{Tw}(\mathcal{C}_p)$ f and g are represented by diagrams

$$A \xdashleftarrow{p} A' \xrightarrow{\sigma} B \qquad\qquad B \xdashleftarrow{q} B' \xrightarrow{\tau} A$$

respectively. As $gf = 1_A$ we must have $p \circ \sigma^*(q)$ be invertible, so in particular p must be an isomorphism; thus we can pick a diagram representing f such that $p = 1_A$ (which is in particular a covering sub-map). Applying the analogous argument to g we can see that we can pick a diagram representing g such that $q = 1_B$. In that case, it is easy to see that we must have $\tau = \sigma^{-1}$ in $\mathrm{Tw}(\mathcal{C}_p)_h$, so σ and τ must be invertible. From this we see that any isomorphism is both a cofibration and a weak equivalence, as desired. $\square$

Now we move on to proving some of the slightly more complicated axioms defining a Waldhausen category. We check that pushouts along cofibrations exist, and that they preserve cofibrations. In fact, in $\mathrm{SC}(\mathcal{C})$ pushouts not only preserve cofibrations; they also preserve weak equivalences.

Lemma 3.4.6. *Given any diagram*

$$C \longleftarrow A \xhookrightarrow{f} B$$

the pushout $C \cup_A B$ of this diagram exists, and the morphism $C \longrightarrow C \cup_A B$ is a cofibration. If f were also a weak equivalence, then this map would also be a weak equivalence.

Proof. The diagram above is represented by the following diagram in $\mathrm{Tw}(\mathcal{C}_p)$:

$$C \xleftarrow{\tau} A'' \dashleftarrow{q} A \dashrightarrow{p} A' \xrightarrow{\sigma} B$$

Complete the middle part of this diagram to a pullback square

$$
\begin{array}{ccc}
\widetilde{A} & \dashrightarrow{q'} & A' \\
p' \downarrow & & \downarrow p \\
A'' & \dashrightarrow{q} & A
\end{array}
$$

Now we define $s\colon C' \dashrightarrow C$ to be $\tau_*(p')$. As $\tau^*\tau_*(p')$ must factor through p', $q \circ \tau^*\tau_*(p')$ must factor through p, so we can write $q \circ \tau^*\tau_*(p') = p \circ r'$. We then define $r\colon B' \dashrightarrow B$ to be $\sigma_*(r')$. (Note that if τ has an injective set map then $r = \sigma_*(q')$.) Now we have the following diagram in $\mathrm{Tw}(\mathcal{C}_p)$:

$$
\begin{array}{ccccccc}
C' & \xleftarrow{\tau'} & A''' & === & A''' & \xrightarrow{\sigma'} & B' \\
s \downarrow & & \downarrow & & \downarrow & & \downarrow r \\
C & \xleftarrow{\tau} & A'' & & A' & \xrightarrow{\sigma} & B
\end{array}
$$

where both squares are pullback squares. The top row of this diagram consists only of maps in $\mathrm{Tw}(\mathcal{C}_h)$. As $\mathcal{C}_h$ is a groupoid it in particular has all pushouts, and so by lemma 2.2.4 the pushout $C' \cup_{A'''} B'$ exists in $\mathrm{Tw}(\mathcal{C}_p)$; we claim that gives us the pushout of the original diagram. Note that the set-map of the shuffle $C' \longrightarrow C' \cup_{A'''} B'$ will be injective (and bijective, if σ's was) because it is the pushout of σ', so the pushout of a cofibration (resp. weak equivalence) is another cofibration (resp. weak equivalence).

To check that this is indeed the pushout, suppose that we are given any commutative square

$$
\begin{array}{ccc}
A & \xhookrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
$$

The diagonal morphism $A \longrightarrow D$ is represented by a diagram $A \xdashleftarrow{t} \widehat{A} \xrightarrow{\rho} D$ in $\mathrm{Tw}(\mathcal{C}_p)$. Considering the composition around the top, we see that $\widehat{A}$ factors through p, and considering the composition around the bottom it must factor through q. In addition, as t comes from the bottom composition we know that $\sigma^*\sigma_* t = t$ and thus t must factor through A'''. We can now apply lemma 3.4.1 to see that we indeed get a unique factorization through our pushout, as desired. □

We have now shown that $\mathrm{SC}(\mathcal{C})$ is a category with cofibrations which is equipped with a subcategory of weak equivalences, and we move on to proving that all cofibration sequences split canonically. Given a cofibration $f\colon A \hookrightarrow B$ we say that the *cofiber* of f is the pushout of f along the morphism $A \longrightarrow *$. We will denote such a map by

$$B \longrightarrow\!\!\!\!\!\to B/A$$

Corollary 3.4.7. *Any cofiber map has a canonical section; this section is a cofibration.*

Proof. Suppose that we are given a cofiber map $B \twoheadrightarrow B/A$. Suppose that this map is represented by the diagram

$$B \overset{p}{\dashleftarrow} B' \overset{\sigma}{\longrightarrow} B/A$$

From the computation in the proof of lemma 3.4.6 it is easy to see that if we write $B = \{b_j\}_{j \in J}$ then $B' = \{b_j\}_{j \in J'}$ where $J' \subseteq J$ and σ has a bijective set-map. Thus we can define a pure shuffle $\sigma^{-1} \colon B/A \hookrightarrow B$ which will be our section. If we change the diagram representing the fiber map by a reindexing then σ^{-1} changes exactly by this reindexing, so this construction is well-defined. $\square$

Remark. This construction is canonical in the twisted arrow category whose objects are cofibrations of $\mathrm{SC}(\mathcal{C})$. It is not canonical in the ordinary arrow category.

Now it only remains to show that the weak equivalences of $\mathrm{SC}(\mathcal{C})$ satisfy all of the axioms we desire of a Waldhausen category.

Lemma 3.4.8. *For any two composable morphisms f and g, if gf and f are weak equivalences then so is g. If $\mathcal{C}$ satisfies the extra condition*

(G) The empty family is not a covering family for any polytope of $\mathcal{C}$. Given a family $\mathcal{A} = \{X_\alpha \dashrightarrow X\}_{\alpha \in A}$ and covering families $\{X_{\alpha\beta} \dashrightarrow X_\alpha\}_{\beta \in B_\alpha}$, if the refined family

$$\{X_{\alpha\beta} \dashrightarrow X\}_{(\alpha,\beta) \in \coprod_{\alpha \in A} B_\alpha}$$

is a covering family then so is $\mathcal{A}$.

then if gf and g are weak equivalences, then so is f. (In other words, if (G) is satisfied then $\mathrm{SC}(\mathcal{C})$ satisfies the Saturation Axiom.)

Proof. Let $f \colon A \longrightarrow B$ and $g \colon B \longrightarrow C$ be morphisms in $\mathrm{SC}(\mathcal{C})$. As weak equivalences form a subcategory of $\mathrm{SC}(\mathcal{C})$ we already know that if f and g are both weak equivalences then so is gf. So it suffices for us to focus on the other two cases. In the following discussion we will be considering the following diagram

where the middle square is a pullback.

First suppose that gf and f are weak equivalences. Then we know that pq' is a covering sub-map, which means that q' must be a covering sub-map as well. We know that $q' = \sigma^* q$, and as σ is an isomorphism we must have $q = (\sigma^{-1})^* q'$. As covering sub-maps are preserved by pullback q must also be a covering sub-map. In addition, by lemma 3.4.3 as σ is an isomorphism so is σ', and thus (as $\tau\sigma'$ is an isomorphism) τ must be as well. Thus we see that q is a covering sub-map and τ an isomorphism, so g is a weak equivalence as desired.

Now, suppose that gf and g are weak equivalences and that (G) is satisfied. We know that both τ and $\tau\sigma'$ are isomorphisms, which means that σ' must be as well. In addition, pq' is a covering sub-map and so is q' (as covers are preserved by pullbacks) which means (by (G)) that p must be as well. As q is a cover with surjective set-map (as by (G) there are no empty covers), by lemma 3.4.3 σ therefore must also be an isomorphism. As p is a covering sub-map and σ is an isomorphism, f is also a weak equivalence. $\square$

Lemma 3.4.9. $\mathrm{SC}(\mathcal{C})$ *satisfies the extension axiom. In other words, given any diagram*

$$
\begin{array}{ccccc}
A & \lhook\joinrel\longrightarrow & B & \longtwoheadrightarrow & B/A \\
\downarrow \wr & & \downarrow f & & \downarrow \wr \\
A' & \lhook\joinrel\longrightarrow & B' & \longtwoheadrightarrow & B'/A'
\end{array}
$$

f is also a weak equivalence.

Proof. It is easy to see that the two sections given by corollary 3.4.7 are going to be consistent in the sense that they will split the above diagram into two diagrams

$$
\begin{array}{ccc}
A & \overset{\sim}{\lhook\joinrel\longrightarrow} & \widetilde{A} \\
\downarrow \wr & & \downarrow f_A \\
A' & \overset{\sim}{\lhook\joinrel\longrightarrow} & \widetilde{A}'
\end{array}
\qquad\qquad
\begin{array}{ccc}
B_0 & \overset{\cong}{\longrightarrow} & B/A \\
f_B \downarrow & & \downarrow \wr \\
B_0' & \overset{\cong}{\longrightarrow} & B'/A'
\end{array}
$$

where f will equal $f_A \sqcup f_B$ up to isomorphism. Thus it suffices to show that both f_A and f_B are weak equivalences. That f_B is a weak equivalence is obvious from the diagrams. The fact that f_A is a weak equivalence follows from lemma 3.4.8. $\qquad\square$

Lemma 3.4.10. $\mathrm{SC}(\mathcal{C})$ *satisfies the gluing axiom. In other words, given any diagram*

$$
\begin{array}{ccccc}
C & \longleftarrow & A & \lhook\joinrel\longrightarrow & B \\
\downarrow \wr & & \downarrow \wr & & \downarrow \wr \\
C' & \longleftarrow & A' & \lhook\joinrel\longrightarrow & B'
\end{array}
$$

the induced morphism $C \cup_A B \longrightarrow C' \cup_{A'} B'$ is also a weak equivalence.

Proof. It is a simple calculation to see that it suffices to consider diagrams represented in $\mathrm{Tw}(\mathcal{C}_p)$ by

$$
\begin{array}{ccccccc}
C & \overset{\sigma}{\longleftarrow} & A_C & \dashrightarrow & A & \dashleftarrow\!\!\dashleftarrow & B \\
\uparrow p & & \uparrow & & \uparrow & & \uparrow \\
C' & \overset{\sigma'}{\longleftarrow} & A_C' & \dashrightarrow & A' & \dashleftarrow\!\!\dashleftarrow & B'
\end{array}
$$

By pulling back p along σ we get a cover $A_C' \dashrightarrow\!\!\twoheadrightarrow A_C$. Then we have a cover $A_C' \times_{A'} B' \dashrightarrow\!\!\twoheadrightarrow A_C \times_A B$, and thus a diagram

$$
\begin{array}{ccccc}
C & \overset{\sigma}{\longleftarrow} & A_C & \dashleftarrow & A_C \times_A B \\
\uparrow & & \uparrow & & \uparrow \\
C' & \overset{\sigma'}{\longleftarrow} & A_C' & \dashleftarrow & A_C' \times_{A'} B'
\end{array}
$$

41

to which we can apply lemma 3.4.2, resulting in a cover

$$\sigma'_*(A'_C \times_{A'} B') \dashrightarrow \sigma_*(A_C \times_A B).$$

However, a simple computation using the definition of the pushout shows us that this is exactly the morphism between the pushouts of the top and bottom row. As it is represented in $\mathrm{Tw}(\mathcal{C}_p)$ by a covering sub-map, it must be a weak equivalence, as desired. $\square$

Lastly we prove that our construction is in fact functorial.

Proposition 3.4.11. *A polytope functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ induces an exact functor of Waldhausen categories $\mathrm{SC}(F)\colon \mathrm{SC}(\mathcal{C}) \longrightarrow \mathrm{SC}(\mathcal{D})$.*

Proof. As Tw is functorial, F induces a double functor $\mathrm{Tw}(F)\colon \mathrm{Tw}(\mathcal{C}) \longrightarrow \mathrm{Tw}(\mathcal{D})$. Then we define the functor $\mathrm{SC}(F)\colon \mathrm{SC}(\mathcal{C}) \longrightarrow \mathrm{SC}(\mathcal{D})$ to be the one induced by $\mathrm{Tw}(F)$. This is clearly well-defined on objects. As F_v preserves pullbacks and the initial object, $\mathrm{Tw}(F)_v$ takes sub-maps to sub-maps, and thus $\mathrm{SC}(F)$ is well-defined on morphisms. Composition in $\mathrm{SC}(\mathcal{C})$ is defined by pulling back vertical morphisms along horizontal morphisms, which commutes with F as F is a polytope functor, so $\mathrm{SC}(F)$ is a well-defined functor. If f is a (vertical or horizontal) morphism in $\mathrm{Tw}(\mathcal{C})$ then $F(f)$ and f have the same set-map, so $\mathrm{Tw}(F)$ preserves injectivity and bijectivity of set-maps, so to see that $\mathrm{SC}(F)$ preserves cofibrations and weak equivalences it suffices to check that $\mathrm{Tw}(F)$ preserves covering sub-maps, which it must as F_v is continuous. So in order to have exactness it suffices to show that $\mathrm{SC}(F)$ preserves pushouts along cofibrations.

Suppose that we are given the diagram

$$C \longleftarrow A \lhook\joinrel\longrightarrow B$$

in $\mathrm{SC}(\mathcal{C})$, which is represented by the diagram

in $\mathrm{Tw}(\mathcal{C})$. The pushout of this is computed by computing the pullback of the two sub-maps, pushing forward the result along τ, pulling it back along τ, and then pushing it forward along σ. Thus in order to see that $\mathrm{SC}(F)$ preserves pushouts it suffices to show that $\mathrm{Tw}(F)$ commutes with pullbacks of sub-maps, and pulling back or pushing forward a sub-map along a shuffle. By considering the formulas for pullbacks and pushforwards we see they they consist entirely of pulling back squares in $\mathcal{C}$ and taking vertical pullbacks in $\mathcal{C}$, so we are done. $\square$

Chapter 4

Suspensions, cofibers, and approximation

4.1   Thickenings

Definition 4.1.1. Let $\mathcal{C}$ be a polytope complex. The polytope complex $\mathcal{C}^{\bowtie}$ is the full subcategory of $\mathrm{Tw}(\mathcal{C}_p)$ containing all objects $\{a_i\}_{i \in I} \in \mathrm{Tw}(\mathcal{C}_p)$ such that for all distinct $i, j \in I$ there exists an $a \in \mathcal{C}$ such that $a_i \times_a a_j = \emptyset$. The topology on $\mathcal{C}_v^{\bowtie}$ is defined pointwise. More precisely, let $X = \{x_i\}_{i \in I}$, and $X_\alpha = \{x_j^{(\alpha)}\}_{j \in J_\alpha}$. We say that $\{p^\alpha : X_\alpha \dashrightarrow X\}_{\alpha \in A}$ is a covering family if for each $i \in I$ the family $\{P_j^\alpha : x_j^{(\alpha)} \dashrightarrow x_i\}_{j \in (p^\alpha)^{-1}(i), \alpha \in A}$ is a covering family in $\mathcal{C}$.

It is easy to check that $-^{\bowtie}$ is in fact a functor $\mathbf{PolyCpx} \longrightarrow \mathbf{PolyCpx}$. It will turn out that $-^{\bowtie}$ is a monad on $\mathbf{PolyCpx}$, and that SC : $\mathbf{PolyCpx} \longrightarrow \mathbf{WaldCat}$ factors through the inclusion $\mathbf{PolyCpx} \longrightarrow \mathrm{Kl}(-^{\bowtie})$ (the Kleisli category of this monad). This factorization provides us with extra morphisms between polytopes, which will be exactly the morphisms we need later when we start doing calculations with face maps in the $S.$ construction.

We start by considering the monad structure of $-^{\bowtie}$. We have a natural inclusion $\eta_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C}^{\bowtie}$ which includes $\mathcal{C}$ into $\mathcal{C}^{\bowtie}$ as the singleton sets; these assemble into a natural transformation $\eta : 1 \Longrightarrow -^{\bowtie}$. This transformation is not a natural isomorphism, even through, morally speaking, $\mathcal{C}^{\bowtie}$ ought to have the same K-theory as $\mathcal{C}$ (as it contains objects which are formal sums of objects of $\mathcal{C}$). It turns out that once we pass to $\mathbf{WaldCat}$ by SC we can find a natural "almost inverse": an exact left adjoint.

Lemma 4.1.2. *The functor $-^{\bowtie}$ is a monad on $\mathbf{PolyCpx}$.*

Proof. In order to make $-^{\bowtie}$ into a monad, we need to define a unit and a multiplication. The unit $\eta : 1_{\mathbf{PolyCpx}} \longrightarrow (-^{\bowtie})$ will be the natural transformation defined on each polytope complex $\mathcal{C}$ by the natural inclusion $\mathcal{C} \longrightarrow \mathcal{C}^{\bowtie}$ given by including $\mathcal{C}$ as the singleton sets indexed by the set $\{*\}$. The multiplication $\mu : (-^{\bowtie})^{\bowtie} \longrightarrow (-^{\bowtie})$ is given by the functor $\mathcal{C}^{\bowtie\bowtie} \longrightarrow \mathcal{C}^{\bowtie}$ given on objects by

$$\{\{a_j^{(i)}\}_{j \in J_i}\}_{i \in I} \longmapsto \{a_j^{(i)}\}_{(i,j) \in \coprod_{i \in I} J_i}.$$

In order for these definitions to make $-^{\bowtie}$ into a monad, we need to make sure that the way we choose the coproduct of indexing sets satisfies the following conditions:

$$\coprod_{\coprod_{k \in K} J_k} I_{j,k} = \coprod_{k \in K} \coprod_{j \in J_k} I_{j,k} \qquad \text{and} \qquad \coprod_{i \in I} \{*\} = \coprod_{\{*\}} I = I.$$

We do this in the following manner. For any elements a, b in a finite set I, we define $a \circ b$ to be the tuple (a, b) if neither a nor b is itself a tuple. If $a = (x_1, \ldots, x_n)$ and b is not a tuple, then $a \circ b = (x_1, \ldots, x_n, b)$; if a is not a tuple and b is a tuple $(y_1, \ldots, y_m)$ then $a \circ b = (a, y_1, \ldots, y_m)$. If both a and b are tuples, as above, then $a \circ b = (x_1, \ldots, x_n, y_1, \ldots, y_m)$. We then define

$$\coprod_{i \in I} J_i = \begin{cases} I & \text{if } J_i = \{*\} \text{ for all } i \in I, \\ J_i & \text{if } I = \{*\}, \\ \{i \circ j \mid i \in I, \ j \in J_i\} & \text{otherwise.} \end{cases}$$

It is easy to check that this satisfies the conditions we need. $\square$

Lemma 4.1.3. *There exists a natural transformation $\nu : \mathrm{SC}(-^{\bowtie}) \Longrightarrow \mathrm{SC}(-)$ which for every polytope complex $\mathcal{C}$ is exactly left adjoint to $\mathrm{SC}(\eta_\mathcal{C}) : \mathrm{SC}(\mathcal{C}) \longrightarrow \mathrm{SC}(\mathcal{C}^{\bowtie})$. The counit of this adjunction will be the identity transformation.*

Proof. Fix a polytope complex $\mathcal{C}$, and let $G = \mathrm{SC}(\eta_\mathcal{C})$. To show that G has a left adjoint it suffices to show that for any $B \in \mathrm{SC}(\mathcal{C}^{\bowtie})$, $(B \downarrow G)$ has an initial object. If we write $B = \{B_j\}_{j \in J}$, where $B_j = \{b_k^j\}_{k \in K_j}$ then the pure covering sub-map

$$\{B_j\}_{j \in J} \twoheadleftarrow\!\!-\!-\!-\!-\! \{\{b_k^j\}\}_{(j,k) \in \coprod_{j \in J} K_j}$$

is the desired object; we define $\nu_\mathcal{C}$ to be the adjoint where $\nu_C(B) = \{b_k^j\}_{(j,k) \in \coprod_j K_j}$. Then the unit is objectwise a pure covering sub-map — thus a weak equivalence — and the counit is the identity, as desired. To see that these assemble into a natural transformation, note that $\nu_\mathcal{C}$ "flattens" each set of sets by covering it with a set of singletons. By purely set-theoretic observations it is clear that this commutes with applying a functor pointwise to each set element, so ν does, indeed, assemble into a natural transformation.

It remains to show that $\nu_\mathcal{C}$ is exact. As left adjoints commute with colimits and $\mathrm{SC}(\mathcal{C})$ has all pushouts, $\nu_\mathcal{C}$ preserves all pushouts. The fact that F preserves cofibrations and weak equivalences follows from the definition of F and the fact that covering sub-maps in $\mathcal{C}^{\bowtie}$ are defined pointwise. $\square$

Now consider the Kleisli category of this monad, $\mathrm{Kl}(-^{\bowtie})$. We have a natural inclusion $\iota : \mathbf{PolyCpx} \longrightarrow \mathrm{Kl}(-^{\bowtie})$ which is the identity on objects, and takes a polytope functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ to the functor $\eta_\mathcal{D} F$. Informally speaking, $\mathrm{Kl}(-^{\bowtie})$ is the category of sets of polytopes that can be "added", in the sense that we can think of a covering sub-map $\{a_i\}_{i \in I} \dashrightarrow\!\!\!\twoheadrightarrow \{b_j\}_{j \in J}$ as expressing the relation $\sum_{i \in I} a_i = \sum_{j \in J} b_j$. Using the functor given by lemma 4.1.3 we can extend SC to a functor on $\mathrm{Kl}(-^{\bowtie})$ rather than just on $\mathbf{PolyCpx}$.

Lemma 4.1.4. *The functor $\mathrm{SC} : \mathbf{PolyCpx} \longrightarrow \mathbf{WaldCat}$ factors through ι.*

Proof. We define a functor $\widetilde{\mathrm{SC}} : \mathrm{Kl}(-^{\bowtie}) \longrightarrow \mathbf{WaldCat}$ by setting $\widetilde{\mathrm{SC}}(\mathcal{C}) = \mathrm{SC}(\mathcal{C})$ on polytope complexes $\mathcal{C} \in \mathrm{Kl}(-^{\bowtie})$, and by

$$\widetilde{\mathrm{SC}}(F : \mathcal{C} \longrightarrow \mathcal{D}) = \nu_\mathcal{D} \mathrm{SC}(F) : \mathrm{SC}(\mathcal{C}) \longrightarrow \mathrm{SC}(\mathcal{D}^{\bowtie}) \longrightarrow \mathrm{SC}(\mathcal{D}).$$

Note that given any polytope functor $F : \mathcal{C} \longrightarrow \mathcal{D}$,

$$\widetilde{\mathrm{SC}}(\iota(F)) = \nu_\mathcal{D} \mathrm{SC}(\eta_\mathcal{D}) \mathrm{SC}(F) = \mathrm{SC}(F),$$

as $\nu_\mathcal{D}$ is left adjoint to $\mathrm{SC}(\eta_\mathcal{D})$ and the counit of the adjunction is the identity. Thus $\widetilde{\mathrm{SC}} \iota = \mathrm{SC}$, as desired. $\square$

44

By abusing notation we will write SC for the extension $\text{Kl}(-^{\bowtie}) \longrightarrow \textbf{WaldCat}$.

We finish up with an example of a polytope complex which is an algebra over $-^{\bowtie}$, and a polytope complex which is not an algebra over $-^{\bowtie}$. Consider $\mathcal{G}_{E^n}$, the polytope complex of n-dimensional Euclidan polytopes. We can define a functor $\mathcal{G}_{E^n}^{\bowtie} \longrightarrow \mathcal{G}_{E^n}$ by mapping any set of pairwise disjoint polytopes to the union of that set (which is well-defined if we define a polytope to be a nonempty union of simplices). It is easy to check that this does, in fact, make $\mathcal{G}_{E^n}$ into an algebra over $-^{\bowtie}$.

Now let $\mathcal{C}$ be the polytope complex of rectangles in $\mathbb{R}^2$ whose sides are parallel to the coordinate axes, with the group of translations acting on it. We claim that this is *not* an algebra over $-^{\bowtie}$. Indeed, suppose that it were, so we have a functor $F : \mathcal{C}^{\bowtie} \longrightarrow \mathcal{C}$. Consider a rectangle R split into four sub-rectangles:

R_1	R_2
R_3	R_4

We know that $F(\{R\}) = R$ and $F(\{R_i\}) = R_i$. Now consider $F(\{R_1, R_4\})$. This must sit inside R, and also contain both R_1 and R_4, so it must be R. Similarly, $F(\{R_2, R_3\}) = R$. But then

$$R = F(\{R_1, R_4\}) \times_{F(\{R\})} F(\{R_2, R_3\}) = F(\{R_1, R_4\} \times_{\{R\}} \{R_2, R_3\}) = F(\emptyset) = \emptyset.$$

Contradiction. So $\mathcal{C}$ is not an algebra over $-^{\bowtie}$.

4.2   Filtered Polytopes

The $S_{\bullet}$ construction considers sequences of objects included into one another. In this section we will look at filtered objects where all of the cofibrations are actually acyclic cofibrations.

Let $W_n\text{SC}(\mathcal{C})$ be the full subcategory of $F_n\text{SC}(\mathcal{C})$ which contains all objects

$$A_1 \overset{\sim}{\hookrightarrow} A_2 \overset{\sim}{\hookrightarrow} \cdots \overset{\sim}{\hookrightarrow} A_n.$$

We can make $W_n\text{SC}(\mathcal{C})$ into a Waldhausen category by taking the structure induced from $F_n\text{SC}(\mathcal{C})$. Then $W_n\text{SC}(\mathcal{C})$ contains $\widetilde{W}_n\text{SC}(\mathcal{C})$ — the full subcategory of $W_n\text{SC}(\mathcal{C})$ of all such objects which can be represented by only pure sub-maps — as an equivalent subcategory (by lemma 2.3.5).

Our goal for this section is to define a polytope complex $f_n\mathcal{C}$ such that $\text{SC}(f_n\mathcal{C})$ is equivalent (as a Waldhausen category) to $W_n\text{SC}(\mathcal{C})$.

Definition 4.2.1. Let $f_n\mathcal{C}$ be the following polytope complex. An object $A \in f_n\mathcal{C}$ is a diagram

$$A_1 \twoheadleftarrow\text{-----} A_2 \twoheadleftarrow\text{-----} \cdots \twoheadleftarrow\text{-----} A_n$$

in $\text{Tw}(\mathcal{C}_p)$ such that each $A_i \in \mathcal{C}^{\bowtie}$ and A_1 is a singleton set. The vertical morphisms $p : A \dashrightarrow B$ are diagrams

$$
\begin{array}{ccccccc}
A_1 & \twoheadleftarrow\text{----} & A_2 & \twoheadleftarrow\text{----} & \cdots & \twoheadleftarrow\text{----} & A_n \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & & & \downarrow{\scriptstyle p_n} \\
B_1 & \twoheadleftarrow\text{----} & B_2 & \twoheadleftarrow\text{----} & \cdots & \twoheadleftarrow\text{----} & B_n
\end{array}
$$

in $\mathcal{C}^{\bowtie}$, and the horizontal morphisms are defined analogously. We put a topology on $f_n\mathcal{C}$ by defining a family $\{X_\alpha \dashrightarrow X\}_{\alpha \in A}$ to be a covering family if for each $i = 1, \ldots, n$ the family $\{X_{\alpha i} \dashrightarrow X_i\}_{\alpha \in A}$ is a covering family in $\mathcal{C}^{\bowtie}$.

Now we construct the functors which give an isomorphism between $\widetilde{W}_n\mathrm{SC}(\mathcal{C})$ and $\mathrm{SC}(f_n\mathcal{C})$. The functor $H : \mathrm{SC}(f_n\mathcal{C}) \longrightarrow \widetilde{W}_n\mathrm{SC}(\mathcal{C})$ simply takes an object of $\mathrm{SC}(f_n\mathcal{C})$ to the sequence of its levelwise unions. More formally, given an object $\{a_i\}_{i \in I}$ in $\mathrm{SC}(f_n\mathcal{C})$, where for each $i \in I$ we have

$$a_i = a_i^1 \leftarrow\!\!-\!\!-\!\!-\!\!- a_i^2 \leftarrow\!\!-\!\!-\!\!-\!\!- \cdots \leftarrow\!\!-\!\!-\!\!-\!\!- a_i^n,$$

with $a_i^j \in \mathcal{C}^{\bowtie}$, we define an object $H(\{a_i\}_{i \in I}) \in \widetilde{W}_n\mathrm{SC}(\mathcal{C})$ by

$$A_1 \lhook\joinrel\xrightarrow{\sim} A_2 \lhook\joinrel\xrightarrow{\sim} \cdots \lhook\joinrel\xrightarrow{\sim} A_n$$

where $A_j = \coprod_{i \in I} a_i^j \in \mathrm{Tw}(\mathcal{C}_p)$. In other words, we consider each object a_i to be a diagram in $\mathrm{Tw}(\mathcal{C}_p)$ and we take the coproduct of all of these diagrams.

To construct an inverse $G : \widetilde{W}_n\mathrm{SC}(\mathcal{C}) \longrightarrow \mathrm{SC}(f_n\mathcal{C})$ to this functor we take a diagram in $\widetilde{W}_n\mathrm{SC}(\mathcal{C})$ and turn it into a coproduct of pure covering sub-maps in $\mathrm{Tw}(\mathcal{C}_p)$. It will turn out that each of these diagrams represents an object of $\mathrm{SC}(f_n\mathcal{C})$, which will give us the desired functor. Given an object $A \in \widetilde{W}_n\mathrm{SC}(\mathcal{C})$ represented by

$$A_1 \lhook\joinrel\xrightarrow{\sim} A_2 \lhook\joinrel\xrightarrow{\sim} \cdots \lhook\joinrel\xrightarrow{\sim} A_n$$

we know that we can write every acyclic cofibration in this diagram as a pure covering sub-map. When a morphism can be represented in this way the representation is unique, so we can in fact consider this object to be a diagram

$$A_1 \leftarrow\!\!-\!\!-\!\!-\!\!- A_2 \leftarrow\!\!-\!\!-\!\!-\!\!- \cdots \leftarrow\!\!-\!\!-\!\!-\!\!- A_n$$

in $\mathrm{Tw}(\mathcal{C}_p)$. This sits above an analogous diagram in **FinSet**. Given any such diagram in **FinSet** we can write it as a coproduct of fibers over the indexing set I of A_1. Consequently we can write A as

$$\coprod_{i \in I} \left(A_1^i \leftarrow\!\!-\!\!-\!\!-\!\!- A_2^i \leftarrow\!\!-\!\!-\!\!-\!\!- \cdots \leftarrow\!\!-\!\!-\!\!-\!\!- A_n^i \right).$$

We will show that each of these component diagrams actually represents an object of $f_n\mathcal{C}$. Indeed, we know by definition that A_1^i is a singleton set $\{a_i\}$. Thus if we write A_j^i as $\{b_k\}_{k \in K}$, from the fact that each of the morphisms in the diagram is a sub-map we know that for $K, k' \in K$, $b_k \times_{a_i} b_{k'} = \emptyset$, so each A_j^i is an object of $\mathcal{C}^{\bowtie}$. Thus this diagram is an object of $f_n\mathcal{C}$ as desired. This definition extends directly to the morphisms as well.

We need to prove that these functors are exact and inverses. It is easy to see that they are inverses on objects, so we focus our attention on the morphisms in the categories. To this end we define two projection functors $\pi_1 : W_n\mathrm{SC}(\mathcal{C}) \longrightarrow \mathrm{SC}(\mathcal{C})$ and $P_1 : f_n\mathcal{C} \longrightarrow \mathcal{C}$ which will help us analyze the situation.

Lemma 4.2.2. *Let* $P_1 : f_n\mathcal{C} \longrightarrow \mathcal{C}$ *take a diagram* $A_1 \leftarrow\!\!-\!\!- \cdots \leftarrow\!\!-\!\!- A_n$ *to the unique element of* A_1. *Then the functor* $\mathrm{SC}(P_1)$ *is faithful.*

Proof. It suffices to show that given any diagram

$$A_1 \overset{\sim}{\hookrightarrow} A_2$$

with $f : A_1 \to B_1$ downward, and

$$B_1 \overset{\sim}{\hookrightarrow} B_2$$

there exists at most one morphism $g : A_2 \longrightarrow B_2$ that makes the diagram commute. In particular, if we consider the diagram in $\mathrm{Tw}((f_n\mathcal{C})_p)$ representing such a commutative square, we have

$$
\begin{array}{ccccc}
A_2 & \dashleftarrow & A_2' & \longrightarrow & B_2 \\
\downarrow & & \downarrow & & \downarrow \\
A_1 & \dashleftarrow & A_1' & \overset{\sigma}{\longrightarrow} & B_1
\end{array}
$$

where the morphism $A_2' \dashrightarrow A_1'$ is a covering sub-map because the square commutes. In particular, this means that $A_2' = \sigma^* B_2$. Thus we can complete the square exactly when we have a sub-map $\sigma^* B_2 \dashrightarrow A_2$ which makes the left-hand square commute, of which there is at most one. $\qquad\square$

And, completely analogously, we can prove a symmetric statement about π_1.

Lemma 4.2.3. *Let $\pi_1 : \widetilde{W}_n\mathrm{SC}(\mathcal{C}) \longrightarrow \mathrm{SC}(\mathcal{C})$ be the exact functor which takes an object $A_1 \overset{\backsim}{\hookrightarrow} \cdots \overset{\backsim}{\hookrightarrow} A_n$ to A_1. Then π_1 is faithful.*

We can now prove the main result of this section.

Proposition 4.2.4. $W_n\mathrm{SC}(\mathcal{C})$ *is exactly equivalent to* $\mathrm{SC}(f_n\mathcal{C})$.

Proof. We will show that G and H induce isomorphisms between $\widetilde{W}_n\mathrm{SC}(\mathcal{C})$ and $\mathrm{SC}(f_n\mathcal{C})$, which will show the result as $\widetilde{W}_n\mathrm{SC}(\mathcal{C})$ is exactly equivalent to $W_n\mathrm{SC}(\mathcal{C})$.

It is clear that GH and HG are the identity on objects, so it remains to show that they are inverses on morphisms. From the definitions it is easy to see that $\mathrm{SC}(P_1)G = \pi_1$ and that $\pi_1 H = \mathrm{SC}(P_1)$, so that

$$\mathrm{SC}(P_1)GH = \pi_1 H = \mathrm{SC}(P_1) \qquad \text{and} \qquad \pi_1 HG = \mathrm{SC}(P_1)G = \pi_1.$$

As $\mathrm{SC}(P_1)$ and π_1 are both faithful, if we consider these on hom-sets we see that G and H are mutual inverses on any hom-set. Thus $\widetilde{W}_n\mathrm{SC}(\mathcal{C})$ is isomorphic to $\mathrm{SC}(f_n\mathcal{C})$.

It remains to show that G and H are exact functors. We already know that they preserve pushouts, so all it remains to show is that they preserve cofibrations and weak equivalences. Note that we know by definition that π_1 and $\mathrm{SC}(P_1)$ are exact functors; thus in order to show that G and H are exact it suffices to show that π_1 and $\mathrm{SC}(P_1)$ reflect cofibrations and weak equivalences.

For both of these cases it suffices to show that in $\mathrm{Tw}(\mathcal{C}_p)$ if

$$
\begin{array}{ccccc}
A_1 & \overset{p}{\dashleftarrow} & A_1' & \overset{\sigma}{\longrightarrow} & B_1 \\
\uparrow i & & \uparrow & & \uparrow j \\
A_2 & \underset{q}{\dashleftarrow} & \sigma^* B_2 & \overset{\widetilde{\sigma}}{\longrightarrow} & B_2
\end{array}
$$

commutes and σ has an injective set-map, then q is a covering sub-map and $\widetilde{\sigma}$ has an injective set-map. The first of these is true because q is the pullback along i of jp, which is a covering sub-map; the second of these is true because pullbacks preserve injectivity of set-maps. So we are done. $\square$

Remark. If we define P_n and π_n analogously to P_1 and π_1 we see that $\mathrm{SC}(P_n)$ and π_n are exact equivalences of categories. Thus $\mathrm{SC}(f_n\mathcal{C})$ and $W_n\mathrm{SC}(\mathcal{C})$ are exactly equivalent, as they are both equivalent to $\mathrm{SC}(\mathcal{C})$. We do not use these functors because they are not compatible with the simplicial maps of $S_n\mathrm{SC}(\mathcal{C})$, and thus will not give inverse equivalences on the K-theory.

4.3   Combing

Let $f : A \hookrightarrow B \in \mathrm{SC}(\mathcal{C})$ be a cofibration. We define the *image* of f to be the cofiber of the canonical cofibration $B/A \hookrightarrow B$ (see corollary 3.4.7). We will write the image of f as $\mathrm{im}(f)$; when the cofibration is clear from context we will often write is as $\mathrm{im}_B(A)$. Note that we have an acyclic cofibration

$$A \overset{\sim}{\hookrightarrow} \mathrm{im}_B(A)$$

More concretely, if we write $A = \{a_i\}_{i \in I}$ and $B = \{b_j\}_{j \in J}$, and if f can be represented by covering sub-map p and the shuffle σ, $\mathrm{im}_B(A) = \{b_j\}_{j \in \mathrm{im}\,\sigma}$.

   Now suppose that we are given an object $A = (A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_n) \in F_n\mathrm{SC}(\mathcal{C})$. Then we define the *i*-th *strand* of A, $\mathrm{St}_i(A)$ to be the diagram

$$A_i/A_{i-1} \overset{\sim}{\hookrightarrow} \mathrm{im}_{A_{i+1}}(A_i/A_{i-1}) \overset{\sim}{\hookrightarrow} \cdots \overset{\sim}{\hookrightarrow} \mathrm{im}_{A_n}(A_i/A_{i-1}).$$

We can consider $\mathrm{St}_i(A)$ to be an object of $F_n\mathrm{SC}(\mathcal{C})$ by padding the front with sufficiently many copies of the zero object; then we can canonically write $A = \coprod_{i=1}^{n} \mathrm{St}_i(A)$.

Definition 4.3.1. We will say that a morphism $f : A \longrightarrow B \in F_n\mathrm{SC}(\mathcal{C})$ is *layered* if for all $1 \le i < k \le n$ the diagram

$$
\begin{array}{ccc}
A_k/A_i & \hookrightarrow & A_k \\
{\scriptstyle f_k/f_i}\downarrow & & \downarrow{\scriptstyle f_k} \\
B_k/B_i & \hookrightarrow & B_k
\end{array}
$$

commutes. We define $L_n\mathrm{SC}(\mathcal{C})$ to be the subcategory of $F_n\mathrm{SC}(\mathcal{C})$ containing all layered morphisms.

   Not all morphisms are layered. For example, let X be a nonzero object, and let $g : X \hookrightarrow Y$ be any cofibration in $\mathrm{SC}(\mathcal{C})$. Then $\emptyset \hookrightarrow Y$ and $X \hookrightarrow Y$ are both objects of $F_2\mathrm{SC}(\mathcal{C})$ and we have a non-layered morphism

$$
\begin{array}{ccc}
\emptyset & \hookrightarrow & Y \\
\uparrow & & \| \\
X & \hookrightarrow & Y
\end{array}
$$

between them. As all cofibers of acyclic cofibrations are trivial, all morphisms of $W_n\mathrm{SC}(\mathcal{C})$ are layered. In fact, if we let $I_{ni} : F_{n-i+1}\mathrm{SC}(\mathcal{C}) \longrightarrow F_n\mathrm{SC}(\mathcal{C})$ be the functor which pads a diagram with i copies of $\emptyset$ at the beginning, then the restriction of I_{ni} to $L_{n-i+1}\mathrm{SC}(\mathcal{C})$ has its image in $L_n\mathrm{SC}(\mathcal{C})$.

Lemma 4.3.2.

1. f is layered if and only if for all $1 \leq i < n$, the morphism

$$f_{i,i+1} : (A_i \hookrightarrow A_{i+1}) \longrightarrow (B_i \hookrightarrow B_{i+1}) \in F_2\mathrm{SC}(\mathcal{C})$$

   is layered.

2. Given any commutative square $(A_1 \hookrightarrow A_2) \longrightarrow (B_1 \hookrightarrow B_2)$ we have an induced commutative square $(\mathrm{im}_{A_2}(A_1) \hookrightarrow A_2) \longrightarrow (\mathrm{im}_{B_2}(B_1) \hookrightarrow B_2)$. Thus if the commutative square $(A_1 \hookrightarrow A_2) \longrightarrow (B_1 \hookrightarrow B_2)$ is layered then so is the commutative square $(A_2/A_1 \hookrightarrow A_2) \longrightarrow (B_2/B_1 \hookrightarrow$

Proof.

1. The forwards direction is trivial, so it suffices to prove the backwards direction. We will prove it by induction on k. For $k = i + 1$ this is given. Now suppose that it is true up to k. Then we have the following diagram

   in which we know that every face other than the front one commutes; we want to show that the front face also commutes. Let

$$\alpha : A_{k+1}/A_i \hookrightarrow A_{k+1} \longrightarrow B_{k+1}$$

   and

$$\beta : A_{k+1}/A_i \longrightarrow B_{k+1}/B_i \hookrightarrow B_{k+1};$$

   we want to show that $\alpha = \beta$. As A_{k+1}/A_i is the pushout of the diagram

$$A_{k+1} \longleftarrow A_k \longrightarrow A_k/A_i,$$

   it suffices to show that $f\alpha = f\beta$ and $g\alpha = g\beta$ for $f : A_k/A_i \hookrightarrow A_{k+1}/A_o$ and $g : A_{k+1} \longrightarrow A_{k+1}/A_i$. The first of these follows directly from the fact that all faces of the cube but the front one commute. For the second of these, note that we have a weak equivalence $A_k \amalg A_{k+1}/A_k \xrightarrow{\sim} A_{k+1}$ and weak equivalences are epimorphisms, so in fact it suffices to show that $g_1\alpha = g_1\beta$ and $g_2\alpha = g_2\beta$ for

$$g_1 = A_k \hookrightarrow A_{k+1} \longrightarrow A_{k+1}/A_i$$

and

$$g_2 = A_{k+1}/A_k \hookrightarrow A_{k+1} \longrightarrow A_{k+1}/A_i.$$

The first of these follows from a simple diagram chase, keeping in mind that all horizontal cofibrations in this cube are actually sections of cofiber maps. The second of these also turns into a simple diagram chase after noting that for any sequence of cofibrations $X \hookrightarrow Y \hookrightarrow Z$ in $\mathrm{SC}(\mathcal{C})$ we have

$$Z/Y \hookrightarrow Z \longrightarrow Z/X \hookrightarrow Z = Z/Y \hookrightarrow Z.$$

2. Note that if we have a commutative square $(A_1 \hookrightarrow A_2) \longrightarrow (B_1 \hookrightarrow B_2)$ it can be represented by the following commutative diagram in $\mathrm{Tw}(\mathcal{C}_p)$:

$$
\begin{array}{ccccc}
A_1 & \twoheadleftarrow\!\!-\!-\!-\!- & A_1' & \longrightarrow & A_2 \\
\big\uparrow & & \big\uparrow & \star & \big\uparrow \\
A_1'' & \twoheadleftarrow\!\!-\!-\!-\!- & X & \longrightarrow & A_2' \\
\big\downarrow & \star & \big\downarrow & & \big\downarrow \\
B_1 & \twoheadleftarrow\!\!-\!-\!-\!- & B_1' & \longrightarrow & B_2
\end{array}
$$

where the starred squares are pullbacks. We know $A_1' \cong \mathrm{im}_{A_2}(A_1)$ and $B_1' \cong \mathrm{im}_{B_2}(B_1)$ and the middle column in the diagram represents a morphism between them. In fact, the right-hand half of this diagram is — up to isomorphism — exactly the square that the lemma states exists. The second part of the statement follows because the cofiber of $A_2/A_1 \hookrightarrow A_2$ is exactly $\mathrm{im}_{A_2}(A_1)$.

$\square$

Lemma 4.3.3. $L_n\mathrm{SC}(\mathcal{C})$ *is a Waldhausen category. The cofibrations (resp. weak equivalences) in $L_n\mathrm{SC}(\mathcal{C})$ are exactly the morphisms which are levelwise cofibrations (resp. weak equivalences). $L_n\mathrm{SC}(\mathcal{C})$ is a simplification of $F_n\mathrm{SC}(\mathcal{C})$.*

We postpone the proof of this lemma until section 4.7 as it is technical and not particularly illuminating.

Lemma 4.3.4. St_i *is an exact functor $L_n\mathrm{SC}(\mathcal{C}) \longrightarrow W_{n-i+1}\mathrm{SC}(\mathcal{C})$. We have a natural transformation $\eta_i : I_{ni}\mathrm{St}_i \longrightarrow \mathrm{id}$ given by the natural inclusions $\mathrm{im}_{A_k}(A_i/A_{i-1}) \hookrightarrow A_k$. On $W_{n-i+1}\mathrm{SC}(\mathcal{C})$,*

$$\mathrm{St}_i I_{ni} = \mathrm{id} \qquad and \qquad \mathrm{St}_j I_{ni}\mathrm{St}_i = 0$$

for $i \neq j$.

Proof. Let $f : A \longrightarrow B \in L_n\mathrm{SC}(\mathcal{C})$. We claim that the morphism

$$
\begin{array}{ccccccc}
A_i/A_{i-1} & \hookrightarrow & A_{i+1} & \hookrightarrow & \cdots & \hookrightarrow & A_n \\
{\scriptstyle f_i/f_{i-1}}\big\downarrow & & {\scriptstyle f_{i+1}}\big\downarrow & & & & {\scriptstyle f_n}\big\downarrow \\
B_i/B_{i-1} & \hookrightarrow & B_{i+1} & \hookrightarrow & \cdots & \hookrightarrow & B_n
\end{array}
$$

in $F_{n-i+1}\mathrm{SC}(\mathcal{C})$ is also layered. By lemma 4.3.2(1) we know that it suffices to check that each square in this diagram satisfies the layering condition. All squares but the first one satisfy it because f is layered. The first square can be factored as

$$\begin{array}{ccccc}
A_i/A_{i-1} & \hookrightarrow & A_i & \hookrightarrow & A_{i+1} \\
\downarrow{\scriptstyle f_i/f_{i-1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i+1}} \\
B_i/B_{i-1} & \hookrightarrow & B_i & \hookrightarrow & B_{i+1}
\end{array}$$

The right-hand square satisfies the layering condition because f is layered; the left-hand square satisfies it by lemma 4.3.2(2). If we let $T_i : L_n\mathrm{SC}(\mathcal{C}) \longrightarrow L_{n-i+1}\mathrm{SC}(\mathcal{C})$ be the functor taking an object to this truncation then T_i is exact, as by lemma 4.3.3 layered cofibrations are exactly levelwise. Note that $T_i I_{ni} = \mathrm{id}$ and we have a natural transformation $\eta' : I_{ni}T_i \longrightarrow \mathrm{id}$.

We can write $\mathrm{St}_i = \mathrm{St}_1 T_i$; thus if we can prove the lemma for $i = 1$ we will be done. The fact that f is layered implies that St_1 is a functor $L_n\mathrm{SC}(\mathcal{C}) \longrightarrow W_n\mathrm{SC}(\mathcal{C})$ (as it is obtained by taking levelwise cofibers in a commutative diagram). As colimits commute past one another, we see that this preserves pushouts along cofibrations. Thus to see that St_1 is exact it remains to show that it preserves cofibrations and weak equivalences, which is true because both weak equivalences and cofibrations are preserved by taking cofibers, and St_1 simply takes two successive cofibers.

The natural transformation η_1 is obtained by factoring each cofibration $A_1 \hookrightarrow A_k$ through the weak equivalence $A_1 \overset{\sim}{\hookrightarrow} \mathrm{im}_{A_k}(A_1)$. By the discussion in the proof of lemma 4.3.2(2) this will in fact be a natural transformation.

Now we show the last part of the lemma. It is a simple computation to see that $\mathrm{St}_i|_{W_m\mathrm{SC}(\mathcal{C})}$ is the identity if $i = 1$, and 0 otherwise. Thus $\mathrm{St}_i I_{ni} = \mathrm{St}_1 T_i I_{ni} = \mathrm{St}_1$ is the identity. If $j < i$ then the j-th component of $I_{ni}\mathrm{St}_i$ is $\emptyset$, so $\mathrm{St}_j I_{ni}\mathrm{St}_i = 0$ trivially. If $j > i$ then $\mathrm{St}_j I_{ni}\mathrm{St}_i = \mathrm{St}_{j-i+1}T_i I_{ni}\mathrm{St}_i = \mathrm{St}_{j-i+1}\mathrm{St}_i = 0$ because $j - i + 1 > 1$. Thus we are done. $\qquad\square$

Proposition 4.3.5. *Let $CP : \prod_{m=1}^{n} W_m\mathrm{SC}(\mathcal{C}) \longrightarrow L_n\mathrm{SC}(\mathcal{C})$ be the functor which takes an n-tuple $(X_1.\ldots,X_n)$ to $\coprod_{i=1}^{n} I_{n(n-i+1)}(X_i)$. We have an exact equivalence of categories*

$$\mathrm{St} : L_n\mathrm{SC}(\mathcal{C}) \rightleftarrows \prod_{m=1}^{n} W_m\mathrm{SC}(\mathcal{C}) : CP,$$

where St is induced by the functors St_m for $m = 1, \ldots, n$.

Proof. We first show that these form an equivalence of categories. From lemma 4.3.4 above, we know that the composition $\mathrm{St} \circ CP$ is the identity on each component (as $\mathrm{St}_i\mathrm{St}_j A$ is the zero object for $i \neq j$), and thus the identity functor. On the other hand, the composition $CP \circ \mathrm{St}$ has a natural transformation $\eta = \coprod_{m=1}^{n} \eta_{n-m+1} : CP \circ \mathrm{St} \longrightarrow \mathrm{id}$; it remains to show that η is in fact a natural isomorphism. However, for every object A, η_A is simply the natural morphism $\coprod_{i=1}^{n} \mathrm{St}_i(A) \longrightarrow A$, which is clearly an isomorphism. So these are in fact inverse equivalences.

As each component of CP is exact (as cofibrations and weak equivalences in $L_n\mathrm{SC}(\mathcal{C})$ are levelwise) we know that CP is exact. On the other hand, St_i is exact for all i, so St is exact. So we are done. $\qquad\square$

The functor St "combs" an object of $L_n\mathrm{SC}(\mathcal{C})$ by separating all of the strands of different lengths.

4.4 Simplicial Polytope Complexes

Our goal for this section is to assemble the $f_i\mathcal{C}$ into a simplicial polytope complex which will mimic Waldhausen's $S.$ construction.

Given $1 \leq i \leq n$ we define a morphism $\partial_i^{(n)} : f_n\mathcal{C} \longrightarrow f_{n-1}\mathcal{C}$ in $\mathrm{Kl}(-^{\bowtie})$ induced by skipping the i-th term. If $i > 1$ this functor comes from **PolyCpx**; if $i = 1$ then we cut off the singleton element from the front, and therefore have to split the rest of the object into fibers over the different polytopes in the (newly) first set. (This is why $\partial_i^{(n)}$ is a morphism in $\mathrm{Kl}(-^{\bowtie})$ rather than in **PolyCpx**.) We define the morphism $\sigma_i^{(n)} : f_n\mathcal{C} \longrightarrow f_{n+1}\mathcal{C}$ to be the morphism of $\mathrm{Kl}(-^{\bowtie})$ given by the polytope functor which repeats the i-th stage. For $i \leq 0$ we define the morphisms $\sigma_i^{(n)} : f_n\mathcal{C} \longrightarrow f_n\mathcal{C}$ and $\partial_i^{(n)} : f_n\mathcal{C} \longrightarrow f_n\mathcal{C}$ to be the identity on $f_n\mathcal{C}$. Note that the only one of these morphisms that does not come from **PolyCpx** is $\partial_1^{(n)}$.

Definition 4.4.1. Let $s_n\mathcal{C} = \bigvee_{i=1}^{n} f_i\mathcal{C}$. We define simplicial structure maps between these by

$$\partial_0 = (0 : f_n\mathcal{C} \longrightarrow s_{n-1}\mathcal{C}) \vee \bigvee_{i=1}^{n-1} (1 : f_i\mathcal{C} \longrightarrow f_i\mathcal{C}),$$

where 0 is the polytope functor sending everything to the initial object $\emptyset$,

$$\partial_i = \bigvee_{i=1}^{n} \partial_{n-j+i}^{(j)} \qquad \text{for } i \geq 1,$$

and

$$\sigma_i : s_n\mathcal{C} \longrightarrow s_{n+1}\mathcal{C} = \bigvee_{j=1}^{n} \sigma_{n-j+i}^{(j)} \qquad \text{for } i \geq 0.$$

It is easy to see that with the ∂_i's as the face maps and the σ_i's as the degeneracy maps, $s.\mathcal{C}$ becomes a simplicial polytope complex.

Putting proposition 4.3.5 together with proposition 4.2.4 we see that we have an exact equivalence $L_n\mathrm{SC}(\mathcal{C}) \longrightarrow \prod_{m=1}^{n} \mathrm{SC}(f_m\mathcal{C})$. However, $\prod_{m=1}^{n} \mathrm{SC}(f_m\mathcal{C})$ is exactly equivalent to $\mathrm{SC}(\bigvee_{m=1}^{n} f_m\mathcal{C}) = \mathrm{SC}(s_n\mathcal{C})$. Thus we have proved the following:

Corollary 4.4.2. *Let* $H_m : \mathrm{SC}(f_m\mathcal{C}) \longrightarrow \widetilde{W}_m\mathrm{SC}(\mathcal{C})$ *be the functor in proposition 4.2.4,* $\iota_m : \widetilde{W}_m\mathrm{SC}(\mathcal{C}) \longrightarrow W_m\mathrm{SC}(\mathcal{C})$ *be the natural inclusion, and* CP_n *be the functor from proposition 4.3.5. Then* $F_n = CP_n \circ (\prod_{m=1}^{n} \iota_m \circ H_m)$ *is an exact equivalence of categories.*

Now we know that $L.\mathrm{SC}(\mathcal{C})$ is a simplicial Waldhausen category, and $\mathrm{SC}(s.\mathcal{C})$ is a simplicial Waldhausen category. $F.$ is a levelwise exact equivalence; we would like to show that it commutes with the simplicial maps, and therefore assembles to a functor of simplicial Waldhausen categories. This will allow us to conclude that the two constructions give equivalent K-theory spectra, and thus that we can work directly with the $\mathrm{SC}(s.\mathcal{C})$ definition.

Proposition 4.4.3. *The functor* $F. : \mathrm{SC}(s.\mathcal{C}) \longrightarrow L.\mathrm{SC}(\mathcal{C})$ *is an exact equivalence of simplicial Waldhausen categories.*

Proof. First we will show that $F.$ is, in fact, a functor of simplicial Waldhausen categories. In particular, it suffices to show that the following two diagrams commute for each i:

$$\begin{array}{ccc}
\mathrm{SC}(s_n\mathcal{C}) & \xrightarrow{\;\sigma_i\;} & \mathrm{SC}(s_{n+1}\mathcal{C}) \\
{\scriptstyle F_n}\downarrow & & \downarrow{\scriptstyle F_{n+1}} \\
L_n\mathrm{SC}(\mathcal{C}) & \xrightarrow{\;\sigma_i\;} & L_{n+1}\mathrm{SC}(\mathcal{C})
\end{array}$$

and

$$\begin{array}{ccc}
\mathrm{SC}(s_n\mathcal{C}) & \xrightarrow{\;\partial_i\;} \mathrm{SC}((s_{n-1}\mathcal{C})^{\bowtie}) \xrightarrow{\;\nu_{s_{n-1}\mathcal{C}}\;} & \mathrm{SC}(s_{n-1}\mathcal{C}) \\
{\scriptstyle F_n}\downarrow & & \downarrow{\scriptstyle F_{n-1}} \\
L_n\mathrm{SC}(\mathcal{C}) & \xrightarrow{\qquad\qquad\partial_i\qquad\qquad} & L_{n-1}\mathrm{SC}(\mathcal{C})
\end{array}$$

where the first diagram is a square because all σ_i's come from morphisms in **PolyCpx**. Both of these diagrams commute by simple computations, since F_n takes "levelwise unions".

Now by corollary 4.4.2 we know that levelwise F_n is an exact equivalence of Waldhausen categories. In addition, propositions 4.3.5 and 4.2.4 give us formulas for the levelwise inverse equivalences; an analogous proof shows that these also assemble into a functor of simplicial Waldhausen categories. Thus $F.$ is an equivalence of simplicial Waldhausen categories, as desired. $\qquad\square$

Suppose that $\mathcal{C}.$ is a simplicial polytope complex. We define the K-theory spectrum of $\mathcal{C}.$ by

$$K(\mathcal{C}.)_n = \left| NwS_.^{(n)}\mathrm{SC}(\mathcal{C}.) : (\Delta^{\mathrm{op}})^{n+2} \longrightarrow \mathbf{Sets} \right|.$$

(Note that this definition is compatible with the K-theory of a polytope complex, if we consider a polytope complex as a constant simplicial complex.)

Lemma 4.4.4. $K(\mathcal{C}.)$ *is a spectrum, which is an Ω-spectrum above level 0.*

In the proof of this lemma we use the following obvious generalization of lemma 5.2 in [22]. A *fiber sequence* of multisimplicial categories is a sequence which is a fibration sequence up to homotopy after geometric realization of the nerves.

Lemma 4.4.5. *([22], 5.2) Let*

$$X_{...} \longrightarrow Y_{...} \longrightarrow Z_{...}$$

be a diagram of n-simplicial categories. Suppose that the following three conditions hold:

- *the composite morphism is constant,*

- *$Z_{...m}$ is connected for all $m \geq 0$, and*

- *$X_{...m} \longrightarrow Y_{...m} \longrightarrow Z_{...m}$ is a fiber sequence for all $m \geq 0$.*

Then $X_{...} \longrightarrow Y_{...} \longrightarrow Z_{...}$ is a fiber sequence.

We now prove lemma 4.4.4.

Proof of lemma 4.4.4. Suppose that $X_{...}$ is an n-simplicial object; we will write $PX_{...}$ for the n-simplicial object in which $PX_{m_1\cdots m_n} = X_{(m_1+1)m_2\cdots m_n}$.

Consider the following sequence of functors.

$$wS_1S_.^{(n-1)}\mathrm{SC}(\mathcal{C}.) \longrightarrow PwS_.^{(n)}\mathrm{SC}(\mathcal{C}.) \longrightarrow wS_.^{(n)}\mathrm{SC}(\mathcal{C}.),$$

where the first functor is the constant inclusion as the 0-space, and the second is the contraction induced by ∂_0 on the outermost simplicial level. As $S_0\mathcal{E}$ is constant for any Waldhausen category $\mathcal{E}$, the composite of the diagram is constant. Similarly, for any $m \geq 0$ $wS_.^{(n)}\mathrm{SC}(\mathcal{C}_m)$ is connected, as if we plug in 0 to any of the $S_.$-directions we get a constant category. In addition, by proposition 1.5.3 of [23], this is a fiber sequence if $n \geq 2$. Thus by lemma 4.4.5 for $n + 1$-simplicial categories, the original diagram was a fiber sequence.

As $S_1\mathcal{E} = \mathcal{E}$ for all Waldhausen categories $\mathcal{E}$, this fiber sequence gives us, for every $n \geq 2$, an induced map $K(\mathcal{C}.)_{n-1} \longrightarrow \Omega K(\mathcal{C}.)_n$ which is a weak equivalence. It remains to show that we have a morphism $K(\mathcal{C}.)_0 \longrightarrow \Omega K(\mathcal{C}.)_1$. Considering the above sequence for $n = 1$ we have

$$wS_1\mathrm{SC}(\mathcal{C}.) \longrightarrow PwS_.\mathrm{SC}(\mathcal{C}.) \longrightarrow wS_.\mathrm{SC}(\mathcal{C}.).$$

While the third criterion from lemma 4.4.5 no longer applies, the composition is still constant and $PwS_.\mathrm{SC}(\mathcal{C}.)$ is still contractible, so we have a well-defined (up to homotopy) morphism $K(\mathcal{C}.)_0 \longrightarrow K(\mathcal{C}.)_1$, as desired. $\square$

For all n $L_.\mathrm{SC}(\mathcal{C}.)$ is a simplification of $S_.\mathrm{SC}(\mathcal{C}.)$, so $K(\mathcal{C}.)_n = |NwL_.^{(n)}\mathrm{SC}(\mathcal{C}.)|$. Now let

$$\widetilde{K}(\mathcal{C}.)_n = \left| Nw\mathrm{SC}(s_.^{(n)}\mathcal{C}.) : (\Delta^{\mathrm{op}})^{n+2} \longrightarrow \mathbf{Sets} \right|.$$

(This is clearly a spectrum, as the proof of 4.4.4 translates directly to this case.) By proposition 4.4.3 we have a morphism $\widetilde{K}(\mathcal{C}.) \longrightarrow K(\mathcal{C}.)$ induced by $F_.$, which is levelwise an equivalence (and thus an equivalence of spectra). In particular we can take $\widetilde{K}(\mathcal{C}.)$ to be the definition of the K-theory of a simplicial polytope complex.

The main advantage of passing to simplicial polytope complexes is that it allows us to start the $S_.$-construction at any level, and thus compute deloopings of our K-theory spectra on the polytope complex level.

Corollary 4.4.6. *Let $\mathcal{C}.$ be a simplicial polytope complex, and let $\sigma\mathcal{C}.$ be the simplicial polytope complex with*

$$(\sigma\mathcal{C}.)_k = s_k\mathcal{C}_k.$$

Then $\Omega K(\sigma\mathcal{C}.) \simeq K(\mathcal{C}.)$.

Proof. Geometric realizations on multisimplicial sets simply look at the diagonal, so

$$\widetilde{K}(\mathcal{C}.)_n = \left| [k] \longmapsto Nw\mathrm{SC}(s_k^{(n)}\mathcal{C}_k) \right|.$$

Thus

$$\widetilde{K}(\sigma\mathcal{C}.)_n = \left| [k] \longmapsto Nw\mathrm{SC}(s_k^{(n)}(s_k\mathcal{C}_k)) \right| = \left| [k] \longmapsto Nw\mathrm{SC}(s_k^{(n+1)}\mathcal{C}_k) \right| = \widetilde{K}(\mathcal{C}.)_{n+1}.$$

Thus $\widetilde{K}(\sigma\mathcal{C}.)$ is a spectrum which is a shift of $\widetilde{K}(\mathcal{C}.)$, so $\widetilde{K}(\mathcal{C}.) \simeq \Omega\widetilde{K}(\sigma\mathcal{C}.)$. As $\widetilde{K}(\mathcal{C}.) \simeq K(\mathcal{C}.)$, the desired result follows. $\square$

Using this corollary we can compute a polytope complex model of every sphere. From the examples in section 3.3 we know that the polytope complex $\mathbb{S} = \emptyset \dashrightarrow *$ has $K(\mathbb{S})$ equal to the sphere spectrum (up to stable equivalence). In order to get S^1 we need to deloop $\mathbb{S}$. Note that $f_n\mathbb{S} = \mathbb{S}$ for all n, so $s_n\mathbb{S} = \mathbb{S}^{\vee n}$. So the simplicial polytope complex which gives S^1 on K-theory is

$$(\mathbb{S}^{\vee 0}, \mathbb{S}^{\vee 1}, \mathbb{S}^{\vee 2}, \ldots, \mathbb{S}^{\vee n}, \ldots).$$

Since $f_n(\mathcal{C} \vee \mathcal{D}) = f_n\mathcal{C} \vee f_n\mathcal{D}$ we know that $f_n\mathbb{S}^{\vee m} = \mathbb{S}^{\vee m}$, so we compute that the simplicial polytope complex which gives S^2 on K-theory is

$$(\mathbb{S}^{\vee 0}, \mathbb{S}^{\vee 1}, \mathbb{S}^{\vee 2^2}, \mathbb{S}^{\vee 3^2}, \ldots, \mathbb{S}^{\vee n^2}, \ldots).$$

In general we obtain S^k as the K-theory of

$$(\mathbb{S}^{\vee 0^k}, \mathbb{S}^{\vee 1^k}, \mathbb{S}^{\vee 2^k}, \mathbb{S}^{\vee 3^k}, \ldots, \mathbb{S}^{\vee n^k}, \ldots).$$

Note that in fact this works for $k = 0$ as well, as long as we interpret 0^0 to be 1.

4.5 Cofibers

Waldhausen's cofiber lemma (see [23], corollary 1.5.7) gives the following formula for the cofiber of a functor $G : \mathcal{E} \longrightarrow \mathcal{E}'$. We define S_nG to be the pullback of the diagram

$$S_n\mathcal{E} \xrightarrow{\ S_nG\ } S_n\mathcal{E}' \xleftarrow{\ \partial_0\ } S_{n+1}\mathcal{E}'.$$

Define $K(S.G)$ by

$$K(S.G)_n = \left| wS_.^{(n)} S.G \right|.$$

Then the sequence $K(\mathcal{E}) \longrightarrow K(\mathcal{E}') \longrightarrow K(S.G)$ is a homotopy cofiber sequence.

Our goal for this section is to compute a version of this for polytope complexes.

Definition 4.5.1. Let $g : \mathcal{C} \longrightarrow \mathcal{D}$ be a morphism in $\mathrm{Kl}(-^{\bowtie})$. We define $\mathcal{D}/g$ to be the simplicial polytope complex with $(\mathcal{D}/g)_n = f_{n+1}\mathcal{D} \vee s_n\mathcal{C}$ and the following structure maps. For all $i > 0$, $\partial_i : (\mathcal{D}/g)_n \longrightarrow (\mathcal{D}/g)_{n-1}$ is induced by the two morphisms

$$\partial_{i+1}^{(n+1)} : f_{n+1}\mathcal{D} \longrightarrow f_n\mathcal{D} \qquad \text{and} \qquad \partial_i : s_n\mathcal{C} \longrightarrow s_{n-1}\mathcal{C}.$$

Similarly, for all $i \geq 0$, $\sigma_i : (\mathcal{D}/g)_n \longrightarrow (\mathcal{D}/g)_{n+1}$ is induced by the morphisms

$$\sigma_{i+1}^{(n+1)} : f_{n+1}\mathcal{D} \longrightarrow f_{n+2}\mathcal{D} \qquad \text{and} \qquad \sigma_i : \sigma_n\mathcal{C} \longrightarrow \sigma_{n+1}\mathcal{C}.$$

∂_0, on the other hand, is induced by the three morphisms

$$\partial_1^{(n+1)} : f_{n+1}\mathcal{D} \longrightarrow f_n\mathcal{D} \qquad f_ng : f_n\mathcal{C} \longrightarrow f_n\mathcal{D} \qquad 1 : s_{n-1}\mathcal{C} \longrightarrow s_{n-1}\mathcal{C}.$$

When g is clear from context we will often write $\mathcal{D}/\mathcal{C}$ instead of $\mathcal{D}/g$. For every $n \geq 0$ we have a diagram of polytope complexes

$$\mathcal{D} \longrightarrow (\mathcal{D}/g)_n \longrightarrow s_n\mathcal{C}$$

given by the inclusion $\mathcal{D} \longrightarrow f_{n+1}\mathcal{D}$ (as the constant objects) and the projection down to $s_n\mathcal{C}$. Then $\mathrm{SC}(\mathcal{D}/g)$ is the pullback of

$$\mathrm{SC}(s.\mathcal{C}) \xrightarrow{\mathrm{SC}(s.g)} \mathrm{SC}(s.\mathcal{D}) \xleftarrow{\partial_0} PSC(s.\mathcal{D}),$$

which exactly mirrors the construction of $S.G$. (This is clear from an analysis of $S.\mathrm{SC}(g)$ analogous to that of section 4.3.) In particular we have from [23] proposition 1.5.5 and corollary 1.5.7 that

$$wS_.^{(n)}\mathrm{SC}(\mathcal{C}) \longrightarrow wS_.^{(n)}\mathrm{SC}(\mathcal{D}) \longrightarrow wS_.^{(n)}\mathrm{SC}(\mathcal{D}/g) \longrightarrow wS_.^{(n)}\mathrm{SC}(s.\mathcal{C})$$

is a fiber sequence of $n+1$-simplicial categories.

Generalizing this to simplicial polytope complexes, we have the following proposition.

Proposition 4.5.2. *Let $g. : \mathcal{C}. \longrightarrow \mathcal{D}.$ be a morphism of simplicial polytope complexes, and write $(\mathcal{D}/g).$ for the simplicial polytope complex where $(\mathcal{D}/g)_n = (\mathcal{D}_n/g_n)_n$. Then we have a cofiber sequence of spectra*

$$K(\mathcal{C}.) \longrightarrow K(\mathcal{D}.) \longrightarrow K((\mathcal{D}/g).),$$

where the first map is induced by $g.$, and the second is induced for each n by the inclusion $\mathcal{D}_n \longrightarrow (\mathcal{D}_n/g_n)_n$ as the constant objects of $f_{n+1}\mathcal{D}_n$.

Proof. As all cofiber sequences in spectra are also fiber sequences, it suffices to show that this is a fiber sequence. As homotopy pullbacks in spectra are levelwise (see, for example, [12], section 18.3), it suffices to show that for all $n \geq 0$, $K(\mathcal{C}.)_n \longrightarrow K(\mathcal{D}.)_n \longrightarrow K((\mathcal{D}/g).)_n$ is a homotopy fiber sequence. However, as we know that above level 0 all of these are Ω-spectra it in fact suffices to show this for $n > 0$.

Thus in particular we want to show that for all $n > 0$ the sequence

$$wS_.^{(n)}\mathrm{SC}(\mathcal{C}.) \longrightarrow wS_.^{(n)}\mathrm{SC}(\mathcal{D}.) \longrightarrow wS_.^{(n)}\mathrm{SC}((\mathcal{D}/g).)$$

is a homotopy fiber sequence of $n+1$-simplicial categories. Let $\mathcal{D}./g.$ be the bisimplicial polytope complex where the (k,ℓ)-th polytope complex is $(\mathcal{D}_k/g_k)_\ell$. It will suffice to show that

$$wS_.^{(n)}\mathrm{SC}(\mathcal{D}.) \longrightarrow wS_.^{(n)}\mathrm{SC}(\mathcal{D}./g.) \longrightarrow wS_.^{(n)}\mathrm{SC}(s.\mathcal{C}.)$$

is a fiber sequence of $n+2$-simplicial categories (where $\mathcal{D}.$ is now considered a bisimplicial polytope complex); in this diagram the second morphism is induced by the projection $(\mathcal{D}_k/g_k)_\ell \longrightarrow s_\ell\mathcal{C}_k$ for all pairs (k,ℓ). Then by comparing this sequence for the functor $1 : \mathcal{C}. \longrightarrow \mathcal{C}.$ to the functor $g.$ we will be able to conclude the desired result. (In this approach we follow Waldhausen in [23], 1.5.6.)

We show this by applying lemma 4.4.5, where we fix the index of the simplicial direction of $\mathcal{C}.$ and $\mathcal{D}.$. The composition of the two functors is constant, as we first include $\mathcal{D}.$ and then project away from it, and as we do not fix any of the $S.$ indices the last space will be connected. Thus we want

$$wS_.^{(n)}\mathrm{SC}(\mathcal{D}_k) \longrightarrow wS_.^{(n)}\mathrm{SC}(\mathcal{D}_k/g_k) \longrightarrow wS_.^{(n)}\mathrm{SC}(s.\mathcal{C}_k)$$

to be a fiber sequence, which holds by our discussion above. So we are done. $\qquad\square$

56

4.6 Wide and tall subcategories

We now take a slight detour into a more computational direction. Consider the case of a polytope complex $\mathcal{D}$, together with a subcomplex $\mathcal{C}$. We know that the inclusion $\mathcal{C} \longrightarrow \mathcal{D}$ induces a map $K(\mathcal{C}) \longrightarrow K(\mathcal{D})$. The goal of this section is to give sufficient conditions on $\mathcal{C}$ which will ensure that this map is an equivalence.

We start off the section with an easy computational result which will make later proofs much simpler.

Lemma 4.6.1. *For any object $Y \in w\mathrm{SC}(\mathcal{C})$, $(Y \downarrow w\mathrm{SC}(\mathcal{C}))$ is a cofiltered preorder.*

Proof. In order to see that $(Y \downarrow w\mathrm{SC}(\mathcal{C})$ is a preorder it suffices to show that given any diagram

$$A \xleftarrow{\;\;f\;\;}_{\sim} Y \xrightarrow{\;\;g\;\;}_{\sim} B$$

in $w\mathrm{SC}(\mathcal{C})$ there exists at most one morphism $A \xrightarrow{\sim} B$ that makes the diagram commute. This diagram is represented by a diagram in $\mathrm{Tw}(\mathcal{C}_p)$

$$
\begin{array}{ccccccc}
& & Y' & \dashrightarrow^{p} & & \xleftarrow{q} Y'' & \\
& \swarrow^{\sigma} & & & \searrow & & \searrow^{\tau} \\
A & & & & Y & & B
\end{array}
$$

where σ and τ are isomorphisms. Then morphisms $h : A \longrightarrow B$ such that $g = hf$ correspond exactly to factorizations of q through p; as $(\mathrm{Tw}(\mathcal{C}_p)^{\mathrm{Sub}} \downarrow T)$ is a preorder, there is at most one of these and we are done.

Thus it remains to show that this preorder is cofiltered; in particular, we want to find an object below A and B under Y. Given a shuffle σ', let $f_{\sigma'} \in \mathrm{SC}(\mathcal{C})$ be the pure shuffle defined by σ'; similarly, for a sub-map p' let $f^{p'} \in \mathrm{SC}(\mathcal{C})$ be the pure sub-map defined by p'. Let $Z = Y' \times_Y Y''$ be the vertical pullback of p and q. Then, the pullback of

$$A \xrightarrow{\;\;\sigma^{-1}\;\;} Y' \xleftarrow{\;\;\;\;\;\;} Z$$

gives a weak equivalence $A \xrightarrow{\sim} Z$, and analogously we have a weak equivalence $B \xrightarrow{\sim} Z$. As these commute under Y we see that $(Y \downarrow w\mathrm{SC}(\mathcal{C}))$ is cofiltered, as desired. $\qquad\square$

The first condition that we need in order to have an equivalence on K-theory is that we must have the same K_0; more specifically, we need every object of $\mathrm{SC}(\mathcal{D})$ to be weakly equivalent to something in $\mathrm{SC}(\mathcal{C})$. As a condition on polytope complexes, this turns into the following definition.

Definition 4.6.2. Suppose that $\mathcal{D}$ is a polytope complex and $\mathcal{C} \longrightarrow \mathcal{D}$ is an inclusion of polytope complexes. We say that $\mathcal{C}$ has *sufficiently many covers* if for every object $B \in \mathcal{D}$ there exists a finite covering family $\{B_\alpha \dashrightarrow B\}_{\alpha \in A}$ such that the B_α are pairwise disjoint, and such that every B_α is horizontally isomorphic to an object of $\mathcal{C}$.

Our first approximation result is almost obvious: if we cover all weak equivalence classes of objects, and all morphisms between these objects, then we must have an equivalence on K-theory. More formally, we have the following:

Lemma 4.6.3. *Suppose that $\mathcal{C}$ has sufficiently many covers, and that $\mathrm{SC}(\mathcal{C})$ sits inside $\mathrm{SC}(\mathcal{D})$ as a full subcategory. Then the induced map $|w\mathrm{SC}(\mathcal{C})| \longrightarrow |w\mathrm{SC}(\mathcal{D})|$ is a homotopy equivalence.*

Proof. Using Quillen's Theorem A (from [17]) it suffices to show that for all $Y \in w\mathrm{SC}(\mathcal{D})$ the category $(Y \downarrow w\mathrm{SC}(\mathcal{C}))$ is contractible. Now as $(Y \downarrow w\mathrm{SC}(\mathcal{D}))$ is a preorder (by lemma 4.6.1) and $\mathrm{SC}(\mathcal{C})$ is a subcategory of $\mathrm{SC}(\mathcal{D})$, we know that $(Y \downarrow w\mathrm{SC}(\mathcal{C}))$ is also a preorder; thus to show that it is contractible we only need to know that it is cofiltered. In addition, as $\mathrm{SC}(\mathcal{C})$ is a full subcategory of $\mathrm{SC}(\mathcal{D})$, it in fact suffices to show that we have enough objects for it to be cofiltered, so it suffices to show that this category is nonempty for all Y.

So let us show that for all $Y \in w\mathrm{SC}(\mathcal{D})$ the category $(Y \downarrow w\mathrm{SC}(\mathcal{C}))$ is nonempty. We need to show that for any $Y \in w\mathrm{SC}(\mathcal{D})$ there exists a $Z \in w\mathrm{SC}(\mathcal{C})$ an a weak equivalence $Y \xrightarrow{\sim} Z$. Write $Y = \{y_i\}_{i \in I}$. For each $i \in I$, let $\{y_\alpha^{(i)} \dashrightarrow y_i\}_{\alpha \in A_i}$ be the cover guaranteed by the sufficient covers condition, and let $\beta_\alpha^{(i)} : y_\alpha^{(i)} \longrightarrow z_\alpha^{(i)}$ be the horizontal isomorphisms guaranteed by the sufficient covers condition. Then the induced vertical morphism $\{y_\alpha^{(i)}\}_{i \in I, \alpha \in A_i} \dashrightarrow \{y_i\}_{i \in I}$ is a covering submap and the vertical morphism $\beta : \{y_\alpha^{(i)}\}_{i \in I, \alpha \in A_i} \longrightarrow \{z_\alpha^{(i)}\}_{i \in I, \alpha \in A_i}$ is a horizontal isomorphism, so the morphism in $\mathrm{SC}(\mathcal{D})$ represented by this is a weak equivalence. But by definition $\{z_\alpha^{(i)}\}_{i \in I, \alpha \in A_i}$ is in $\mathrm{SC}(\mathcal{C})$, so we are done. $\qquad\square$

In the statement of the previous lemma we had two conditions. One was a condition on $\mathcal{C}$, and one was a condition on $\mathrm{SC}(\mathcal{C})$. We would like to get those conditions down to conditions just about $\mathcal{C}$, as that will make using this kind of results easier. In order for a morphism of $\mathrm{SC}(\mathcal{D})$ to be in $\mathrm{SC}(\mathcal{C})$ we need some representative of the morphism to come from a diagram in $\mathrm{Tw}(\mathcal{C})$; in particular, this means that both the representing object, and the morphisms which are the components of the vertical and horizontal components, must be in $\mathcal{C}$.

If $\mathcal{C}$ is not a full subcomplex of $\mathcal{D}$ then much of this analysis becomes much more difficult, so for the rest of this section we will assume that $\mathcal{C}$ is a full subcomplex of $\mathcal{D}$. This means that as long as we know that a representing object of the morphism is in $\mathrm{Tw}(\mathcal{C})$, it is sufficient to conclude that the morphism will be in $\mathrm{SC}(\mathcal{C})$. In particular, we want to be able to conclude that just because the source and target of a morphism are in $\mathrm{SC}(\mathcal{C})$ then the morphism must be, as well. We can translate this into the following condition.

Definition 4.6.4. Suppose that $\mathcal{C}$ is a full subcomplex of $\mathcal{D}$. We say that $\mathcal{C}$ is *wide* (respectively, *tall* if for any horizontal (resp. vertical) morphism $A \longrightarrow B \in \mathcal{D}$, if B is in $\mathcal{C}$ then so is A.

If $\mathcal{C}$ is a full subcomplex of $\mathcal{D}$ then we know that $\mathrm{Tw}(\mathcal{C})$ is a full subcategory of $\mathrm{Tw}(\mathcal{D})$. If $\mathcal{C}$ happens to also be wide, we know something even stronger: given any horizontally connected component of $\mathrm{Tw}(\mathcal{D})$, either that entire component is in $\mathrm{Tw}(\mathcal{C})$, or nothing in the component is in $\mathrm{Tw}(\mathcal{C})$. Analogously, if $\mathcal{C}$ is tall we can say the same thing for vertically connected components. This lets us conclude that $\mathrm{SC}(\mathcal{C})$ is a full subcategory of $\mathrm{SC}(\mathcal{D})$.

Lemma 4.6.5. *Let $\mathcal{C}$ be a full subcomplex of $\mathcal{D}$. If $\mathcal{C}$ is wide or tall then $\mathrm{SC}(\mathcal{C})$ is a full subcategory of $\mathrm{SC}(\mathcal{D})$.*

Proof. Let $\{a_i\}_{i \in I}, \{b_j\}_{j \in J} \in \mathrm{SC}(\mathcal{C})$, and let $f : \{a_i\}_{i \in I} \longrightarrow \{b_j\}_{j \in J}$ be a morphism in $\mathrm{SC}(\mathcal{D})$. This morphism is represented by a diagram

$$\{a_i\}_{i \in I} \xleftarrow{\ \ p\ \ } \{a'_k\}_{k \in K} \xrightarrow{\ \ \sigma\ \ } \{b_j\}_{j \in J}$$

In order for f to be in $\mathrm{SC}(\mathcal{C})$ it suffices to show that each a'_k is in $\mathcal{C}$, as $\mathcal{C}$ is a full subcategory of $\mathcal{D}$. Now if $\mathcal{C}$ is wide then for all $k \in K$ we have a horizontal morphism $\Sigma_k : a_k \longrightarrow b_{\sigma(k)}$. As $\mathcal{C}$ is wide and each $b_j \in \mathcal{C}$ we must have $a'_k \in \mathcal{C}$ for all k. So $\mathrm{SC}(\mathcal{C})$ is a full subcategory of $\mathrm{SC}(\mathcal{D})$. If, on the other hand, $\mathcal{C}$ is tall then for each $k \in K$ we consider the vertical morphism $P_k : a'_k \dashrightarrow a_{p(k)}$.

As $a_i \in \mathcal{C}$ for all $i \in I$ we must also have $a'_k \in \mathcal{C}$ for all $k' \in K$. So $SC(\mathcal{C})$ is a full subcategory of $SC(\mathcal{D})$, and we are done. $\qquad \square$

Which leads us to the following approximation result.

Proposition 4.6.6. *Suppose that $\mathcal{C}$ is a subcomplex of $\mathcal{D}$ with sufficiently many covers. If $\mathcal{C}$ is wide or tall, the inclusion $\mathcal{C} \longrightarrow \mathcal{D}$ induces an equivalence $K(\mathcal{C}) \longrightarrow K(\mathcal{D})$.*

Proof. Lemma 4.6.3 shows that $K(\mathcal{C})_i \longrightarrow K(\mathcal{D})_i$ is an equivalence for $i = 0$. If we can show that for all n, $s_n\mathcal{C}$ is a wide or tall subcomplex of $s_n\mathcal{D}$ with sufficiently many covers we will be done, as we will be able to induct on i to see that the induced morphism is an equivalence on all levels. In fact, note that it suffices to show that $f_n\mathcal{C}$ is a wide (resp. tall) subcomplex of $f_n\mathcal{D}$ with sufficiently many covers.

First we show that $f_n\mathcal{C}$ has sufficiently many covers in $f_n\mathcal{D}$. Consider an object D

$$D_1 \xleftarrow{} \cdots \xleftarrow{} D_n$$

of $f_n\mathcal{D}$. As $\mathcal{C}$ has sufficiently many covers in $\mathcal{D}$ there exists a covering family $\{B_\alpha \dashrightarrow D_n\}_{\alpha \in A}$ of D_n in which every object is horizontally isomorphic to an object of $\mathcal{C}$. Given an object $X \in \mathcal{D}$, let $\overline{X} \in f_n\mathcal{D}$ be the constant object where $\overline{X}_k = \{X\}$. Then the family $\{\overline{B}_\alpha \dashrightarrow D\}_{\alpha \in A}$ is a covering family of D. As each B_α was horizontally isomorphic to an object in $\mathcal{C}$, each $\overline{B}_\alpha$ is horizontally isomorphic to something in $f_n\mathcal{C}$, and we are done.

The fact that if $\mathcal{C}$ was a tall (resp. wide) subcomplex of $\mathcal{D}$ then $f_n\mathcal{C}$ is a tall (resp. wide) subcomplex of $\mathcal{D}$ follows directly from the definition of $f_n\mathcal{C}$ and $f_n\mathcal{D}$. $\qquad \square$

Finally, we can generalize this result to simplicial polytope complexes.

Corollary 4.6.7. *Suppose that $\mathcal{C}. \longrightarrow \mathcal{D}.$ is a morphism of simplicial polytope complexes. If for each n, the morphism $\mathcal{C}_n \longrightarrow \mathcal{D}_n$ is an inclusion of $\mathcal{C}_n$ as a subcomplex into $\mathcal{D}_n$ and satisfies the conditions of lemma 4.6.6, then the induced map $K(\mathcal{C}.) \longrightarrow K(\mathcal{D}.)$ is an equivalence.*

We finish up this section with a couple of applications of this result.

More explicit formula for suspensions and cofibers.

For any polytope complex $\mathcal{C}$ and any positive integer n we have a polytope functor $\mathcal{C} \longrightarrow f_n\mathcal{C}$ given by including and object a as the constant object

$$\overline{a} = \{a\} \xleftarrow{} \{a\} \xleftarrow{} \cdots \xleftarrow{} \{a\}$$

This includes $\mathcal{C}$ as a wide subcomplex of $f_n\mathcal{C}$. In fact, $\mathcal{C}$ also has sufficiently many covers. Given any object

$$A = A_1 \xleftarrow{} A_2 \xleftarrow{} \cdots \xleftarrow{} A_n,$$

write $A_n = \{a_i\}_{i \in I}$. Then the family $\{\overline{a}_i \dashrightarrow A\}_{i \in I}$ is a covering family, and each $\overline{a}_i \in \mathcal{C}$. Thus we have an inclusion $\mathcal{C}^{\vee n} \longrightarrow s_n\mathcal{C}$ which induces an equivalence on K-theory.

In fact, this is an equivalence on the K-theory of simplicial polytope complexes, as this inclusion commutes with the simplicial structure maps. Thus $s.\mathcal{C}$ can be considered to be a bar construction on $\mathcal{C}$, as the structure maps of $s.\mathcal{C}$, when restricted to the constant objects, exactly mirror the morphisms of the bar construction. (The 0-th face map forgets the first one, the next $n-1$ glue successive copies of $\mathcal{C}$ together, and the n-th one forgets the last one, exactly as the bar construction does. The degeneracies each skip one of the $\mathcal{C}$'s in $s_{n+1}\mathcal{C}$.)

Generalizing to simplicial polytope complexes, this gives the following simplifications of the formulas for $\sigma\mathcal{C}.$ and $(\mathcal{D}/g).$ from corollary 4.4.6 and proposition 4.5.2:

Corollary 4.6.8. *Let* $g : \mathcal{C}. \longrightarrow \mathcal{D}.$ *be a morphism of simplicial polytope complexes. Let* $\sigma\mathcal{C}.$ *and* $(\mathcal{D}/g).$ *be the simplicial polytope complexes defined by*

$$(\sigma\mathcal{C}.)_n = \mathcal{C}_n^{\vee n} \qquad and \qquad (\mathcal{D}/g)_n = \mathcal{D}_n \vee \mathcal{C}_n^{\vee n}.$$

Then $\Omega K(\sigma\mathcal{C}.) \simeq K(\mathcal{C}.)$ *and*

$$K(\mathcal{C}.) \longrightarrow K(\mathcal{D}.) \longrightarrow K((\mathcal{D}/g).)$$

is a cofiber sequence of spectra.

It is necessary to check that these inclusions commute with the simplicial maps, but it is easy to see that they do. Note that on $(\mathcal{D}/g)_n$, ∂_0 is induced by the three morphisms

$$\partial_0 : \mathcal{D}_n \longrightarrow \mathcal{D}_{n-1} \qquad g\partial_0 : \mathcal{C}_n \longrightarrow \mathcal{D}_{n-1} \qquad \partial_0^{\vee n-1} : \mathcal{C}_n^{\vee n-1} \longrightarrow \mathcal{C}_{n-1}^{\vee n-1}.$$

Local data on homogeneous manifolds.

Let X be a geodesic n-manifold with a preferred open cover $\{U_\alpha\}_{\alpha \in A}$ such that for any $\alpha \in A$ and any two points $x, y \in U_\alpha$ there exists a unique geodesic connecting x and y. (For example, $X = E^n$, S^n, or $\mathbb{H}^n$ are examples of such X. In the first and third case we take our open cover to be the whole space; in the second case we take it to be the set of open hemispheres.) We then define a polytope complex $\mathcal{C}_X$ in the following manner. Define a *simplex* of X to be a convex hull of $n+1$ points all sitting inside some U_α with nonempty interior, and a *polytope* of X to be a finite union of simplices. We then define $\mathcal{C}_{Xv}$ to be the poset of polytopes of X under inclusion with the obvious topology. Given two polytopes P and Q, we define a *local isometry of P onto Q* to be a triple (U, V, φ) such that U and V are open subsets of X with $P \subseteq U$ and $Q \subseteq V$, $\varphi : U \longrightarrow V$ is an isometry of U into V, and $\varphi(P) = Q$. Then we define a horizontal morphism $P \longrightarrow Q$ to be an equivalence class of local isometries of P onto Q, with $(U, V, \varphi) \sim (U', V', \varphi')$ if $\varphi|_{U \cap U'} = \varphi'|_{U \cap U'}$. Under these definitions it is clear that $\mathcal{C}_X$ is a polytope complex.

Now let $U \subseteq X$ be any preferred open subset of X with the preferred cover $\{U\}$. Then $\mathcal{C}_U$ is also a polytope complex and we have an obvious inclusion map $\mathcal{C}_U \longrightarrow \mathcal{C}_X$.

Lemma 4.6.9. *If X is homogeneous then the inclusion $\mathcal{C}_U \hookrightarrow \mathcal{C}_X$ induces an equivalence $K(\mathcal{C}_U) \longrightarrow K(\mathcal{C}_X)$.*

Proof. Clearly $\mathcal{C}_U$ is a tall subcomplex of $\mathcal{C}_X$. Given any polytope $P \subseteq \mathcal{C}_X$ we can triangulate it by triangles small enough to be in a single chart. Once we are in a single chart we can subdivide each triangle by barycentric subdivision until the diameter of every triangle in the triangulation is small enough that the triangle can fit inside U. As X is homogeneous there is a local isometry of any such triangle into U, and thus $\mathcal{C}_U$ has sufficiently many covers. Thus by proposition 4.6.6 the induced map $K(\mathcal{C}_U) \longrightarrow K(\mathcal{C}_X)$ is an equivalence. $\square$

Any isometry $X \longrightarrow Y$ which takes preferred open sets into preferred open sets induces a polytope functor $\mathcal{C}_X \longrightarrow \mathcal{C}_Y$ (which is clearly an isomorphism). Thus the statement of proposition 1.0.4 is exactly that all morphisms in the diagram

$$\mathcal{C}_X \longleftarrow \mathcal{C}_U \xrightarrow{\;\mathcal{C}_\varphi\;} \mathcal{C}_V \longrightarrow \mathcal{C}_Y$$

are equivalences, which follows easily from the above lemma.

Non-examples

We conclude this section with a couple of non-examples. First, take any polytope complex $\mathcal{C}$ and consider the polytope complex $\mathcal{C} \vee \mathcal{C}$. $\mathcal{C}$ sits inside this (as the left copy, for example) and is tall by definition, but the K-theories of these are not equivalent as the left copy of $\mathcal{C}$ does not contain sufficiently many covers. (In particular, it can't cover anything in the right copy of $\mathcal{C}$.) However, if we added "twist" isomorphisms — horizontal isomorphisms between corresponding objects in the left and right copies of $\mathcal{C}$ — then the left $\mathcal{C}$ would contain sufficiently many covers, and the K-theories of these would be equal.

As our second non-example we will look at ideals of a number field. Let K be a number field with Galois group G. Let the objects of $\mathcal{C}$ be the ideals of K. We will have a vertical morphism $I \dashrightarrow J$ whenever $I | J$, and we will have our horizontal morphisms induced by the action of G. The K-theory of this will be countably many spheres wedged together, one for each prime power ideal of K. (See section 3.3.5 for a more detailed exploration of this example.) The prime ideals sit inside $\mathcal{C}$ as a wide subcomplex, but they do not give an equivalence because if $\mathfrak{p}^k$ is a prime power ideal for $k > 1$ then it can't be covered by prime ideals. The K-theories of these two will in fact be equivalent, since they are both countably many spheres wedged together, but the inclusion does not induce an equivalence.

4.7 Proof of lemma 4.3.3

This section concerns the proof of lemma 4.3.3:

Lemma 4.3.3. $L_n\mathrm{SC}(\mathcal{C})$ *is a Waldhausen category. The cofibrations (resp. weak equivalences) in $L_n\mathrm{SC}(\mathcal{C})$ are exactly the morphisms which are levelwise cofibrations (resp. weak equivalences). $L_n\mathrm{SC}(\mathcal{C})$ is a simplification of $F_n\mathrm{SC}(\mathcal{C})$.*

A morphism $A \longrightarrow B \in F_n\mathrm{SC}(\mathcal{C})$ is represented by a diagram

$$
\begin{array}{ccccccc}
A_1 & \hookrightarrow & A_2 & \hookrightarrow & \cdots & \hookrightarrow & A_n \\
\downarrow & & \downarrow & & & & \downarrow \\
B_1 & \hookrightarrow & B_2 & \hookrightarrow & \cdots & \hookrightarrow & B_n
\end{array}
$$

and by lemma 4.3.2(1) will be layered exactly when for each $i = 1, \ldots, n-1$ the diagram

$$
\begin{array}{ccc}
A_i/A_{i-1} & \hookrightarrow & A_i \\
\downarrow & & \downarrow \\
B_i/B_{i-1} & \hookrightarrow & B_i
\end{array}
$$

commutes. As each square is considered separately, for all of the proofs in this section we will assume that $n = 2$, as for all other values of n the proofs will be equivalent, and it saves on having an extra variable floating around.

Lemma 4.7.1. *Any layered morphism which is levelwise a cofibration is a cofibration.*

Proof. We want to show that if

$$f_1, f_2 : (A_1 \xhookrightarrow{i} A_2) \longrightarrow (B_1 \xhookrightarrow{j} B_2)$$

is layered, then the induced morphism $\varphi : A_2 \cup_{A_1} B_1 \longrightarrow B_2$ is a cofibration. We know that $A_2 \cup_{A_1} B_1 \cong (A_2/A_1) \amalg B_1$, and that $\varphi = j \amalg (f_2/f_1)$ (where the second part follows directly from the layering condition). Thus it suffices to show that f_2/f_1 is a cofibration. This follows from the more general statement that in $\mathrm{SC}(\mathcal{C})$, given two composable morphisms g, h, if h and hg are cofibrations then so is g. As $A_2/A_1 \hookrightarrow B_2$ and $B_2/B_1 \hookrightarrow B_2$ are both cofibrations, it follows that f_2/f_1 must be one as well. $\qquad\square$

Now we turn our attention to showing that $L_n\mathrm{SC}(\mathcal{C})$ is a simplification of $F_n\mathrm{SC}(\mathcal{C})$. We first develop a little bit of computational machinery for layering, which will allow us to work with cofibrations more easily.

Given any object $A = \{a_i\}_{i \in I} \in \mathrm{SC}(\mathcal{C})$, we say that A' is a *subobject* of A if $A' = \{a_i\}_{i \in I'}$ for some subset $I' \subseteq I$. If A', A'' are two subobjects of A, we will write $A' \cap A''$ for $\{a_i\}_{i \in I' \cap I''}$, and we will write $A' \subseteq A''$ if $I' \subseteq I''$. Suppose that $f : A \longrightarrow B$ is a morphism in $\mathrm{SC}(\mathcal{C})$. Pick a representation of this by a sub-map p and a shuffle σ, and write $B = \{b_j\}_{j \in J}$. Then $\mathrm{im}_B A = \{b_j\}_{j \in \mathrm{im}\,\sigma}$. Note that this agrees with the previous definition of image when f is a cofibration, and $\mathrm{im}_B A$ is a subobject of B. If we write $A = A_1 \amalg A_2$ then A_1 and A_2 are subobjects of A, and $\mathrm{im}_B A = \mathrm{im}_B A_1 \cup \mathrm{im}_B A_2$. If f were a cofibration, we also have $\mathrm{im}_B A_1 \cap \mathrm{im}_B A_2 = \emptyset$; if f were a weak equivalence then $\mathrm{im}_B A = B$. (For example, $\mathrm{im}_B A \cap (B/A) = \emptyset$.) Given a second morphism $g : B \longrightarrow C$, $\mathrm{im}_C A \subseteq \mathrm{im}_C B$.

Now consider a commutative square

$$f_1, f_2 : (A_1 \hookrightarrow A_2) \longrightarrow (B_1 \hookrightarrow B_2).$$

This square satisfies the layering condition exactly when

$$\mathrm{im}_{B_2}(A_2/A_1) \subseteq \mathrm{im}_{B_2}(B_2/B_1) = B_2/B_1,$$

or equivalently when $\mathrm{im}_{B_2}(A_2/A_1) \cap \mathrm{im}_{B_2} B_1 = \emptyset$. We will use this restatement in our computations.

Lemma 4.7.2. *Cofibrations are layered.*

Proof. If $A \hookrightarrow B$ is a cofibration, then by definition $A_2 \cup_{A_1} B_1 \hookrightarrow B_2$ is a cofibration. But we have an acyclic cofibration $(A_2/A_1) \amalg B_1 \xhookrightarrow{\sim} A_2 \cup_{A_1} B_1$, so $\mathrm{im}_{B_2}(A_2/A_1) \cap \mathrm{im}_{B_2} B_1 = \emptyset$, as desired. $\quad\square$

Lemma 4.7.3. *Layered morphisms are closed under pushouts. More precisely, given any commutative square*

$$\begin{array}{ccc} A & \xhookrightarrow{} & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in which all morphisms are layered, the induced morphisms

$$C \hookrightarrow B \cup_A C \qquad\qquad B \longrightarrow B \cup_A C \qquad\qquad B \cup_A C \longrightarrow D$$

are all layered.

Proof. The first of these is clearly layered as it is a cofibration.

Write $X_i = B_i \cup_{A_i} C_i$. Keep in mind that for all i, we have an acyclic cofibration $(B_i/A_i) \amalg C_i \overset{\sim}{\hookrightarrow} X_i$.

For the second, we need to show that $\mathrm{im}_{X_2}(B_2/B_1) \cap \mathrm{im}_{X_2}(X_1) = \emptyset$. We have

$$
\begin{aligned}
\mathrm{im}_{X_2}(B_2/B_1) \cap \mathrm{im}_{X_2} X_1 &= \mathrm{im}_{X_2}(B_2/B_1) \cap (\mathrm{im}_{X_2} C_1 \cup \mathrm{im}_{X_2}(B_1/A_1)) \\
&= (\mathrm{im}_{X_2}(B_2/B_1) \cap \mathrm{im}_{X_2} C_1) \cup (\mathrm{im}_{X_2}(B_2/B_1) \cap \mathrm{im}_{X_2}(B_1/A_1)).
\end{aligned}
$$

Consider the first of the two sets we are unioning. By the definition of X_2, $\mathrm{im}_{X_2} C_2 \cap \mathrm{im}_{X_2} B_2 = \mathrm{im}_{X_2} A_2$. Thus

$$
\mathrm{im}_{X_2}(B_2/B_1) \cap \mathrm{im}_{X_2} C_1 \subseteq \mathrm{im}_{X_2}(B_2/B_1) \cap \mathrm{im}_{X_2} C_2 \subseteq \mathrm{im}_{X_2} A_2.
$$

On the other hand,

$$
\mathrm{im}_{X_2}(C_1) \cap \mathrm{im}_{X_2}(A_2) = \mathrm{im}_{X_2}(\mathrm{im}_{C_2} C_1 \cap \mathrm{im}_{C_2} A_2) = \mathrm{im}_{X_2}(\mathrm{im}_{C_2} A_1) = \mathrm{im}_{X_2} A_1
$$

as $A \longrightarrow C$ is layered. Thus we want to show that $\mathrm{im}_{X_2}(B_2/B_1) \cap \mathrm{im}_{X_2} A_1 = \emptyset$. It suffices to show this inside B_2, where it is obvious. Now consider the second part. As $B_1/A_1 \longrightarrow X_2$ and $B_2/B_1 \longrightarrow X_2$ both factor through B_2, it suffices to show that $(B_2/B_1) \cap \mathrm{im}_{B_2}(B_1/A_1) = \emptyset$, which is clear by definition.

It remains to show that the last of these morphisms is layered. In particular, we need to show that

$$
\mathrm{im}_{D_2}(X_2/X_1) \cap \mathrm{im}_{D_2}(D_1) = \emptyset.
$$

But it is easy to see that

$$
\begin{aligned}
\mathrm{im}_{D_2}(X_2/X_1) \cap \mathrm{im}_{D_2}(D_1) &= \mathrm{im}_{D_2}((C_2 \amalg B_2/A_2)/(C_1 \amalg B_1/A_1)) \cap \mathrm{im}_{D_2}(D_1) \\
&= \mathrm{im}_{D_2}(C_2/C_1 \amalg (B_2/A_2)/(B_1/A_1)) \cap \mathrm{im}_{D_2}(D_1) \\
&= (\mathrm{im}_{D_2}(C_2/C_1) \cap \mathrm{im}_{D_2}(D_1)) \cup \\
&\qquad \cup (\mathrm{im}_{D_2}((B_2/A_2)/(B_1/A_1)) \cap \mathrm{im}_{D_2}(D_1)).
\end{aligned}
$$

the first of these is empty because $C \longrightarrow D$ is layered. The second is empty because $\mathrm{im}_{D_2}((B_2/A_2)/(B_1/A_1)) \subseteq \mathrm{im}_{D_2}(B_2/B_1)$, and the intersection of this with $\mathrm{im}_{D_2}(D_1)$ is empty because $B \longrightarrow D$ is layered. So we are done. $\qquad\square$

Now we are ready to prove lemma 4.3.3.

Proof of lemma 4.3.3. Firstly we will show that all weak equivalences of $F_n \mathrm{SC}(\mathcal{C})$ are layered. In particular, it suffices to show that any weak equivalences of $F_n \mathrm{SC}(\mathcal{C})$ is also a cofibration, since we already know by lemma 4.7.2 that all cofibrations are layered. In particular, if we have a a commutative square

$$
(A_1 \hookrightarrow A_2) \overset{\sim}{\longrightarrow} (B_1 \hookrightarrow B_2)
$$

we want to show that the induced morphism $A_2 \cup_{A_1} B_1 \longrightarrow B_2$ is a cofibration. As weak equivalences are preserved under pushouts we know that $A_2 \overset{\sim}{\longrightarrow} A_2 \cup_{A_1} B_1$ is a weak equivalence, as is $A_2 \overset{\sim}{\longrightarrow} B_2$. As weak equivalences in $\mathrm{SC}(\mathcal{C})$ satisfy 2-of-3 in this direction we are done (see lemma 3.4.8).

All morphisms $A \longrightarrow *$ are in $L_n \mathrm{SC}(\mathcal{C})$, as these are trivially layered. As lemma 4.7.3 showed that $L_n \mathrm{SC}(\mathcal{C})$ is closed under pushouts, we see that $L_n \mathrm{SC}(\mathcal{C})$ is, in fact, a simplification of $F_n \mathrm{SC}(\mathcal{C})$. Weak equivalences of $L_n \mathrm{SC}(\mathcal{C})$ are levelwise because weak equivalences in $F_n \mathrm{SC}(\mathcal{C})$ are levelwise, and cofibrations are levelwise by lemma 4.7.1. $\qquad\square$

Chapter 5

Further structures on PolyCpx

5.1 Two Symmetric Monoidal Structures

5.1.1 Cartesian Product

Our goal is to create a polytope complex to mirror the tensor product of abelian groups and the smash products of symmetric spectra. If we think of a finite pairwise-disjoint covering family $\{A_i \dashrightarrow A\}_{i \in I}$ as representing the equation $[A] = \sum_{i \in I} A_i$, then polytope functors exactly represent additive functions between scissors congruence groups. Thus our first step is to create a polytope complex that can represent "bilinear maps" out of a pair of polytope complexes.

Definition 5.1.1. Given two polytope complexes $\mathcal{C}, \mathcal{D}$ we define the *product complex* $\mathcal{C} \times \mathcal{D}$ to be the polytope complex whose underlying category is $\mathcal{C} \times \mathcal{D}$, and whose topology is the "orthogonal" product topology. Specifically, the topology is generated by the coverage given by families

$$\{A_i \times B_j \dashrightarrow A \times B\}_{(i,j) \in I \times J}$$

for all covering families $\{A_i \dashrightarrow A\}_{i \in I}$ and $\{B_j \dashrightarrow B\}_{j \in J}$ in $\mathcal{C}$ and $\mathcal{D}$, respectively.

(For the details of the definition of a coverage see [11].)

It is easy to check that $\mathcal{C} \times \mathcal{D}$ is, in fact, a polytope complex. The only axiom that causes any trouble is axiom (E): showing that if a covering family $\{X_\alpha \dashrightarrow X\}_{\alpha \in A}$ has $X_\beta = \emptyset$ for some $\beta \in A$ then $\{X_\alpha \dashrightarrow X\}_{\alpha \in A \setminus \{\beta\}}$ is also a covering family. As any covering family in the pretopology is generated by finitely many refinements in the coverage (and as $\emptyset \times \emptyset$ has no nontrivial covering families) it suffices to show that we can remove $\emptyset \times \emptyset$ from any family in the coverage. In order to show this we in fact need two refinements, and the required family will be a covering family in the pretopology associated to the coverage, but not actually in the coverage.

Lemma 5.1.2. *If* $\{X_\alpha \times Y_\beta \dashrightarrow X \times Y\}_{(\alpha,\beta) \in A \times B}$ *is a covering family in the topology of* $\mathcal{C} \times \mathcal{D}$ *with* $X_{\alpha'} \times Y_{\beta'} = \emptyset \times \emptyset$ *for some* $\alpha' \in A$ *and* $\beta' \in B$ *then the family* $\{X_\alpha \times Y_\beta \dashrightarrow X \times Y\}_{(\alpha,\beta) \in A \times B \setminus \{(\alpha',\beta')\}}$ *is also a covering family.*

Proof. As $X_{\alpha'} = \emptyset$ we can apply (E) to the covering family $\{X_\alpha \dashrightarrow X\}_{\alpha \in A}$ to conclude that $\{X_\alpha \dashrightarrow X\}_{\alpha \in A \setminus \{\alpha'\}}$ is also a covering family. To get the desired family we will refine the covering family $\{X \times Y_\beta \dashrightarrow X \times Y\}_{\beta \in B}$. For all $\beta \neq \beta'$, cover $X \times Y_\beta$ by the covering family $\{X_\alpha \times Y_\beta \dashrightarrow X \times Y_\beta\}_{\alpha \in A}$, and cover $X \times \emptyset$ by $\{X_\alpha \times \emptyset \dashrightarrow X \times \emptyset\}_{\alpha \in A \setminus \{\alpha'\}}$. This will be a covering family, and will contain all $X_\alpha \times Y_\beta$ for $(\alpha, \beta) \in A \times B \setminus \{(\alpha', \beta')\}$. $\square$

Let $\mathcal{I}$ be the polytope complex with no noninitial objects. For all polytope complexes $\mathcal{C}$, $\mathcal{C} \times \mathcal{I} \cong \mathcal{I} \times \mathcal{C} \cong \mathcal{C}$.

Lemma 5.1.3. *We have a functor* $\times\colon \mathbf{PolyCpx} \times \mathbf{PolyCpx} \longrightarrow \mathbf{PolyCpx}$ *defined as* $(\mathcal{C}, \mathcal{D}) \longmapsto \mathcal{C} \times \mathcal{D}$ *which gives a symmetric monoidal structure* $(\mathbf{PolyCpx}, \cdot \times \cdot, \mathcal{I}, \alpha, \gamma, \lambda, \rho)$.

Proof. We make the following definitions:

$\alpha_{\mathcal{C}, \mathcal{D}, \mathcal{E}}$ the polytope functor $(\mathcal{C} \times \mathcal{D}) \times \mathcal{E} \longrightarrow \mathcal{C} \times (\mathcal{D} \times \mathcal{E})$ defined by taking the pair $(A, (B, C))$ to the pair $((A, B), C)$, and analogously on morphisms.

$\gamma_{\mathcal{C}, \mathcal{D}}$ the polytope functor $\mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{D} \times \mathcal{C}$ which takes the pair (A, B) to the pair (B, A).

$\lambda_{\mathcal{C}}$ the functor which takes an object $(A, \emptyset)$ to A.

$\rho_{\mathcal{C}}$ the functor which takes an object $(\emptyset, A)$ to A.

It is clear from the definitions that these satisfy the desired diagrams for a symmetric monoidal structure, assuming that they are well-defined polytope functors for every $\mathcal{C}, \mathcal{D}, \mathcal{E}$. As pullbacks are defined coordinatewise it is clear that all of these preserve pullbacks; all that we need to show is that they are vertically continuous. In order to do this it suffices to show that they take any covering family in the generating coverage to another covering family. However, this is a definition check: note that if $\{X_\alpha \times (Y_\alpha \times Z_\alpha) \dashrightarrow X \times (Y \times Z)\}_{\alpha \in A}$ is a covering family then we must be able to write $A = A_1 \times A_2 \times A_3$ such that this family is equal to the family $\{X_{\alpha_1} \times (Y_{\alpha_2} \times Z_{\alpha_3}) \dashrightarrow X \times (Y \times Z)\}_{\alpha_j \in A_j}$ in such a way that when restricted to any coordinate we have a covering family. But in that case we can write the indexing set for the covering family $\{(X_\alpha \times Y_\alpha) \times Z_\alpha \dashrightarrow (X \times Y) \times Z\}_{\alpha \in A}$ in an analogous way, so this will also be a covering family. γ is done analogously. $\square$

Remark. Note that each of $\mathcal{C}$ and $\mathcal{D}$ include into $\mathcal{C} \times \mathcal{D}$ as the subcomplexes $\mathcal{C} \times \mathcal{I}$ and $\mathcal{I} \times \mathcal{D}$, respectively. Thus the functor $\cdot \times \cdot$ has a subfunctor $(\mathcal{C}, \mathcal{D}) \longmapsto \mathcal{C} \vee \mathcal{D}$; in fact, it is easy to check that $(\mathbf{PolyCpx}, \cdot \vee \cdot, \mathcal{I}, \alpha, \gamma, \lambda, \rho)$ is also a symmetric monoidal structure on $\mathbf{PolyCpx}$.

It may seem that the product complex $\mathcal{C} \times \mathcal{D}$ classifies bilinear maps out of $\mathcal{C}$ and $\mathcal{D}$, as the topology is designed so that weak equivalences in $\mathrm{SC}(\mathcal{C} \times \mathcal{D})$ are bilinear in $\mathcal{C}$ and $\mathcal{D}$. And, indeed, it is easy to check that (in many cases) $K_0(\mathcal{C} \times \mathcal{D}) \cong K_0(\mathcal{C}) \otimes K_0(\mathcal{D})$, as would be desired in such a structure. (For more details, see example 5.1.6.) Unfortunately, functors out of $\mathcal{C} \times \mathcal{D}$ are not, in fact, "bilinear" functors: if we fix an $A \in \mathcal{C}$ the functor $\mathcal{D} \longrightarrow \mathcal{C} \times \mathcal{D}$ given by $D \longmapsto A \times D$ is not a polytope functor, as it does not take $\emptyset$ to $\emptyset$. If we ould somehow set all of these objects to equal $\emptyset$ this problem would be resolved; to do this we will develop a theory of "relative pairs" of polytope complexes.

5.1.2 Initial Subcomplexes and Smash Products

A relative pair of complexes is, analogously to a relative pair of spaces, a polytope complex together with a subcomplex. Our goal in studying such pairs is to be able to study the K-theory of the "quotient complexes" we can form by collapsing the subcomplex into the initial object. In order for such a collapse to be well-defined we need a couple of extra conditions on the subcomplexes we allow in a relative pair.

Definition 5.1.4. Let $\mathcal{C}$ be a polytope complex and $\mathcal{C}'$ a subcomplex. $\mathcal{C}'$ is *initial* in $\mathcal{C}$ if, given any (vertical or horizontal) morphism $f\colon A \longrightarrow B \in \mathcal{C}$, if $B \in \mathcal{C}'$, so is f. We write a pair of a polytope complex $\mathcal{C}$ together with an initial subcomplex $\mathcal{C}'$ as $(\mathcal{C}, \mathcal{C}')$. A functor of pairs $F\colon (\mathcal{C}, \mathcal{C}') \longrightarrow (\mathcal{D}, \mathcal{D}')$ is a polytope functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ such that $F(\mathcal{C}') \subseteq \mathcal{D}'$. We denote the category of such pairs by $\mathbf{RelPair}$.

Equivalently, in the terminology of section 4.6, $\mathcal{C}'$ is initial if and only if it is wide and tall.

We have an inclusion functor $\iota\colon \mathbf{PolyCpx} \longrightarrow \mathbf{RelPair}$ given by $\mathcal{C} \longmapsto (\mathcal{C}, \mathcal{I})$. This inclusion has a left adjoint, a contraction functor $c\colon \mathbf{RelPair} \longrightarrow \mathbf{PolyCpx}$. We can explicitly construct $c(\mathcal{C}, \mathcal{C}')$, which we will generally write $\mathcal{C}\backslash\mathcal{C}'$, as follows. $\mathcal{C}\backslash\mathcal{C}'$ is the full subcomplex of $\mathcal{C}$ containing all objects which are either (a) vertically initial or (b) not in $\mathcal{C}'$. We will say that a family of morphisms $\{X_\alpha \dashrightarrow X\}_{\alpha \in A}$ is a covering family if there exists a family of morphisms $\{X_\beta \dashrightarrow X\}_{\beta \in B}$ with each $X_\beta \in \mathcal{C}'$ such that $\{X_i \dashrightarrow X\}_{i \in A \cup B}$ is a covering family in $\mathcal{C}$. We also get $c(F\colon (\mathcal{C}, \mathcal{C}') \longrightarrow (\mathcal{D}, \mathcal{D}')) = F|_{\mathcal{C}\backslash\mathcal{C}'}$.

Note that in $\mathcal{C}\backslash\mathcal{C}'$ some objects may have empty covering families; see, for example, example 5.1.5.

Given a relative pair $(\mathcal{C}, \mathcal{C}')$ we have an induced inclusion of K-theories $K(\mathcal{C}') \longrightarrow K(\mathcal{C})$. It is important to note that $K(\mathcal{C}\backslash\mathcal{C}')$ is not necessarily the homotopy cofiber of the inclusion $K(\mathcal{C}') \longrightarrow K(\mathcal{C})$. In order to emphasize this fact we use the notation $\mathcal{C}\backslash\mathcal{C}'$ instead of the (perhaps more natural) $\mathcal{C}/\mathcal{C}'$: in section 4.5 we used $\mathcal{C}/\mathcal{C}'$ to denote the polytope complex representing the homotopy cofiber of the inclusion $K(\mathcal{C}') \hookrightarrow K(\mathcal{C})$. Let us consider two examples:

Example 5.1.5. Recall that a subcomplex $\mathcal{C}' \hookrightarrow \mathcal{C}$ has sufficiently many covers if every object $A \in \mathcal{C}$ has a covering family $\{A_i \dashrightarrow A\}_{i \in I}$ with $A_i \in \mathcal{C}'$ for all $i \in I$. Suppose that $(\mathcal{C}, \mathcal{C}')$ is a relative pair such that the inclusion $\mathcal{C}' \longrightarrow \mathcal{C}$ has sufficiently many covers, so $K(\mathcal{C}') \overset{\sim}{\longrightarrow} K(\mathcal{C})$ (see proposition 4.6.6). In this case is $K(\mathcal{C}\backslash\mathcal{C}')$ contractible, and thus is the homotopy cofiber of the inclusion $K(\mathcal{C}') \hookrightarrow K(\mathcal{C})$. To see this, it suffices to show that $w\mathrm{SC}(\mathcal{C}\backslash\mathcal{C}')$ has a terminal object: the usual 0 object $\{\}$. Note that for all $\{a_i\}_{i \in I} \in \mathrm{SC}(\mathcal{C}\backslash\mathcal{C}')$ for each $i \in I$ we have a cover of a_i by objects in $\mathcal{C}'$, so in particular $\{\} \dashrightarrow\!\!\!\!\rightarrow \{a_i\}_{i \in I}$ is a covering sub-map. Thus the morphism $\{a_i\}_{i \in I} \longrightarrow 0$ is a weak equivalence, and $w\mathrm{SC}(\mathcal{C}\backslash\mathcal{C}')$ has a terminal object, and thus $K(\mathcal{C}\backslash\mathcal{C}')$ is contractible, as desired.

Example 5.1.6. Now consider the inclusion induced from the pair $(\mathcal{C} \times \mathcal{D}, \mathcal{C} \vee \mathcal{D})$. (This is the relative pair we will use to define the smash product of polytope complexes.) Assuming that the two extra conditions

(a) $K_0(\mathcal{C}) \times K_0(\mathcal{D}) \neq 0$, and

(b) for all objects A in $\mathcal{C}$ or $\mathcal{D}$, the family $\{\emptyset \dashrightarrow A, A \dashrightarrow A\}$ is a covering family,

hold, $K(\mathcal{C} \times \mathcal{D}\backslash\mathcal{C} \vee \mathcal{D})$ will not be the homotopy cofiber of this inclusion. Indeed, using theorem 3.2.2 and condition (b) we can easily compute that $K_0(\mathcal{C} \times \mathcal{D}) \cong K_0(\mathcal{C}) \otimes K_0(\mathcal{D}) \cong K_0(\mathcal{C} \times \mathcal{D}\backslash\mathcal{C} \vee \mathcal{D})$ and the map $K(\mathcal{C} \times \mathcal{D}) \longrightarrow K(\mathcal{C} \times \mathcal{D}\backslash\mathcal{C} \vee \mathcal{D})$ induces an isomorphism between them. As $K(\mathcal{C} \vee \mathcal{D}) \simeq K(\mathcal{C}) \times K(\mathcal{D}) \neq 0$ by condition (a), the sequence $K(\mathcal{C} \vee \mathcal{D}) \longrightarrow K(\mathcal{C} \times \mathcal{D}) \longrightarrow K(\mathcal{C} \times \mathcal{D}\backslash\mathcal{C} \vee \mathcal{D})$ is not a homotopy cofiber sequence.

Suppose that $(\mathcal{E}, \otimes, \alpha(, \gamma))$ satisfies the pentagonal (and hexagonal) axiom of a (symmetric) monoida category, but not any of the unit axioms. Then we say that $\otimes$ is a *(symmetric) nonunital-monoidal* structure on $\mathcal{E}$.

Lemma 5.1.7. *Suppose that* $(\mathbf{RelPair}, \cdot \circledast \cdot, \alpha, (, \gamma))$ *is a (symmetric) nonunital-monoidal structure on* $\mathbf{RelPair}$. *If we define* $\cdot \otimes \cdot = c(\iota(\cdot) \circledast \iota(\cdot))$, *then we get a (symmetric) nonunital-monoidal structure* $(\mathbf{PolyCpx}, \otimes, c\alpha_\iota(, c\gamma_\iota))$.

$\otimes$ is the symmetric nonunital-monoidal structure *induced by* $\circledast$.

Proof. We will show this for the pentagon axiom; the hexagon axiom follows analogously. Consider the diagram

67

$$((A \otimes B) \otimes C) \otimes D \longrightarrow (A \otimes B) \otimes (C \otimes D)$$

$$(A \otimes (B \otimes C)) \otimes D) \longrightarrow A \otimes ((B \otimes C) \otimes D) \longrightarrow A \otimes (B \otimes (C \otimes D))$$

Note that this diagram is exactly c applied to the diagram

$$((\iota A \circledast \iota B) \circledast \iota C) \circledast \iota D \longrightarrow (\iota A \circledast \iota B) \circledast (\iota C \circledast \iota D)$$

$$(\iota A \circledast (\iota B \circledast \iota C)) \circledast \iota D) \longrightarrow \iota A \circledast ((\iota B \circledast \iota C) \circledast \iota D) \longrightarrow \iota A \circledast (\iota B \circledast (\iota C \circledast \iota D))$$

which commutes because $\circledast$ is monoidal. $\qquad\square$

Now consider the functor $\bar{\wedge} \colon ((\mathcal{C}, \mathcal{C}'), (\mathcal{D}, \mathcal{D}')) \longmapsto (\mathcal{C} \times \mathcal{D}, \mathcal{C} \times \mathcal{D}' \cup \mathcal{C}' \times \mathcal{D})$. It is easy to check that this is a functor $\mathbf{RelPair} \times \mathbf{RelPair} \longrightarrow \mathbf{RelPair}$. Letting $\alpha, \gamma, \lambda, \rho$ be the stucture maps of the symmetric monoidal category $(\mathbf{PolyCpx}, \times, \mathcal{I})$, we can extend them to a symmetric monoidal structure $(\mathbf{RelPair}, \bar{\wedge}, (\mathcal{I}, \mathcal{I}), \alpha, \gamma, \lambda, \rho)$.

We want to construct a monoidal structure on $\mathbf{PolyCpx}$ using this product in $\mathbf{RelPair}$. Let $\wedge$ be the symmetric nonunital-monoidal structure induced by $\bar{\wedge}$. Note that we are going to need a new unit, since $\mathcal{C} \times \mathcal{I} \cong \mathcal{C} \vee \mathcal{I}$, and so the unit of $\bar{\wedge}$ will not be a unit once we pass it down to $\mathbf{PolyCpx}$. However, by applying lemma 5.1.7 we get associator and commutator natural isomorphisms, so all that we will need in order to make this structure symmetric monoidal is a unit together with natural transformations λ and ρ which make the necessary diagrams commute. It turns out that the unit we want is exactly $\mathbb{S}$; it remains to construct λ and ρ.

Consider the following two natural transformations: $\Lambda \colon (\mathbb{S}, \mathcal{I}) \bar{\wedge} \cdot \Longrightarrow 1$, and $P \colon \cdot \bar{\wedge} (\mathbb{S}, \mathcal{I}) \Longrightarrow 1$ defined as follows. For each relative pair $(\mathcal{C}, \mathcal{C}')$, $\Lambda_{(\mathcal{C}, \mathcal{C}')}$ is the polytope functor that takes each object of the form $(\emptyset, A)$ to $\emptyset$, and each object of the form $(*, A)$ to A. P is defined analogously. With these definitions it is easy to see that for all pairs $(\mathcal{C}, \mathcal{C}')$ and $(\mathcal{D}, \mathcal{D}')$ we have

$$\Lambda_{(\mathcal{C}, \mathcal{C}')} \circ \gamma_{(\mathcal{C}, \mathcal{C}')\mathbb{S}} = P_{\mathcal{C}} \qquad \text{and} \qquad (1_{(\mathcal{C}, \mathcal{C}')} \bar{\wedge} \Lambda_{(\mathcal{C}, \mathcal{C}')}) \circ \alpha_{(\mathcal{C}, \mathcal{C}')\mathbb{S}(\mathcal{D}, \mathcal{D}')} = P_{(\mathcal{C}, \mathcal{C}')} \times 1_{(\mathcal{D}, \mathcal{D}')}.$$

We define $\lambda = c\Lambda_\iota$ and $\rho = cP_\iota$.

Proposition 5.1.8. $(\mathbf{PolyCpx}, \cdot \wedge \cdot, \mathbb{S}, \alpha, \gamma, \lambda, \rho)$ *is a symmetric monoidal structure on* $\mathbf{PolyCpx}$.

Proof. All that remains to show here is to check that λ and ρ are natural isomorphisms, not just natural transformations. (This needs to be checked because Λ and P were not natural isomorphisms to start with.) However, it is easy to check that explicitly $\mathbb{S} \wedge \mathcal{C}$ is exactly $\mathcal{C}$ where every object A is replaced by a pair $(*, A)$ and every morphism is replaced by a pair where the first coordinate is the identity morphism on $*$. On this λ peels off the first coordinate, which is clearly an isomorphism. ρ is checked analogously. $\qquad\square$

To finish up this section we show that the smash product of polytope complexes preserves nice inclusions of subcomplexes. In particular, the following lemma shows that the smash product of polytope complexes preserves relative pairs of polytope complexes.

Lemma 5.1.9. *Suppose that $\mathcal{C} \hookrightarrow \mathcal{D}$ is an inclusion of polytope complexes and $\mathcal{E}$ is any polytope complex. Then if the inclusion is wide (resp. tall, has sufficiently many covers) then so is the induced inclusion $\mathcal{C} \wedge \mathcal{E} \hookrightarrow \mathcal{D} \wedge \mathcal{E}$.*

Proof. First, suppose that the inclusion is wide. This means that for any horizontal morphism $A \longrightarrow B \in \mathcal{D}$, if $B \in \mathcal{C}$ then so is A. Now consider any morphism $A \wedge E \longrightarrow B \wedge E' \in \mathcal{D} \wedge \mathcal{E}$. If $B \wedge E' \in \mathcal{C} \wedge \mathcal{E}$ this means that $B \in \mathcal{C}$, which implies that $A \in \mathcal{C}$ and thus $A \wedge E \in \mathcal{C} \wedge \mathcal{E}$; thus the inclusion is wide. Tallness is proved analogously.

Now suppose that $\mathcal{C} \hookrightarrow \mathcal{D}$ has sufficiently many covers, and consider any object $B \wedge E \in \mathcal{D} \wedge \mathcal{E}$. We know that there exists a covering family $\{B_\alpha \dashrightarrow B\}_{\alpha \in A}$ in $\mathcal{D}$ such that for each B_α there exists a horizontal morphism $C_\alpha \longrightarrow B_\alpha \in \mathcal{D}$ with $\mathcal{C}_\alpha \in \mathcal{C}$. But then the covering family $\{B_\alpha \wedge E \dashrightarrow B \wedge E\}_{\alpha \in A}$ is a covering family of $B \wedge E$ with the desired horizontal morphisms $C_\alpha \wedge E \longrightarrow B_\alpha \wedge E$. Thus the inclusion has sufficiently many covers, as desired. $\qquad\square$

5.2 The K-theory functor is monoidal

5.2.1 Waldhausen categories

The K-theory of a polytope complex is defined to be the following composition:

$$K : \mathbf{PolyCpx} \xrightarrow{\mathrm{SC}} \mathbf{WaldCat} \xrightarrow{K_{\mathrm{Wald}}} \mathbf{Sp}$$

$$\mathcal{C} \longmapsto \mathrm{SC}(\mathcal{C}) \longmapsto K(\mathcal{C})$$

The goal of this section is to show that K is a lax symmetric monoidal functor. It would be convenient if we could just show that each of K_{Wald} and SC are symmetric monoidal; however, since **WaldCat** is not a symmetric monoidal category this is not possible. But **WaldCat** is a symmetric multicategory, so for this section we will use the language of multicategories.

In order to show that K is a symmetric monoidal functor it suffices to show that both SC and K_{Wald} are multifunctors. The fact that K_{Wald} is a multifunctor is well-known; for more details, see 2.3.7 in section 2.3.2. Thus it remains to show:

Lemma 5.2.1. SC *is a multifunctor.*

Proof. In order to show that SC is a multifunctor it suffices to give, for any k-tuple of polytope complexes $\mathcal{C}_1, \dots, \mathcal{C}_k$ a k-exact functor $\varphi \colon \mathrm{SC}(\mathcal{C}_1) \times \dots \times \mathrm{SC}(\mathcal{C}_k) \longrightarrow \mathrm{SC}(\mathcal{C}_1 \wedge \dots \wedge \mathcal{C}_k)$; then for any $k+1$-tuple $\mathcal{C}_1, \dots, \mathcal{C}_k, \mathcal{D}$ and any polytope functor $F \colon \mathcal{C}_1 \wedge \dots \wedge \mathcal{C}_k \to \mathcal{D}$ we would get an induced k-exact functor

$$\mathrm{SC}(\mathcal{C}_1) \times \dots \times \mathrm{SC}(\mathcal{C}_k) \xrightarrow{\quad\varphi\quad} \mathrm{SC}(\mathcal{C}_1 \wedge \dots \wedge \mathcal{C}_k) \xrightarrow{\quad\mathrm{SC}(F)\quad} \mathrm{SC}(\mathcal{D}).$$

Supposing that φ were natural in each of the $\mathcal{C}_i$'s, this would show that SC is a multifunctor.

For any two polytope complexes $\mathcal{C}$ and $\mathcal{D}$, consider the biexact functor

$$\curlywedge : \mathrm{SC}(\mathcal{C}) \times \mathrm{SC}(\mathcal{D}) \longrightarrow \mathrm{SC}(\mathcal{C} \wedge \mathcal{D}),$$

defined as follows: given an object $A = \{a_i\}_{i \in I} \in \mathrm{SC}(\mathcal{C})$ and an object $B = \{b_j\}_{j \in J} \in \mathrm{SC}(\mathcal{D})$ we define $A \curlywedge B = \{a_i \wedge b_j\}_{(i,j) \in I \times J}$. Morphisms are induced in the obvious manner. Thus for any k-tuple of polytope complexes $\mathcal{C}_1, \dots, \mathcal{C}_k$ we have a k-exact functor

$$\mathrm{SC}(\mathcal{C}_1) \times \dots \times \mathrm{SC}(\mathcal{C}_k) \xrightarrow{\cong} ((\mathrm{SC}(\mathcal{C}_1) \times \mathrm{SC}(\mathcal{C}_2)) \times \dots) \times \mathrm{SC}(\mathcal{C}_k) \xrightarrow{((\curlywedge)\curlywedge\dots)\curlywedge}$$
$$\longrightarrow \mathrm{SC}(((\mathcal{C}_1 \wedge \mathcal{C}_2) \wedge \dots) \wedge \mathcal{C}_k)$$
$$\xrightarrow{\cong} \mathrm{SC}(\mathcal{C}_1 \wedge \dots \wedge \mathcal{C}_k).$$

As each of the composed functors is natural in each of the $\mathcal{C}_i$'s this gives the desired functor.

We also need to check that the multifunctor commutes with the Σ_k-action, but this is clear since that simply permutes the $\mathcal{C}_j$'s in both the source and target. $\qquad\square$

Thus we have shown:

Proposition 5.2.2. $K\colon \mathbf{PolyCpx} \longrightarrow \mathbf{Sp}$ *is lax symmetric monoidal.*

We can now define rings and modules in polytope complexes.

Definition 5.2.3. A *ring polytope complex* is a polytope complex $\mathcal{R}$ together with polytope functors $\mu\colon \mathcal{R} \wedge \mathcal{R} \longrightarrow \mathcal{R}$ and $\iota\colon \mathbb{S} \longrightarrow \mathcal{R}$ such that

$$
\begin{array}{ccccc}
(\mathcal{R} \wedge \mathcal{R}) \wedge \mathcal{R} & \xrightarrow{\ \alpha_{\mathcal{R}\mathcal{R}\mathcal{R}}\ } & \mathcal{R} \wedge (\mathcal{R} \wedge \mathcal{R}) & \xrightarrow{\ 1 \times \mu\ } & \mathcal{R} \wedge \mathcal{R} \\
{\scriptstyle \mu \times 1}\downarrow & & & & \downarrow{\scriptstyle \mu} \\
\mathcal{R} \wedge \mathcal{R} & & \xrightarrow{\hspace{4cm}\mu\hspace{4cm}} & & \mathcal{R}
\end{array}
$$

and

$$
\begin{array}{ccccc}
\mathbb{S} \wedge \mathcal{R} & \xrightarrow{\ \iota \wedge 1_{\mathcal{R}}\ } & \mathcal{R} \wedge \mathcal{R} & \xleftarrow{\ 1_{\mathcal{R}} \wedge \iota\ } & \mathcal{R} \wedge \mathbb{S} \\
& {\scriptstyle \lambda_{\mathcal{R}}}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle \rho_{\mathcal{R}}} & \\
& & \mathcal{R} & &
\end{array}
$$

commute. If in addition we have a polytope functor $\tau\colon \mathcal{R} \wedge \mathcal{R} \longrightarrow \mathcal{R} \wedge \mathcal{R}$ such that the $\mu \circ \tau = \mu$ then $\mathcal{R}$ is a *commutative ring polytope complex.*

As K is lax monoidal these diagrams commute in symmetric spectra with $K(\mathcal{R})$ instead of $\mathcal{R}$, which means that for a ring polytope complex $\mathcal{R}$ its K-theory is a symmetric ring spectrum.

5.2.2 Γ-spaces

Recall that a Γ-space is a functor $X\colon \mathbf{FinSet}_* \longrightarrow s\mathbf{Sets}$ that takes $\{0\}$ to the one-point simplicial set. We can assign to a Γ-space a spectrum $\mathbf{B}\Gamma$ in the following manner. Let S^1 be the pointed simplicial set with one non-degenerate simplex in each of dimensions 0 and 1, and no other non-degenerate simplices; we can identify the k-simplices of S^1 with the set $\{0, 1, \ldots, k\}$, which we will write k^+. In this, 0 will be the distinguished point. We can consider S^1 to be a functor $\Delta^{\mathrm{op}} \longrightarrow \mathbf{FinSet}_*$, so that $X \circ S^1$ is a bisimplicial set. Analogously, if we define $S^n = (S^1)^{\wedge n}$ then $X \circ S^n$ will also be a bisimplicial set. $\mathbf{B}X$ is defined to be the spectrum whose n-th space is $X \circ S^n$. In some cases, in particular the one we consider here, $\mathbf{B}X$ will be an Ω-spectrum above level 1. (See [19] or [3] for more details.) For any finite pointed set A we will write A_0 for the set $A\backslash\{*\}$.

Fix a polytope complex $\mathcal{C}$. For any finite based space A, we will write $\mathrm{SC}^A(\mathcal{C})$ for the full subcategory of $\prod_{a \in A_0} \mathrm{SC}(\mathcal{C})$ consisting of all tuples of objects $\{\{b_j\}_{j \in J_a}\}_{a \in A_0}$ such that $J_a \cap J_{a'} = \emptyset$ for all distinct $a, a' \in A_0$. For any morphism $\varphi\colon A \to A' \in \mathbf{FinSet}_*$ we define a functor

$$
\mathrm{SC}^A(\mathcal{C}) \xrightarrow{\hspace{4cm}} \mathrm{SC}^{A'}(\mathcal{C})
$$

$$
\{\{b_j\}_{j \in J_a}\}_{a \in A_0} \longmapsto \{\{b_j\}_{j \in \bigcup_{a \in \varphi^{-1}(a')} J_a}\}_{a' \in A_0'}.
$$

As $\mathrm{SC}^A(\mathcal{C}) \simeq \mathrm{SC}(\mathcal{C})^{|A_0|}$ so $\mathrm{SC}^A(\mathcal{C})$ is also a Waldhausen category. We define the Γ-space $\Gamma_\mathcal{C}$ by $\Gamma_\mathcal{C}(A) = Nw\mathrm{SC}^A(\mathcal{C})$. For ease of notation, we define $X_\mathcal{C}$ to be the functor $\mathbf{FinSet}_* \longrightarrow \mathbf{Cat}$ defined by $A \longmapsto \mathrm{SC}(\mathcal{C})^{A_0}$. When $\mathcal{C}$ is obvious from context we will omit it.

Lemma 5.2.4. *For any polytope complex $\mathcal{C}$,*

$$\mathbf{B}\Gamma_\mathcal{C} \simeq K(\mathcal{C}).$$

Proof. We know that both $\mathbf{B}\Gamma_\mathcal{C}$ and $K(\mathcal{C})$ are Ω-spectra above level 1; thus it suffices to check that they are equivalent at level 1.

We can write

$$(\mathbf{B}\Gamma_\mathcal{C})_1 = |Nw \circ X \circ S^1| \qquad \text{and} \qquad K(\mathcal{C})_1 = |Nw \circ S_n\mathrm{SC}(\mathcal{C})|,$$

so we redirect our attention to the simplicial categories $X \circ S^1$ and $[n] \longmapsto S_n\mathrm{SC}(\mathcal{C})$. From the description above we can see that $X \circ S^1$ is the functor $[n] \longmapsto \mathrm{SC}^{\{1,\dots,n\}}(\mathcal{C})$.

We have the following string of inclusions:

$$\mathrm{SC}^{\{1,\dots,n\}}(\mathcal{C}) \longrightarrow \mathrm{SC}(\mathcal{C})^n \longrightarrow S_n\mathrm{SC}(\mathcal{C}),$$

where the last inclusion maps a tuple $(A_1, \dots, A_n)$ to the object

$$A_1 \longrightarrow A_1 \amalg A_2 \longrightarrow \cdots \longrightarrow A_1 \amalg \cdots \amalg A_n.$$

These induce a string of natural transformations

$$X \circ S^1 \Longrightarrow ([n] \longmapsto \mathrm{SC}(\mathcal{C})^n) \Longrightarrow ([n] \longmapsto S_n\mathrm{SC}(\mathcal{C})).$$

That the second of these is a natural transformation is obvious, but the first requires some thinking. Given a map $A \longrightarrow A' \in \mathbf{FinSet}_*$ we have an induced functor $\mathrm{SC}^A(\mathcal{C}) \longrightarrow \mathrm{SC}^{A'}(\mathcal{C})$, which extends to a functor $\mathrm{SC}(\mathcal{C})^{|A|} \longrightarrow \mathrm{SC}(\mathcal{C})^{|A'|}$ by replacing the union of indexing sets by a disjoint union. This breaks the Γ-space structure, since disjoint union is not strictly commutative, but it does not break the simplicial structure, as the simplicial maps require only strict associativity, not commutativity. Thus in order to make this properly simplicial we need to simply make sure that we have an appropriately associative disjoint union. We can do this by considering the skeleton of $\mathbf{FinSet}$ consisting only of sets $\{1, \dots, k\}$ and defining

$$\{1, \dots, k\} \amalg \{1, \dots, \ell\} := \{1, \dots, k+\ell\}$$

with $\{1, \dots, k\}$ mapping into the first k elements, and $\{1, \dots, \ell\}$ into the last ℓ. Thus we see that the first inclusion induces a natural transformation of functors.

We know that the first natural transformation is a levelwise equivalence of simplicial categories from the discussion above. The second of these is an equivalence of bisimplicial sets after applying $Nw\cdot$ by the proof of corollary 4.6.8. Therefore the composition induces an equivalence $(\mathbf{B}\Gamma_\mathcal{C})_1 \simeq K(\mathcal{C})_1$, as desired. $\qquad\qquad\square$

It follows that an equivalent definition of the K-theory of a polytope complex is the composition

$$K_\Gamma \colon \mathbf{PolyCpx} \xrightarrow{\quad G \quad} \mathbf{\Gamma Sp} \xrightarrow{\quad \mathbf{B} \quad} \mathbf{Sp}$$

$$\mathcal{C} \longmapsto (n^+ \mapsto |w\mathrm{SC}(\mathcal{C})^n|) \longmapsto \mathbf{B}(n^+ \mapsto |w\mathrm{SC}(\mathcal{C})^n|)$$

In [15], Lydakis shows that the category of Γ-spaces is symmetric monoidal; with respect to this monoidal structure, the functor $\mathbf{B}$ is a lax symmetric monoidal functor (see [16]). The smash product of the Γ-spaces X and Y is defined to be the Γ-space $X \wedge Y$ such that for any Z maps $X \wedge Y \longrightarrow Z$ are in natural bijection with maps $X(k^+) \wedge Y(\ell^+) \longrightarrow Z(k^+ \wedge \ell^+)$ which are natural in both k^+ and ℓ^+. If we could show that G is a lax symmetric monoidal functor we would conclude that K_Γ is also a lax symmetric monoidal functor.

Lemma 5.2.5. *G is lax symmetric monoidal.*

Proof. We need to construct a morphism $G(\mathcal{C}) \wedge G(\mathcal{D}) \longrightarrow G(\mathcal{C} \wedge \mathcal{D})$. As discussed above, it is sufficient to construct maps $G(\mathcal{C})(k^+) \wedge G(\mathcal{D})(\ell^+) \longrightarrow G(\mathcal{C} \wedge \mathcal{D})(k^+ \wedge \ell^+)$, natural in both k^+ and ℓ^+.

Let $\curlywedge \colon \mathrm{SC}(\mathcal{C}) \times \mathrm{SC}(\mathcal{D}) \longrightarrow \mathrm{SC}(\mathcal{C} \wedge \mathcal{D})$ be the functor defined in the proof of lemma 5.2.1. We can define a functor (which by abusing notation we will also call $\curlywedge$) $\mathrm{SC}(\mathcal{C})^k \times \mathrm{SC}(\mathcal{D})^\ell \longrightarrow \mathrm{SC}(\mathcal{C} \wedge \mathcal{D})^{kl}$ induced by the functors

$$\mathrm{SC}(\mathcal{C})^k \times \mathrm{SC}(\mathcal{D})^\ell \xrightarrow{\pi_i \times \pi_j} \mathrm{SC}(\mathcal{C}) \times \mathrm{SC}(\mathcal{D}) \xrightarrow{\curlywedge} \mathrm{SC}(\mathcal{C} \wedge \mathcal{D})^{k\ell}_{(i,j)}$$

where $\mathrm{SC}(\mathcal{C} \wedge \mathcal{D})^{k\ell}_{(i,j)}$ is the copy of $\mathrm{SC}(\mathcal{C} \wedge \mathcal{D})$ indexed by $k(i-1) + j$ in $\mathrm{SC}(\mathcal{C} \wedge \mathcal{D})^{k\ell}$. Note that the composition

$$\mathrm{SC}(\mathcal{C})^k \vee \mathrm{SC}(\mathcal{D})^\ell \longrightarrow \mathrm{SC}(\mathcal{C})^k \times \mathrm{SC}(\mathcal{D})^\ell \xrightarrow{\curlywedge} \mathrm{SC}(\mathcal{C} \wedge \mathcal{D})^{k\ell}$$

is 0. Applying $|w \cdot |$ to this diagram we get

$$G(\mathcal{C})(k^+) \vee G(\mathcal{D})(\ell^+) \longrightarrow G(\mathcal{C})(k^+) \times G(\mathcal{D})(\ell^+) \longrightarrow G(\mathcal{C} \wedge \mathcal{D})(k^+ \wedge \ell^+),$$

with trivial composition, so we have an induced map

$$G(\mathcal{C})(k^+) \wedge G(\mathcal{D})(\ell^+) \longrightarrow G(\mathcal{C} \wedge \mathcal{D})(k^+ \wedge \ell^+)$$

and it is easy to see from the definition of this map that it is natural in both k^+ and ℓ^+, giving us a map $\mu_{\mathcal{C} \wedge \mathcal{D}} \colon G(\mathcal{C}) \wedge G(\mathcal{D}) \longrightarrow G(\mathcal{C} \wedge \mathcal{D})$. This will be exactly the monoidal structure map.

Let $\mathbb{S}_\Gamma$ be the unit of the smash product of Γ-spaces; this is the functor which assigns to n^+ the discrete space of n points. The unit of the smash product of polytope complexes is the polytope complex $\mathbb{S}$, which has $\mathrm{SC}(\mathbb{S}) \cong \mathbf{FinSet}_*$ (where the weak equivalences are the isomorphisms). Our unit map $\epsilon \colon \mathbb{S}_\Gamma \longrightarrow G(\mathbb{S})$ will be the natural transformation of functors which takes a point $x \in n^+$ to the point in $|w\mathrm{SC}(\mathcal{C})^n|$ represented by the object which is $\{\}_\emptyset$ everywhere but the x-th coordinate, and which is $\{1\}_{\{1\}}$ on the x-th coordinate. It is easy to check that this is actually a morphism of Γ-spaces.

It is easy to see that ϵ and μ satisfy the coherence axioms for a lax symmetric monoidal functor. $\square$

Thus we have shown the following:

Proposition 5.2.6. *K_Γ is a symmetric monoidal functor.*

We now have two functors $K, K_\Gamma \colon \mathbf{PolyCpx} \longrightarrow \mathbf{Sp}$. We can construct a natural transformation $\eta \colon K_\Gamma \implies K$ in the following manner. For any two functors $F \colon \mathcal{A} \longrightarrow \mathcal{B}$ and $G \colon \mathcal{A}' \longrightarrow \mathcal{B}$ we will write $F \boxtimes G$ for the induced functor $\mathcal{A} \times \mathcal{A}' \longrightarrow \mathcal{B} \times \mathcal{B} \overset{\times}{\longrightarrow} \mathcal{B}$; we will use this notation to distinguish this "box product" from the usual product of simplicial objects. We will write $F^{\boxtimes n}$ for the n-ford box product. As an added bit of notation, we define $S^{\boxtimes n}$ to be the n-simplicial pointed set defined by

$$S^{\boxtimes n}_{i_1 \cdots i_n} = S^1_{i_1} \wedge \cdots \wedge S^1_{i_n},$$

so that $S^n = |S^{\boxtimes n}|$. We can then define a functor $X \circ S^{\boxtimes n} \implies S^{(n)}_\cdot SC(\mathcal{C})$ by defining the functor

$$\xi_{i_1 \cdots i_n} \colon SC(\mathcal{C})^{\{1, \ldots, i_1\} \wedge \cdots \wedge \{1, \ldots, i_n\}} \longrightarrow S_{i_1} \cdots S_{i_n} SC(\mathcal{C})$$

to take a tuple $\{A_i\}_{i \in I}$ where $I = \{*, 1, \ldots, i_1\} \wedge \cdots \wedge \{*, 1, \ldots, i_n\}$ to the diagram where the object indexed by $((j_1, \ldots, j_n) < (k_1, \ldots, k_n))$ is

$$\coprod_{\alpha_1 = j_1}^{k_1} \cdots \coprod_{\alpha_n = j_n}^{k_n} A_{\alpha_1 \cdots \alpha_n},$$

with the obvious inclusions. Here we make sure that the indexing set of the coproduct is the union of the indexing sets (which will be disjoint by construction). The functor $\xi_{i_1 \cdots i_n}$ is the left Kan extension along the functor

$$\{1, \ldots, i_1\} \wedge \cdots \wedge \{1, \ldots, i_n\} \longrightarrow [i_1] \times \cdots \times [i_n],$$

which includes the left-hand set into the objects of the right-hand category. (This is the n-dimensional generalization of the morphism constructed in the proof of lemma 5.2.4.) Note that $\xi_{i_1 \cdots i_n}$ factors through the inclusion

$$F_{i_1} \cdots F_{i_n} SC(\mathcal{C}) \longrightarrow S_{i_1} \cdots S_{i_n} SC(\mathcal{C}).$$

Lemma 5.2.7. $\eta \colon K_\Gamma \implies K$ *is a symmetric monoidal natural transformation.*

Proof. In this discussion, we will be computing the K-theory of a polytope complex using the $F_\cdot$ construction, instead of the $S_\cdot$ construction. Because inside $SC(\mathcal{C})$ we have a well-define functorial cofiber functor that is compatible with the simplicial structure, computing the K-theories in both of these ways are (functorially) equivalent.

We need to show that the following two diagrams commute:

$$
\begin{array}{ccc}
K_\Gamma(\mathcal{C}) \wedge K_\Gamma(\mathcal{D}) & \xrightarrow{\eta_\mathcal{C} \wedge \eta_\mathcal{D}} & K(\mathcal{C}) \wedge K(\mathcal{D}) \\
\downarrow & & \downarrow \\
K_\Gamma(\mathcal{C} \wedge \mathcal{D}) & \xrightarrow{\eta_{\mathcal{C} \wedge \mathcal{D}}} & K(\mathcal{C} \wedge \mathcal{D})
\end{array}
\qquad
\begin{array}{ccc}
& \mathbb{S} & \\
\swarrow & & \searrow \\
K_\Gamma(\mathbb{S}) & \xrightarrow{\eta_\mathbb{S}} & K(\mathbb{S})
\end{array}
$$

The fact that the right-hand diagram commutes is simple: given that $K_\Gamma(\mathbb{S}) = K(\mathbb{S})$ and that the two diagonal morphisms are both ι, our chosen equivalence $\mathbb{S} \longrightarrow QS^0$, it clearly commutes.

Consider the diagram on the left. As the vertical morphisms are given by the universal property for smash products of symmetric spectra, it suffices to show that for all m and n the two morphisms $K_\Gamma(\mathcal{C})_m \wedge K_\Gamma(\mathcal{D})_n \longrightarrow K(\mathcal{C} \wedge \mathcal{D})_{m+n}$ given by going around the top and around the bottom of the diagram are the same. It suffices to show that the diagram of $m + n$-simplicial categories

$$
\begin{array}{ccc}
(X_{\mathcal{C}} \circ S^{\boxtimes m}) \boxtimes (X_{\mathcal{D}} \circ S^{\boxtimes n}) \longrightarrow F_{\cdot}^{(m)}\mathrm{SC}(\mathcal{C}) \boxtimes F_{\cdot}^{(n)}\mathrm{SC}(\mathcal{D}) \longrightarrow S_{\cdot}^{(m)}\mathrm{SC}(\mathcal{C}) \boxtimes S_{\cdot}^{(n)}\mathrm{SC}(\mathcal{D}) \\
\downarrow \qquad\qquad\qquad\qquad \downarrow \qquad\qquad\qquad\qquad \downarrow \\
X_{\mathcal{C} \wedge \mathcal{D}} \circ S^{\boxtimes (m+n)} \longrightarrow F_{\cdot}^{(m+n)}\mathrm{SC}(\mathcal{C} \wedge \mathcal{D}) \longrightarrow S_{\cdot}^{(m+n)}\mathrm{SC}(\mathcal{C} \wedge \mathcal{D})
\end{array}
$$

commutes; the diagram we want can be obtained by applying $|Nw\cdot|$ to this one and checking that the diagonal of the outside commutative square factors through the smash product. Note that the right-hand square clearly commutes, so it suffices to consider the left-hand square. Consider this at level $[k_1] \times \cdots \times [k_m] \times [\ell_1] \times \cdots \times [\ell_n]$. Recall that k_0^+ is the pointed set k^+ without the basepoint, so it is the set $\{1, \ldots, k\}$. We can write the left-hand square as

$$
\left[\prod_{i=1}^{m}(k_i)_0^+ \times \prod_{j=1}^{n}(\ell_j)_0^+, \mathrm{SC}(\mathcal{C}) \times \mathrm{SC}(\mathcal{D}) \right] \xrightarrow{\xi_{k_1\cdots k_m} \times \xi_{\ell_1\cdots \ell_n}} \left[\prod_{i=1}^{m}[k_i] \times \prod_{j=1}^{n}[\ell_j], \mathrm{SC}(\mathcal{C}) \times \mathrm{SC}(\mathcal{D}) \right]
$$
$$
\downarrow \curlywedge \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \curlywedge
$$
$$
\left[\prod_{i=1}^{m}(k_i)_0^+ \times \prod_{i=1}^{n}(\ell_j)_0^+, \mathrm{SC}(\mathcal{C} \wedge \mathcal{D}) \right] \xrightarrow{\xi_{k_1\cdots k_m \ell_1\cdots \ell_n}} \left[\prod_{i=1}^{m}[k_i] \times \prod_{j=1}^{n}[\ell_j], \mathrm{SC}(\mathcal{C} \wedge \mathcal{D}) \right]
$$

For this to commute, we need to show that for $A_i \in \mathrm{SC}(\mathcal{C})$ and $B_j \in \mathrm{SC}(\mathcal{D})$, we have

$$
\left(\coprod_{i \in I} A_i \right) \curlywedge \left(\coprod_{j \in J} B_j \right) = \coprod_{(i,j) \in I \times J} A_i \curlywedge B_j.
$$

Write $A_i = \{a_\iota\}_{\iota \in I_i}$ and $B_j = \{b_\gamma\}_{\gamma \in J_j}$; then we can rewrite the above equation as

$$
\{a_\iota \wedge b_\gamma\}_{(\iota,\gamma) \in \coprod_{i \in I} I_i \times \coprod_{j \in J} J_j} = \{a_\iota \wedge b_\gamma\}_{(\iota,\gamma) \in \coprod_{(i,j) \in I \times J} I_i \times J_j}.
$$

Thus we simply need to show that $\coprod_{i \in I} I_i \times \coprod_{j \in J} J_j = \coprod_{(i,j) \in I \times J} I_i \times J_j$. This is not necessarily the case: consider the case where $I = J = A_i = B_j = \{1,2\}$. However, as all of the indexing sets from the A_i's come from a Γ-space, we know that $I_i \cap I_{i'} = \emptyset$, and similarly for the J_j's. This means that instead of disjoint unions we can take ordinary unions in the equation above; in that case, the two sets are always equal, and we are done.

The conditions for morphisms are the same, and are checked analogously. $\qquad\square$

5.3 Examples

5.3.1 Euclidean Ring Structure

The polytope complex $\mathcal{G}_{E^m} \wedge \mathcal{G}_{E^n}$ includes into the polytope complex $\mathcal{G}_{E^{m+n}}$ by writing $E^{m+n} \cong E^m \times E^n$ and mapping a pair (P_m, P_n) to the polytope $P_m \times P_n$. By extending this to the vertical and the horizontal morphisms this extends to a polytope functor $\mathcal{E}_m \wedge \mathcal{E}_n \longrightarrow \mathcal{E}_{m+n}$. We define $\mathcal{G}_E = \bigvee_{n \geq 0} \mathcal{G}_{E^n}$; then these polytope functors extend to a multiplication $\mu_E \colon \mathcal{G}_E \wedge \mathcal{G}_E \longrightarrow \mathcal{G}_E$. We can also define a unit $\iota_E \colon \mathbb{S} \longrightarrow \mathcal{G}_E$ by taking the polytope in $\mathbb{S}$ to the single 0-polytope in E^0.

As a side note, note that $\mathcal{G}_{E^m} \wedge \mathcal{G}_{E^n}$ is *not* a subcomplex of $\mathcal{G}_{E^{m+n}}$, as there are families in $\mathcal{G}_{E^{m+n}}$ which are covering families in $\mathcal{G}_{E^{m+n}}$ but not covering families in $\mathcal{G}_{E^m} \wedge \mathcal{G}_{E^n}$. The easiest way to see this is to consider the case when $m = n = 1$. Consider the following family of morphisms:
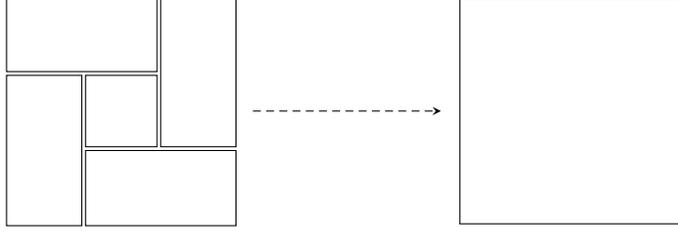
Inside $\mathcal{G}_{E^2}$ this is a covering family, but inside $\mathcal{G}_{E^1} \wedge \mathcal{G}_{E^1}$ this is not, as it cannot be written as a finite composition of refinements in each direction. However, it can be refined to a covering family inside $\mathcal{G}_{E^1} \wedge \mathcal{G}_{E^1}$ by decomposing along the dotted lines:
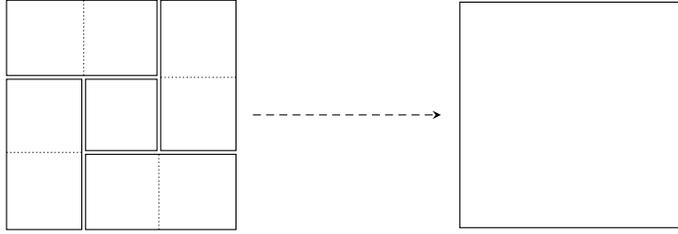
Using this we can check that even though $\mathcal{G}_{E^m} \wedge \mathcal{G}_{E^n}$ is not a subcomplex of $\mathcal{G}_{E^{m+n}}$, $K(\mathcal{G}_{E^m} \wedge \mathcal{G}_{E^n})$ is equivalent to the K-theory of the subcomplex which is the image of $\mathcal{G}_{E^m} \wedge \mathcal{G}_{E^n}$ inside $\mathcal{G}_{E^{m+n}}$.

5.3.2   Spherical Ring Structure

In the spherical geometry case, we also have polytope functors $\mathcal{G}_{S^m} \wedge \mathcal{G}_{S^n} \longrightarrow \mathcal{G}_{S^{m+1+n}}$, given in the following manner. Consider S^n as a subset of $\mathbb{R}^{n+1}$, and a polytope in S^n as a solid cone of rays inside $\mathbb{R}^{n+1}$. We have an inclusion $\mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^{n+1+m+1}$. Then the polytope functor $\mathcal{G}_{S^m} \wedge \mathcal{G}_{S^n} \longrightarrow \mathcal{G}_{S^{m+1+n}}$ is given by mapping a pair of a solid cone $P \subseteq \mathbb{R}^{m+1}$ and a solid cone $Q \subseteq \mathbb{R}^{n+1}$ to the solid cone $P \times Q \subseteq \mathbb{R}^{m+1+n+1}$. (Another way of thinking about this is to include $S^m \subseteq S^{m+1+n}$ as the first m coordinates, S^n as the last n coordinates, and take the join inside S^{m+1+n} of the two polytopes. We then define $\mathcal{G}_S = \mathbb{S} \vee \bigvee_{n \geq 0} \mathcal{G}_{S^n}$, and we have a multiplication $\mu_S \colon \mathcal{G}_S \wedge \mathcal{G}_S \longrightarrow \mathcal{G}_S$. The unit is the formal inclusion of $\mathbb{S}$ as the extra $\mathbb{S}$ factor.

5.3.3   The Burnside Category

Suppose that we have a finite group G. For finite G-sets S and T we define the polytope complex $\mathbf{Burn}_G(S, T)$ to be the polytope complex with

objects diagrams of finite G-sets $S \xleftarrow{f} A \xrightarrow{g} T$, denoted (A, f, g)

morphisms vertical (resp. horizontal) morphisms are diagrams

where the morphism $A \hookrightarrow B$ is an injective G-set map (resp. an isomoprhism).

75

vertical topology a collection of morphisms $\{(A_i, f_i, g_i) \dashrightarrow (A, f, g)\}_{i \in I}$ is a covering family if $\bigcup_{i \in I} A_i = A$.

The ordinary Burnside category $\mathcal{B}_G$ is defined to have as its objects finite G-sets, and as its morphisms $S \longrightarrow T$ the abelian group generated by spans $S \xleftarrow{f} A \xrightarrow{g} T$ of finite G-sets under a scissors congruence type reulation (for more on this, see, for example [14] or [10]). In fact, it is easy to see that $K_0(\mathbf{Burn}_G(S, T)) = \mathcal{B}_G(S, T)$. We can define the enriched Burnside category $\mathbf{Burn}_G$ to have the objects the finite G-sets, and morphism polytope complexes $\mathbf{Burn}_G(S, T)$.

The composition in $\mathcal{B}_G$ is defined on generators by defining the composition of $(A, f, g) : S \longrightarrow T$ and $(B, f', g') : T \longrightarrow U$ to be the span $(A \times_T B, f \circ \pi_A, g' \circ \pi_B) : S \longrightarrow U$. We extend this definition to polytope complexes by defining a polytope functor

$$\mathbf{Burn}_G(T, U) \wedge \mathbf{Burn}_G(S, T) \longrightarrow \mathbf{Burn}_G(S, U)$$

to be the above pullback on objects, and the obvious extension on morphisms. With these definitions $\mathbf{Burn}_G$ is a category enriched over $\mathbf{PolyCpx}$, and therefore (as the K-theory functor is symmetric monoidal) over spectra.

5.4 Simplicial Enrichment

In the previous chapter we considered, not just polytope complexes, but simplicial polytope complexes. We can clearly make the category of simplicial polytope complexes a closed symmetric monoidal category as well, simply from the structure on polytope complexes. Given any pointed set S (with distinguished basepoint $*$), we can construct a free polytope complex on S, denoted FS, by defining

objects: S,

vertical morphisms: the morphisms $* \dashrightarrow s$ for all $s \in S$,

horizontal morphisms: the trivial morphisms, and

topology: the discrete topology.

It is easy to check that F is a functor $\mathbf{Sets}_* \longrightarrow \mathbf{PolyCpx}$. Note that F has a right adjoint, the forgetful functor U that takes a polytope complex to its set of objects.

Lemma 5.4.1. *The adjuction $F \colon \mathbf{Sets}_* \rightleftarrows \mathbf{PolyCpx} \colon U$ is an adjunction of strict monoidal functors.*

Proof. In order to check that F and U are adjoints we will construct a unit and a counit. Note that UF is the identity functor, so our unit will be the identity natural transformation. The counit, $\epsilon \colon FU \longrightarrow 1_{\mathbf{PolyCpx}}$ will be the morphism which is the identity everywhere. Since $FU\mathcal{C}$ is naturally a subcomplex of $\mathcal{C}$ this will be well-defined. It is easy to check that these satisfy the conditions to make F left adjoint to U.

This is clear directly from the definitions of $\wedge$ in pointed sets and $\mathbf{PolyCpx}$. The only important thing to note is that the smash product of two discrete topologies is also discrete. $\square$

Pick a weak equivalence $\iota \colon S^0 \longrightarrow K(\mathbb{S})$. Note that given any pointed set S we have a weak equivalence

$$\Sigma^\infty S \xrightarrow{\ \cong\ } \bigvee_{S\backslash *} S^0 \xrightarrow{\ \iota\ \sim\ } \bigvee_{S\backslash *} K(\mathbb{S}) \xrightarrow{\ \cong\ } K\Big(\bigvee_{S\backslash *} \mathbb{S}\Big) \xrightarrow{\ \cong\ } K(FS)$$

Thus, up to weak equivalence, $K \circ F \simeq \Sigma^\infty$. Considering pointed simplicial sets and simplicial polytoe complexes, we see that the diagram

$$
\begin{array}{ccc}
s\mathbf{Sets}_* & \xrightarrow{\quad F \quad} & s\mathbf{PolyCpx} \\
& {\scriptstyle \Sigma^\infty}\searrow \quad \underset{\mathbf{Sp}}{} \quad \swarrow {\scriptstyle K} &
\end{array}
$$

also commutes up to natural weak equivalence, which we will call ι_*. (Here we are using the fact that $\mathrm{hocolim}_{\Delta^{\mathrm{op}}} QS^0 X_n = \Sigma^\infty X$..) We will use the functor F to induce a simplicial structure on $s\mathbf{PolyCpx}$.

In particular, we will follow lemma II.2.4 in [8].

Lemma 5.4.2. *The functor* $\cdot \otimes \cdot : s\mathbf{PolyCpx} \times s\mathbf{Sets} \longrightarrow s\mathbf{PolyCpx}$ *given by*

$$\mathcal{C}. \otimes K = \mathcal{C}. \wedge F(K_+)$$

makes $s\mathbf{PolyCpx}$ *into a simplicial category.*

Proof. We need to check the three conditions for the functor. First we want to show that for a fixed $K \in s\mathbf{Sets}$ the functor $\cdot \otimes K$ has a right adjoint. We know, however, that for simplicial polytope complexes $\mathcal{C}.$ and $\mathcal{D}.$ we have

$$\mathrm{Hom}(\mathcal{C}. \otimes K, \mathcal{D}.) \cong \mathrm{Hom}(\mathcal{C}. \wedge F(K_+), \mathcal{D}.) \cong \mathrm{Hom}(\mathcal{C}., \mathcal{D}.^{F(K_+)})$$

so the functor $\mathcal{D} \longmapsto \mathcal{D}^{F(K_+)}$ is clearly a right adjoint to $\cdot \otimes K$. Note that for a simplicial set K and simplicial polytope complex $\mathcal{C}.$ we can now define two different mapping objects $\mathcal{C}.^K$: the mapping object defined through the simplicial structure, and the internal mapping object defined from the monoidal structure on $s\mathbf{PolyCpx}$. The above formula shows that they agree, so we have a well-defined notion of mapping object.

Now we want to check that the functor $\mathcal{C}. \otimes \cdot$ commutes with arbitary colimits and $A \otimes * \cong A$. The second part is clear, since $F(*) \cong \mathbb{S}$, the unit of the monoidal structure. To prove the first part we will show that $\mathcal{C}. \otimes \cdot$ is a left adjoint, which means that it obviously commutes with all colimits. For a fixed $\mathcal{C}. \in s\mathbf{PolyCpx}$ we have that $\mathcal{C}. \otimes \cdot$ is the composition

$$s\mathbf{Sets} \longrightarrow s\mathbf{Sets}_* \xrightarrow{\ F\ } s\mathbf{PolyCpx} \xrightarrow{\ \mathcal{C}. \wedge \cdot\ } s\mathbf{PolyCpx}.$$

It suffices to show that this is a left adjoint levelwise. The first of these is clearly a left adjoint; the second is an adjoint by lemma 5.4.1. The third is a left adjoint because $\mathbf{PolyCpx}$ is a closed symmetric monoidal category. Thus $\mathcal{C}. \otimes \cdot$ commutes with all colimits, as desired.

Lastly we want to show that for fixed simplicial sets K, L we have a natural isomorphism $\mathcal{C}. \otimes (K \times L) \longrightarrow (\mathcal{C}. \otimes K) \otimes L$. As before, it suffices to show that for each n we have a natural isomorphism $\mathcal{C}_n \otimes (K_n \times L_n) \longrightarrow (\mathcal{C}_n \otimes K_n) \otimes L_n$. It is easy to check this from the definition. . $\quad\square$

We can now define a simplicial enrichment for **PolyCpx** by defining

$$\mathbf{Hom}(\mathcal{C}_\cdot, \mathcal{D}_\cdot)_n = \mathrm{Hom}(\mathcal{C}_\cdot \otimes \Delta^n, \mathcal{D}_\cdot).$$

Note that for all simplicial polytope complexes $\mathcal{C}_\cdot$ and simplicial sets L we have a natural transformation

$$\eta_{\mathcal{C}_\cdot, L} \colon K(\mathcal{C}_\cdot) \otimes L \Longrightarrow K(\mathcal{C}_\cdot \otimes L)$$

given by the composition

$$K(\mathcal{C}_\cdot) \wedge \Sigma_+^\infty L \xrightarrow{1 \wedge (\iota_*)_L} K(\mathcal{C}_\cdot) \wedge K(F(L_+)) \xrightarrow{\mu} K(\mathcal{C}_\cdot \wedge F(L_+))$$

Note that η is associative, in the sense that the following diagram, and the corresponding diagram with the isomorphisms inverted, commutes for all simplicial sets L and L':

$$
\begin{array}{ccc}
(K(\mathcal{C}_\cdot) \otimes L) \otimes L' & \xrightarrow{\quad\quad\quad\quad \eta \quad\quad\quad\quad} & K(\mathcal{C}_\cdot \otimes L) \otimes L' \\
\cong \downarrow & & \downarrow \eta \\
K(\mathcal{C}_\cdot) \otimes (L \times L') \xrightarrow{\eta} K(\mathcal{C}_\cdot \otimes (L \times L')) & \xrightarrow{\cong} & K((\mathcal{C}_\cdot \otimes L) \otimes L')
\end{array}
$$

Using this we can now show that not only is K lax monoidal, but it is compatible with the simplicial enrichment.

Proposition 5.4.3. $K \colon s\mathbf{PolyCpx} \longrightarrow \mathbf{Sp}$ *is a simplicially enriched functor.*

Proof. We need to show that K gives a map $\mathbf{Hom}(\mathcal{C}_\cdot, \mathcal{D}_\cdot) \longrightarrow \mathbf{Hom}(K(\mathcal{C}_\cdot), K(\mathcal{D}_\cdot))$. For each $n \geq 0$ we have a map

$$
\omega \colon \mathrm{Hom}(\mathcal{C}_\cdot \wedge F(\Delta_+^n), \mathcal{D}_\cdot) \xrightarrow{\quad K \quad} \mathrm{Hom}(K(\mathcal{C}_\cdot \wedge F(\Delta_+^n)), K(\mathcal{D}_\cdot))
$$
$$
\xrightarrow{\circ \eta_{\mathcal{C}, \Delta^n}} \mathrm{Hom}(K(\mathcal{C}_\cdot) \wedge \Sigma_+^\infty \Delta^n, K(\mathcal{D}_\cdot)).
$$

As all of the simplicial maps interact only with the Δ^n in the equations, and all of the morphisms are natural in all three variables, we see that this induces a map of simplicial sets, as desired.

It remains to check that this map is coherent with composition and identities. In order to check composition we need to check that for any n and any $\mathcal{C}_\cdot$, $\mathcal{D}_\cdot$ and $\mathcal{E}_\cdot$ the diagram

$$
\begin{array}{ccc}
\mathrm{Hom}(\mathcal{D}_\cdot \otimes \Delta^n, \mathcal{E}_\cdot) \times \mathrm{Hom}(\mathcal{C}_\cdot \otimes \Delta^n, \mathcal{D}_\cdot) & \xrightarrow{\quad\quad \circ \quad\quad} & \mathrm{Hom}(\mathcal{C}_\cdot \otimes \Delta^n, \mathcal{E}_\cdot) \\
\omega \times \omega \downarrow & & \downarrow \omega \\
\mathrm{Hom}(K(\mathcal{D}_\cdot) \otimes \Delta^n, K(\mathcal{E}_\cdot)) \times \mathrm{Hom}(K(\mathcal{C}_\cdot) \otimes \Delta^n, K(\mathcal{D}_\cdot)) & \xrightarrow{\circ} & \mathrm{Hom}(K(\mathcal{C}_\cdot) \otimes \Delta^n, K(\mathcal{E}_\cdot))
\end{array}
$$

commutes. In particular, (after transposing the diagram) we need that for any polytope functor $F \colon \mathcal{C}_\cdot \otimes \Delta^n \longrightarrow \mathcal{D}_\cdot$, the outside of the following diagram commutes:

$$
\begin{array}{ccccccc}
K(\mathcal{C}_\cdot) \otimes \Delta^n & \xrightarrow{1 \otimes \Delta} & K(\mathcal{C}_\cdot) \otimes (\Delta^n \times \Delta^n) & \longrightarrow & (K(\mathcal{C}_\cdot) \otimes \Delta^n) \otimes \Delta^n & \xrightarrow{K(F)\eta \otimes 1} & K(\mathcal{D}_\cdot) \otimes \Delta^n \\
\eta \downarrow & & \eta \downarrow & & \eta \circ \eta \downarrow & & \downarrow \eta \\
K(\mathcal{C}_\cdot \otimes \Delta^n) & \xrightarrow{K(1 \otimes \Delta)} & K(\mathcal{C}_\cdot \otimes (\Delta^n \times \Delta^n)) & \longrightarrow & K((\mathcal{C}_\cdot \otimes \Delta^n) \otimes \Delta^n) & \xrightarrow{K(F \otimes 1)} & K(\mathcal{D}_\cdot \otimes \Delta^n)
\end{array}
$$

Going around the top of this diagram corresponds to going around the top of the Hom-diagram, and going around the bottom corresponds to going around the bottom. The left- and right-hand squares commute because η is a natural transformation. The middle square commutes because η is associative. $\qquad\square$

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